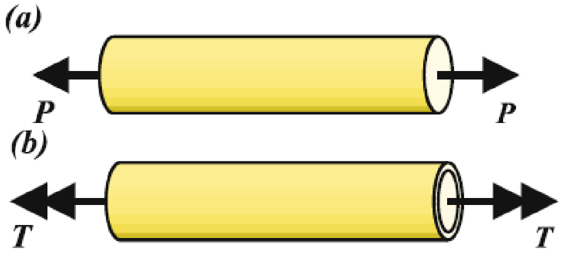
Chapter 3. Strain and Stress - - الفصل الثالث التشوه والإجهاد

3.0 Introduction

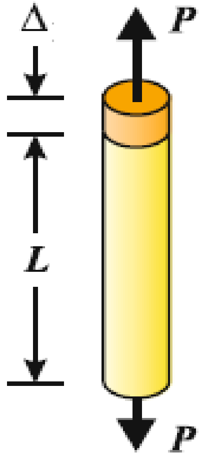
Strain (التشوه) and stress (الإجهاد) allow to judge if a component is stiff enough (not deflect too much) or strong enough (permanently deform, or break into two parts) for its intended application.

Two basic tests help to define strain and stress, and to determine the related material properties:



- The *tension* test: a straight uniform bar is subjected to an axial force
- The *torsion* test: a straight thin circular shaft is subjected to an axial torque.

3.1 The Tension Test – Axial Properties



L, bar's length; *A*, cross-sectional area; *P*, axial applied force; *Δ*, elongation.

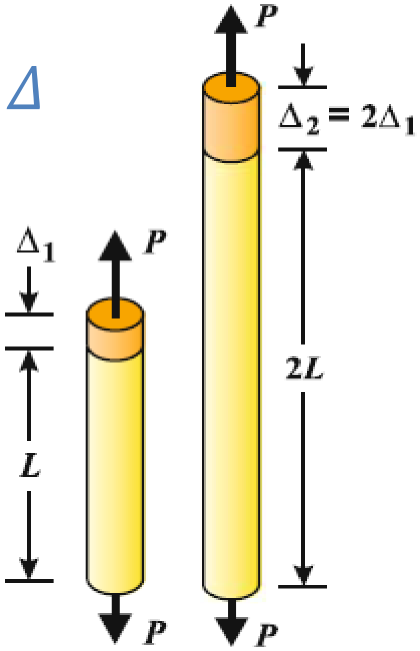
When P is proportional to Δ $\longrightarrow$ response is ***linear***. خطي

If $\Delta = 0$ when P is removed $\longrightarrow$ response is ***Elastic***. مرن

A ***linear– elastic*** response is observed in most materials for small loads and elongations.

The force–elongation (P–Δ) response of the test is generalized by defining the terms strain and stress.

Strain : The ***axial strain*** ε – parallel to the axis – is the ratio of the bar's extension Δ to its original length L. Hence, $\varepsilon = \Delta/L$



It is called the ***normal strain*** since it describes movement normal to the bar's cross-section. ***Engineering strain*** $(\Delta << L)$.

Strain is *dimensionless*. Δ & L are of the same dimension. A typical value is on the order of $\varepsilon = 0.0005$.

Strain is generally expressed as a percentage, 0.05%. Another form is $\varepsilon = 500 \times 10^{-6}$ or 500 micro-strain.

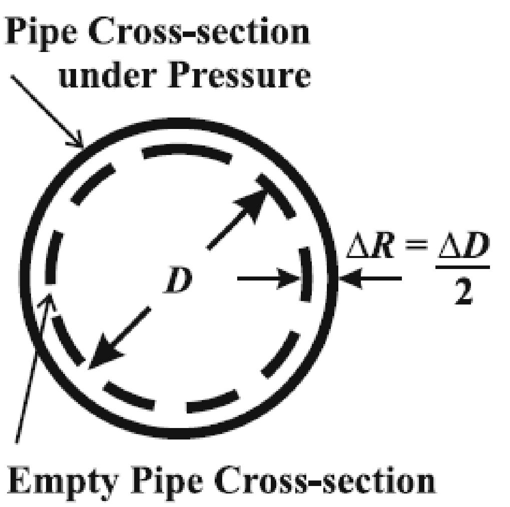
If the length of the bar is doubled, the same load P will elongate the bar by 2Δ. Applying the definition of strain gives: $\varepsilon = 2\Delta/2L = \Delta/L$

If the bar elongates $(\Delta > 0)$, the strain is positive $(\varepsilon > 0)$. The bar is said to be in tension and the force is a tensile force $(P > 0)$.

If the bar shortens $(\Delta < 0)$, the strain is negative $(\varepsilon < 0)$. The bar is said to be in compression and the force is a compressive force $(P < 0)$.

Ex. 3.1 Strain in a Pipe

Given: The Trans-Alaska Pipeline, diameter D=1.22 m, transports oil under pressure. The pressure causes an increase in pipe diameter.
Field measurements at a certain location show that the circumferential strain – the strain around the pipe circumference – is $\varepsilon = 0.05\%$%.

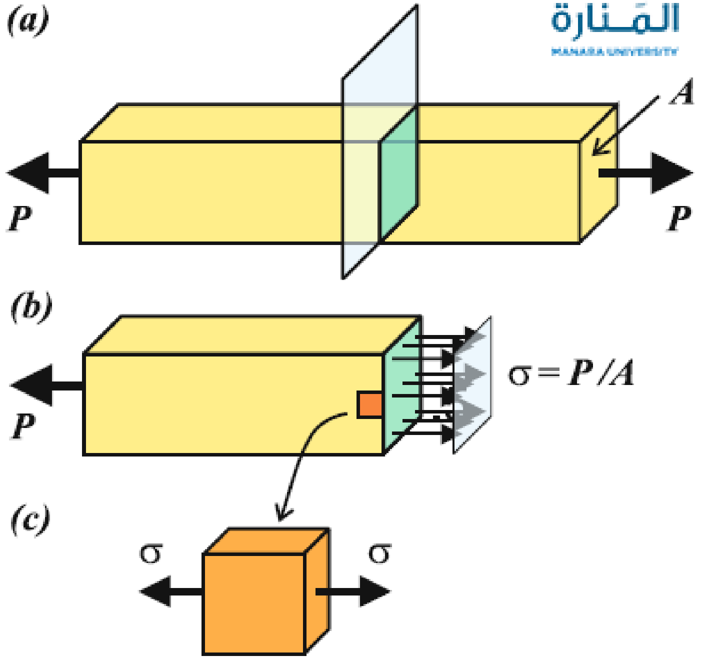


Required: Determine the change in pipe diameter ΔD.

Solution:

Answer : $\Delta D = 0.61$ mm

Stress



The bar in *Fig.* has constant cross-sectional area A (need not be square), is subjected to axial force P, but the load must act through the centroid of the cross-section so that the response at any cross-section, far from the ends, is *uniform* (the same over the entire cross-sectional area).

The ***axial stress or normal stress σ***, in a bar is the axial force divided by the cross-sectional area over which the force acts. Thus, $\sigma = \frac{P}{A}$

In *tension, force and stress* are both *positive* ($P > 0, \sigma > 0$)

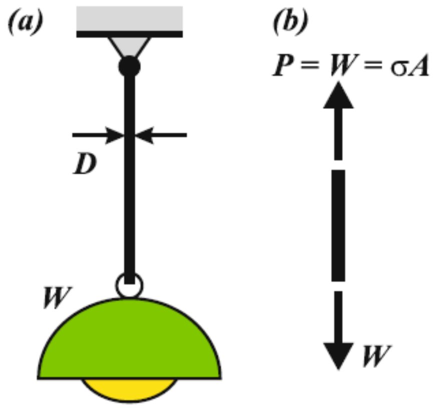
In *compression, the force and stress are* both *negative* ($P < 0, \sigma < 0$)

The dimensions of stress are force per unit area. In SI units, the unit of stress is newton per square meter, called a Pascal (Pa).

In practical situations, stresses are typically on the order of 100×10^6 Pa. For convenience, stress is generally given in *megapascals,* MPa.

Ex. 3.2 Hanging Lamp

Given: A lamp weighing W= 50 N hangs from the ceiling by a steel wire of diamete D= 2.5 mm. (*Fig*.).

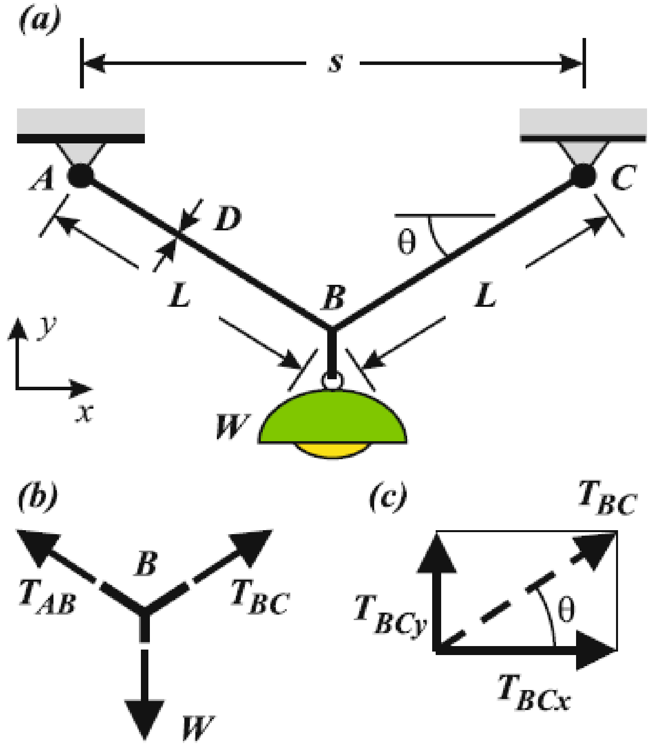


Required: Determine the stress in the wire.

Solution:

Ex. 3.3 Lamp Hanging by Two Wires

Given: A lamp of weight W hangs by two wires, each of length L (Fig.). Each wire has a cross-sectional area A and makes an angle of θ with the horizontal.

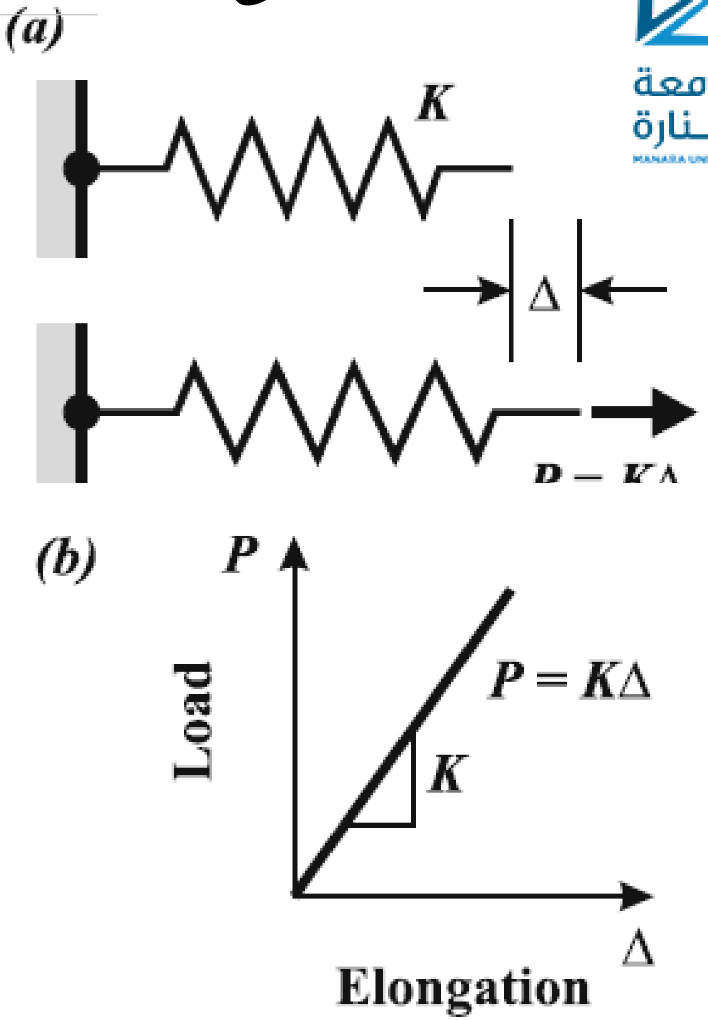


Required: Determine the stress σ in each wire. In terms of W, A, θ .

Then use the same values as in Ex. 3.2. and θ =15º.

Solution:

Young's Modulus



Stiffness : is a measure of a system's resistance to deformation. A rubber band is easily deformed by hand; but a steel wire or rod of the same cross-sectional area is not. The stiffness of steel is greater than that of rubber.

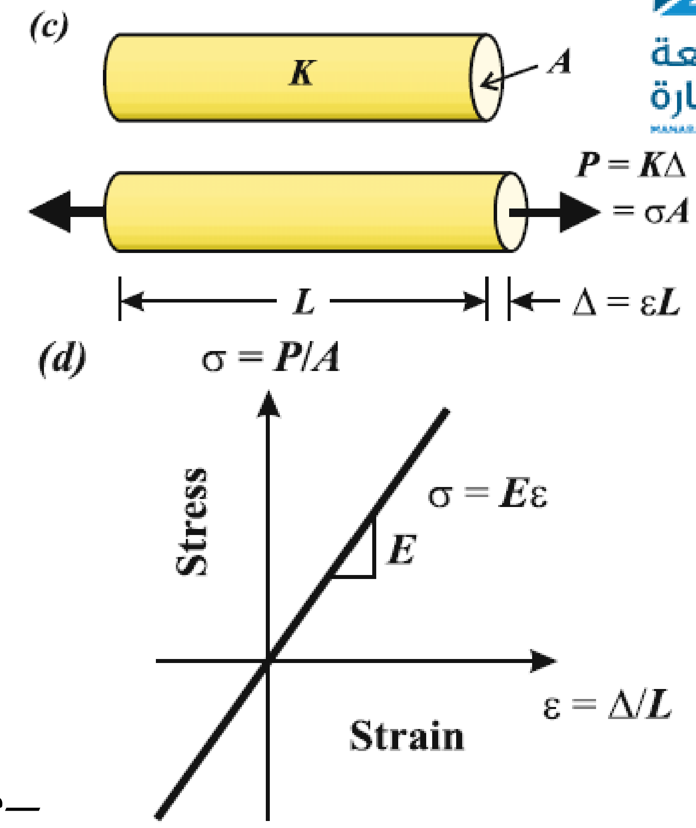
Tensile force P applied to a linear spring causes it to elongate by Δ (*Fig. a*). When the measured values of force and elongation are plotted on a graph, the result is a straight line (*Fig. b*). The line is defined by *Hooke's Law for Springs:* $P = K\Delta$

where K is the spring constant or spring **stiffness**. ***الصلابة***

If the elongation (displacement) is too large, this linear relationship is no longer valid, which can be demonstrated by stretching a spring so much that it does not return to its original shape.

Tensile force P applied to a bar of initial cross–sectional area A and initial length L causes it to elongate by Δ (*Fig. c*).

For small displacements, the bar behaves like a linear spring, i.e., $P = K\Delta$.



Dividing force by area and elongation by length gives the stress and strain:

$\sigma = P/A$ and $\varepsilon = \Delta/L$.

The plot in (*Fig. d*) is known as a ***stress–strain*** curve.

Since A and L are constants, the stress–strain curve is also linear for small displacements.

By normalizing (dividing) force by area, and elongation by length, the *force–elongation*, P–Δ curve becomes a *stress–strain* curve σ–ε.

The σ–ε curve is independent of bar size and depends only on the material.

The *slope* E of the σ–ε curve is the stiffness of the material. It is called ***Young's modulus, the elastic modulus, or the modulus of elasticity***.

***Hook's Law for axial stress-strain is*:** $\sigma = E\varepsilon$

Since strain ε is dimensionless, Young's modulus has the same units as stress σ (Pa=N/m^2).

Representative moduli are given in the next *Table*. The modulus is generally large and expressed in SI units as GPa (gigapascals, 1 GPa =10^9 Pa)

In *compression* ($\sigma < 0$), for metals the initial material response is generally linear with a slope of E.

Young's modulus depends on the nature of the atomic bonds of the material.

Although there are many types of steels, all have a modulus of about 190–215 GPa, steel is primarily iron with a small amount of carbon and a few additional elements.

The amorphous nature of rubbers & polymers, and their various degrees of atomic bonding, results in the low & relatively widespread values of their moduli.

Engineering ceramics have covalent and ionic bonds – the strongest atomic bonds – so their moduli are high.

Material	*E* (SI) Gpa=10^9 Pa
Steels	207
Titanium alloys	115
Aluminum alloys	70
Nickel Alloys	215
Cast irons	180
Douglas fir (parallel to grain)	12.4
Glass	70
Rubbers	0.01-0.1
Polymers	0.1-5
Engineering ceramics	300-450
Carbon fiber/polymer matrix composite	70-200

Because steel has a large modulus, it is used in buildings and structures where deflections must be small. The modulus of aluminum (70 GPa) is one-third that of steel. If aluminum is used in place of steel with the same geometry, the resulting strains and deflections would be three times greater for the same loads. The designer must be aware of this response.

The values of *E* for rubbers and polymers are very small; these materials are seldom used to support large loads as they would have large deflections. Rubber is often used as a cushion. Polymers are used to cover automobile panels and to provide protective enclosures for electronic components.

In high-tech aerospace applications, polymers are used to keep high-strength/high-stiffness fibers in alignment and to protect them. Such a combination of materials is called a *composite*. One class of composites is carbon fiber reinforced polymers (*CFRP*), which have high elastic moduli but low density. This combination of properties is very attractive in aerospace applications, but composite materials are generally expensive due to the cost of manufacture

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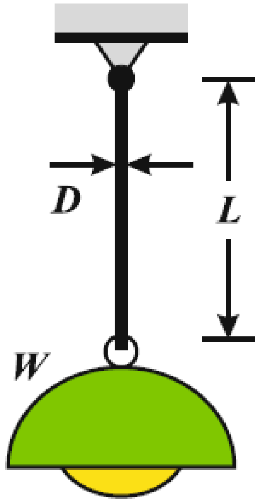
Ex. 3.4 Displacement of Hanging Lamp Due to Weight



Given: The wire supporting the lamp in Ex. 3.2 *is* L =1.5 m long (*Fig.*). The wire is steel with elastic modulus E = 207 GPa.

Required: Determine the elongation Δ of the wire due to the lamp's weight.

Solution:

Stiffness and Flexibility of an Axial Member

The Young's modulus E may be used to calculate the *stiffness* and *flexibility* of any axial bar. Consider a bar of constant cross-sectional area A, length L, and modulus E, subjected to load P applied at each end through the centroid of the cross-section. The bar's *stiffness* K is found using the definitions of stress & strain:

$$P = \sigma A = (E\varepsilon)A = E\,(\Delta / L)\,A = (EA/L)\Delta = K\Delta$$

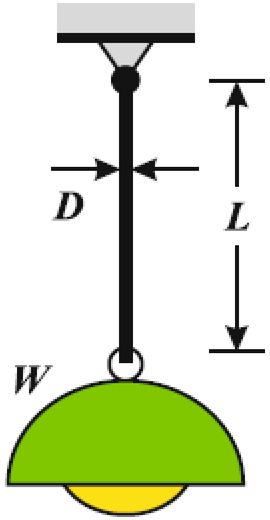
The *stiffness* of an axial bar relating P to D, is therefore: $K = EA/L$

The last equation can be rearranged to solve for displacement in terms of the applied load. The inverse of *stiffness* K is the *flexibility* f, defined by:

$$\Delta = P/K = f\,P = PL/EA \Rightarrow f = 1/K = L/EA$$

The terms *stiffness* and *flexibility* are commonly used in practice. In racing car suspensions, stiffness is high to avoid excessive sway in tight turns at high speed. By contrast, highway riding requires comfort and a more flexible suspension, meaning a lower stiffness or higher flexibility.

Ex. 3.5 Stiffness and Flexibility of an Axial Member

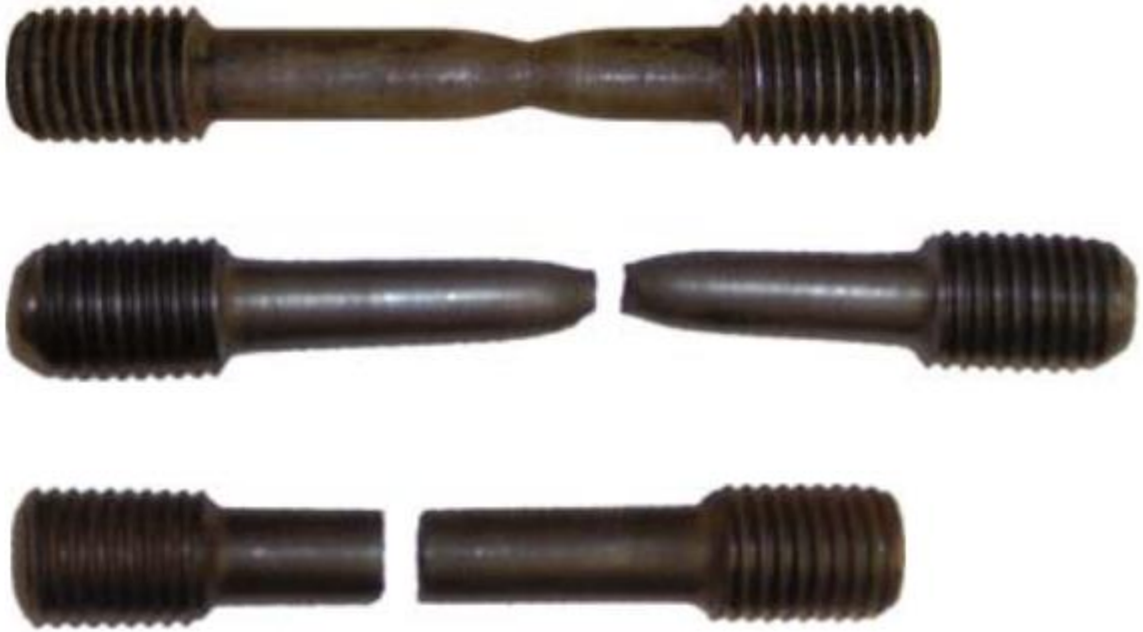


Given: The lamp from *Ex. 3.2* and *3.4* (*Fig.*).

Required:

1. Determine the stiffness K and flexibility f of the 1.5 m long steel wire.
2. Determine the load P needed to produce a displacement D=1.25 mm.
3. What is the displacement if P=50 N.

1. Stress–Strain Curves for Ductile Materials



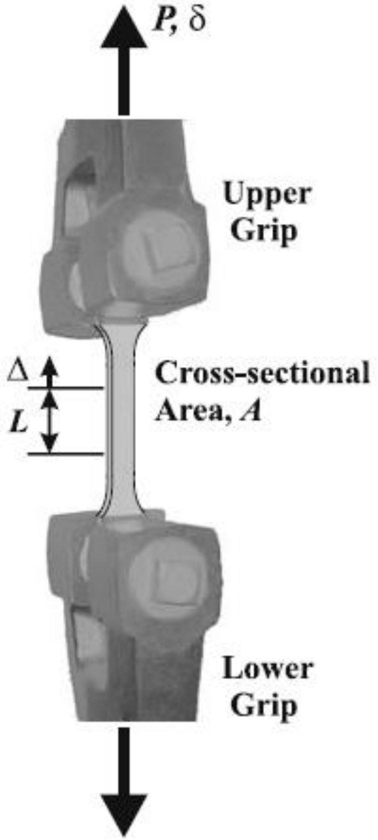
Stress–Strain Curves for Ductile Materials

Stress–strain curves are determined from experiments on axial bars (*Fig.*).

A bar with cross-sectional area A is clamped into a set of grips of a testing machine. One grip is fixed and the other moves by known displacement δ ; the force P required to cause the displacement is measured. Such an experiment is known as a *displacement* or *strain controlled* experiment.

Measurements are taken of the change in length Δ and the associated force P.

By dividing the applied force P by area A and the elongation Δ by gage length L, the stress required to cause a certain strain is found: $\sigma = P/A$; $\varepsilon = \Delta/L$.

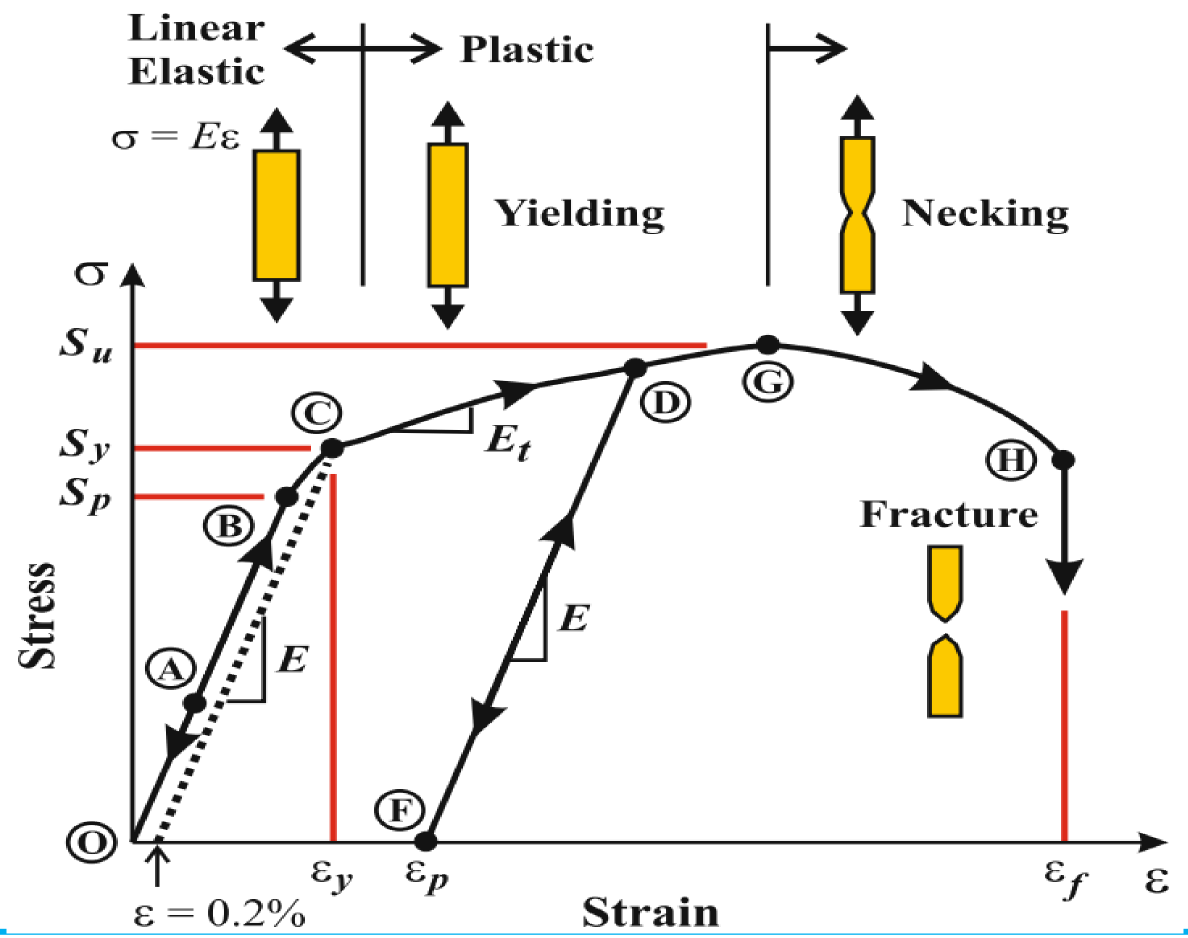


Strain can also be measured with *strain gages,* which are about 10 mm in length. A strain gage is epoxied to the specimen, and is part of an electronic circuit. As the specimen – and thus the strain gage – changes length, the resistance of the strain gage changes, providing an electric signal that indicates the strain of the specimen.

Linear–Elastic Loading and Unloading

For initial loading (line OAB), the stress is linear with slope E : $\sigma = E\varepsilon$.

Removing load along BAO (Unloading), the strain returns to zero. ⟶ Elasticity.



Key to plot:

⟶Arrows indicate direction of $\sigma - \varepsilon$ Plot. B-D & D-H are one-way.
Unloading from a point on CG is linear with slope E.

O. Origin.

A. Point on Linear-Elastic Curve.

B. Proportional Limit.

C. Yield Point ("0.2% Yield").

D. Point on Plastic-Curve.

F. Permanent Strain due to Yielding.

G. Ultimate Strength, onset of Necking.

H. Failure Strain.

E , Et: Young's Modulus, Tangent Modulus.

S_p, S_y, S_u: Proportional Limit, Yield Limit, Ultimate Limit.

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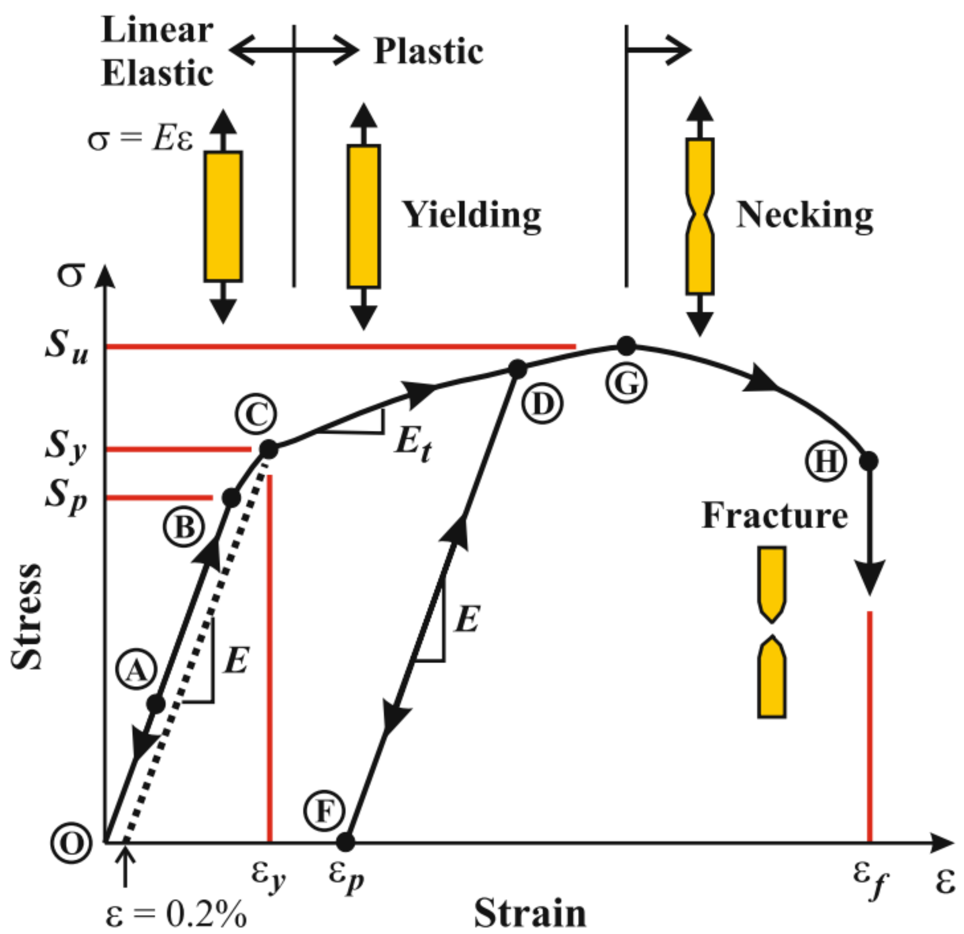
Yielding

Linear–elastic behavior ends at a value of stress called: *proportional limit* S_p (point B), difficult to be determined from experiments.

An engineering convention called *yield strength* S_y (σ_y), replaces S_p by intersecting the curve with the dotted parallel to *OAB*, displaced by 0.2% at the strain axis (point C). $\varepsilon_y = Sy/E$, is the *yield strain*.



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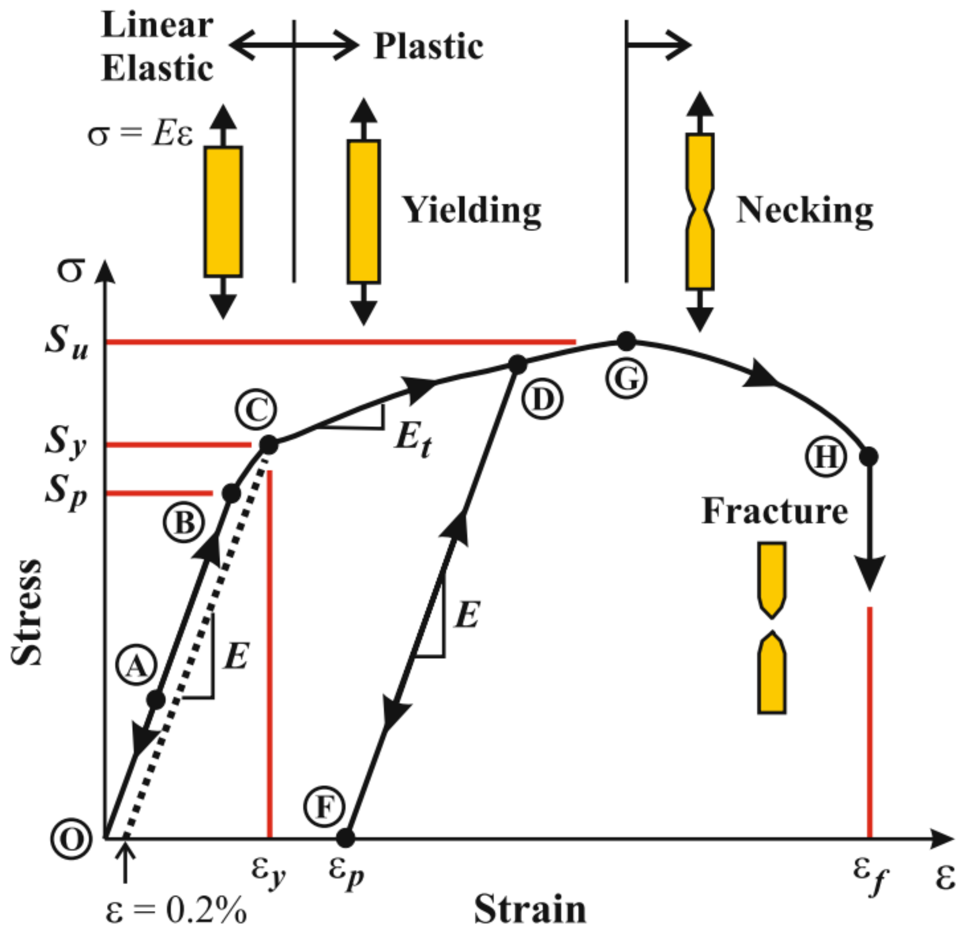
Plastic Deformation, Necking, and Failure

As strain increases from C towards G, the slope of the $\sigma - \varepsilon$ curve, the *tangent modulus* E_t, decreases eventually to zero when the stress reaches a maximum: The *ultimate tensile strength* S_u.



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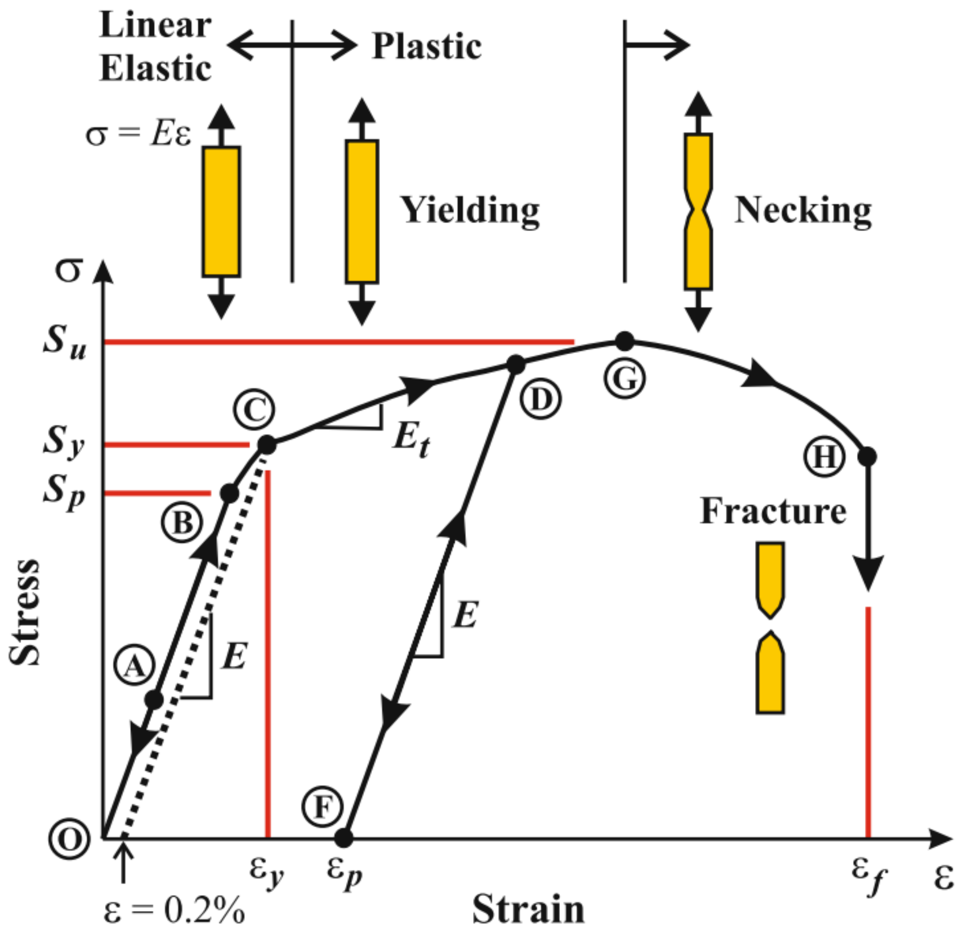
Plastic Deformation, Necking, and Failure

After reaching S_u, stress decreases with increasing strain (from G to H). Force $P = \sigma A$ decreases because somewhere along the bar, its cross-sectional area begins to decrease significantly.



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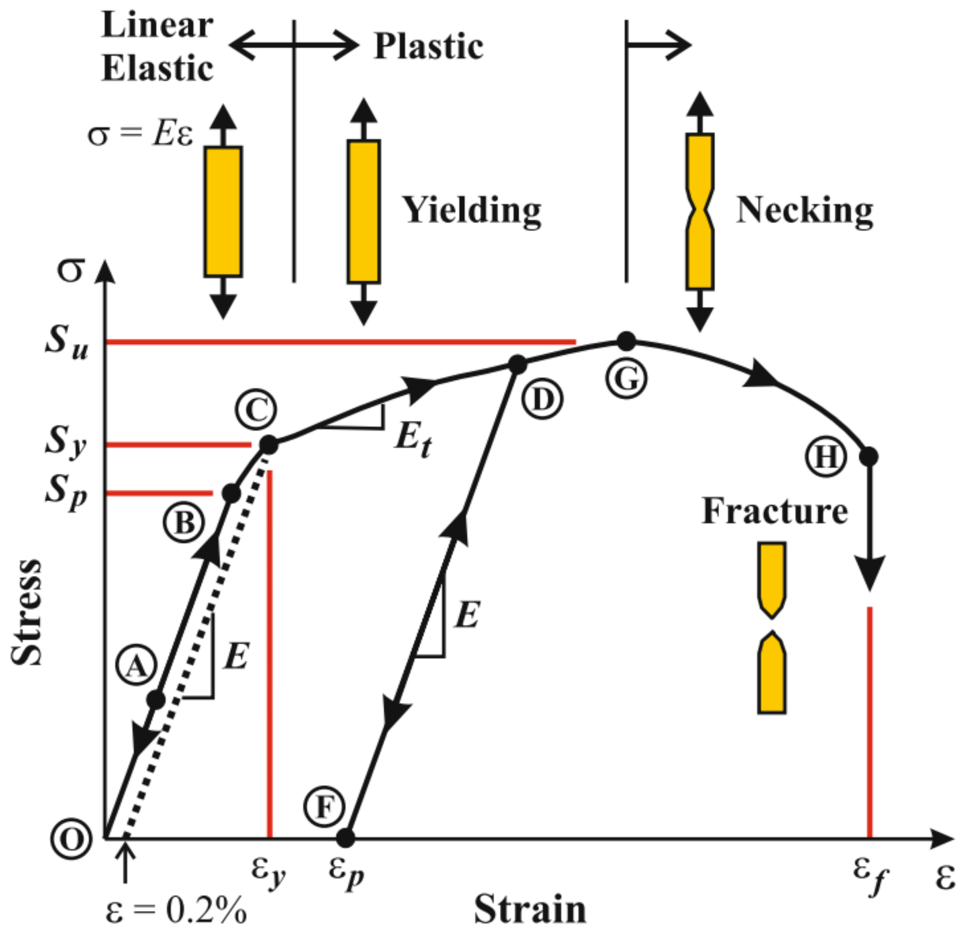
Plastic Deformation, Necking, and Failure

At this localized reduction in area, *necking*, the stress is higher than the nominal stress, so the strain and elongation are concentrated there.

Finally fracture into two pieces occurs in the *neck* at the *failure strain* ε_f.



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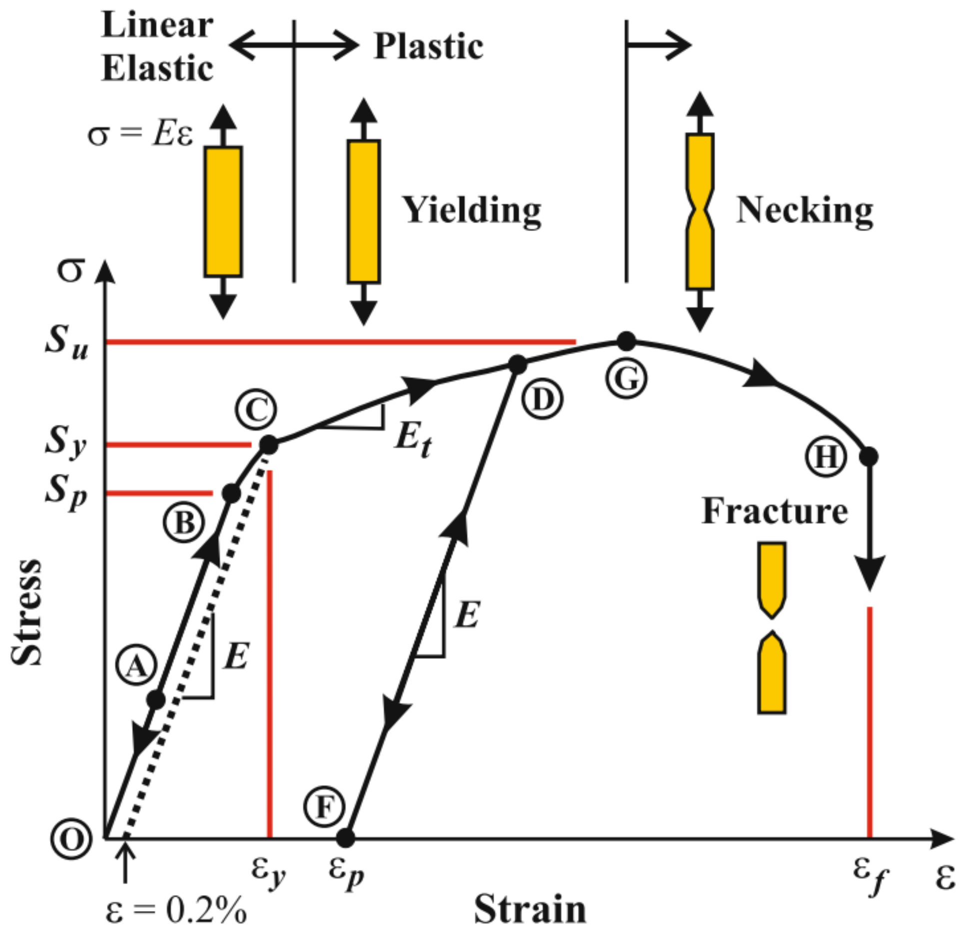
Plastic Deformation, Necking, and Failure

A material is generally classified as *ductile* if the strain to failure $\varepsilon_f \gg \varepsilon_y$ (by an order of magnitude).

Ductile materials typically have failure strains on the order of 15% or more.

The ductility of metals allows them to be bent into various shapes without breaking.



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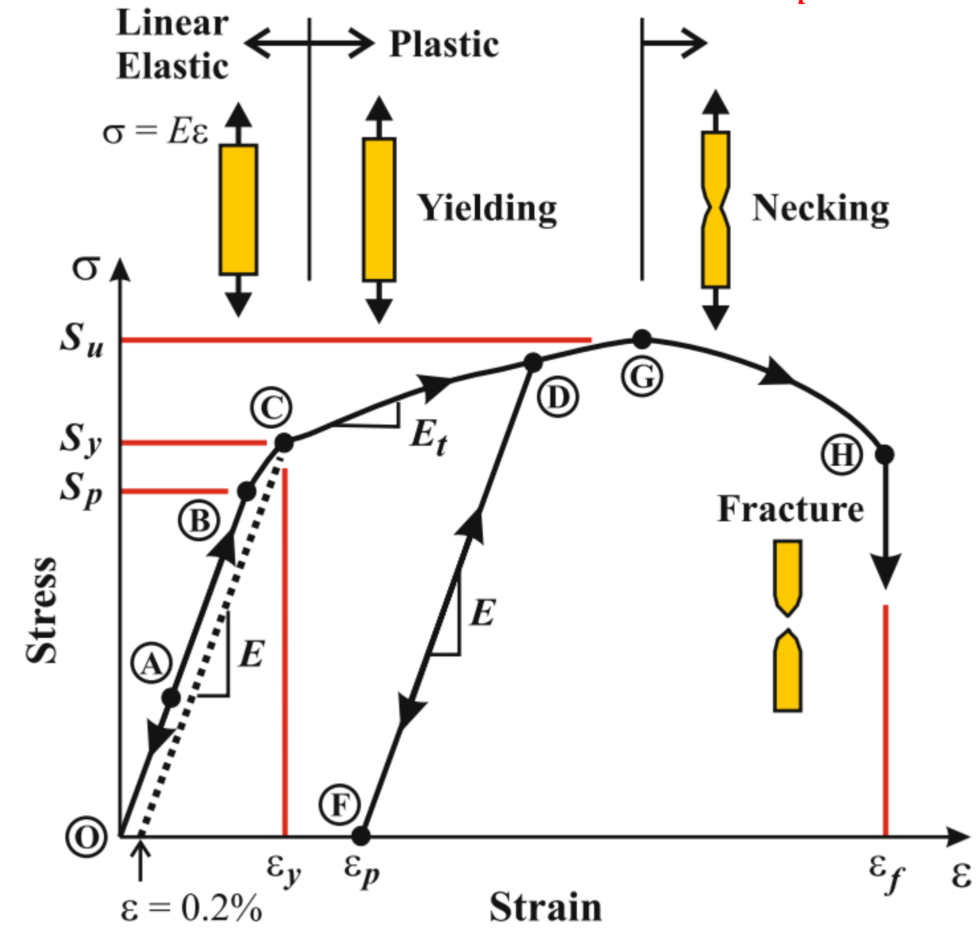
Unloading After Plastic Deformation and Reloading

Removing load after yielding but before necking (between C & G), the $\sigma - \varepsilon$ response follows an *elastic unloading line DF,* having the same slope as the linear–elastic loading line *OAB*.

When the stress is completely removed, the bar does not return to its original length, but suffers a *permanent strain or plastic strain* ε_p.



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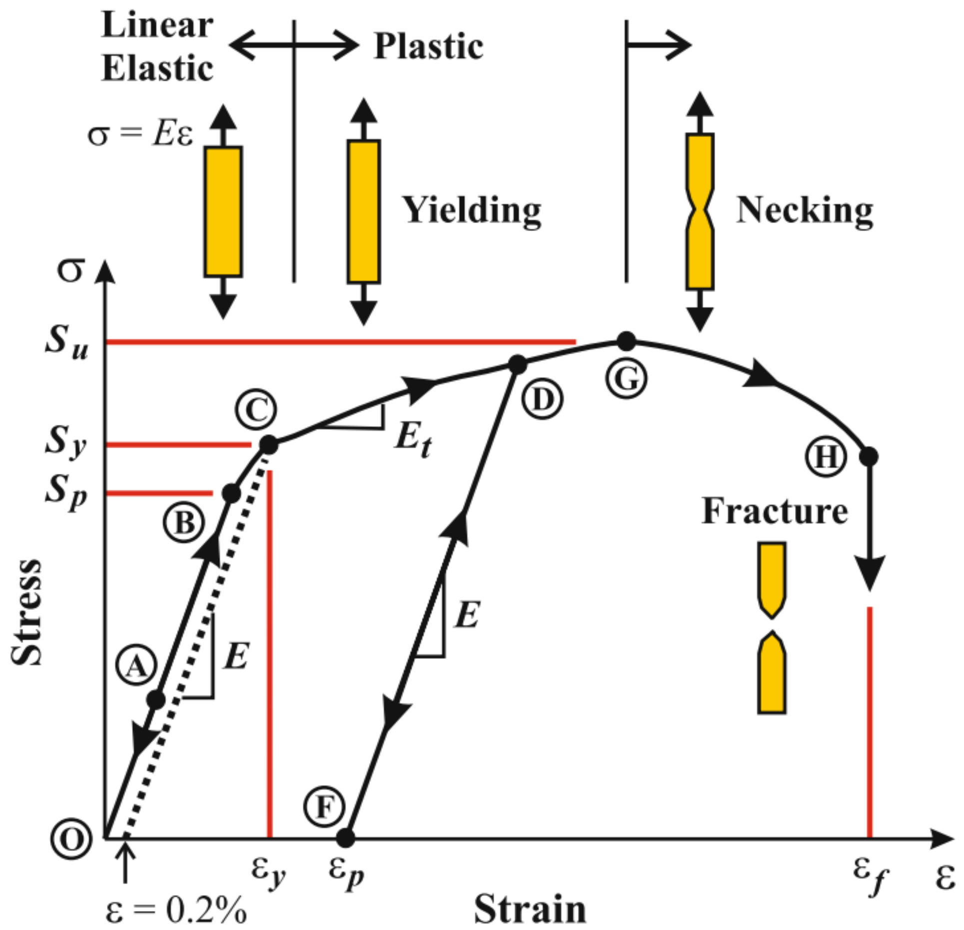
Unloading After Plastic Deformation and Reloading

Reapplying load, the response begins at point *F* and is linear up to a greater stress than the original yield strength S_y at point *D*, where it rejoins the overall curve. This phenomenon is known as *strain-hardening*.



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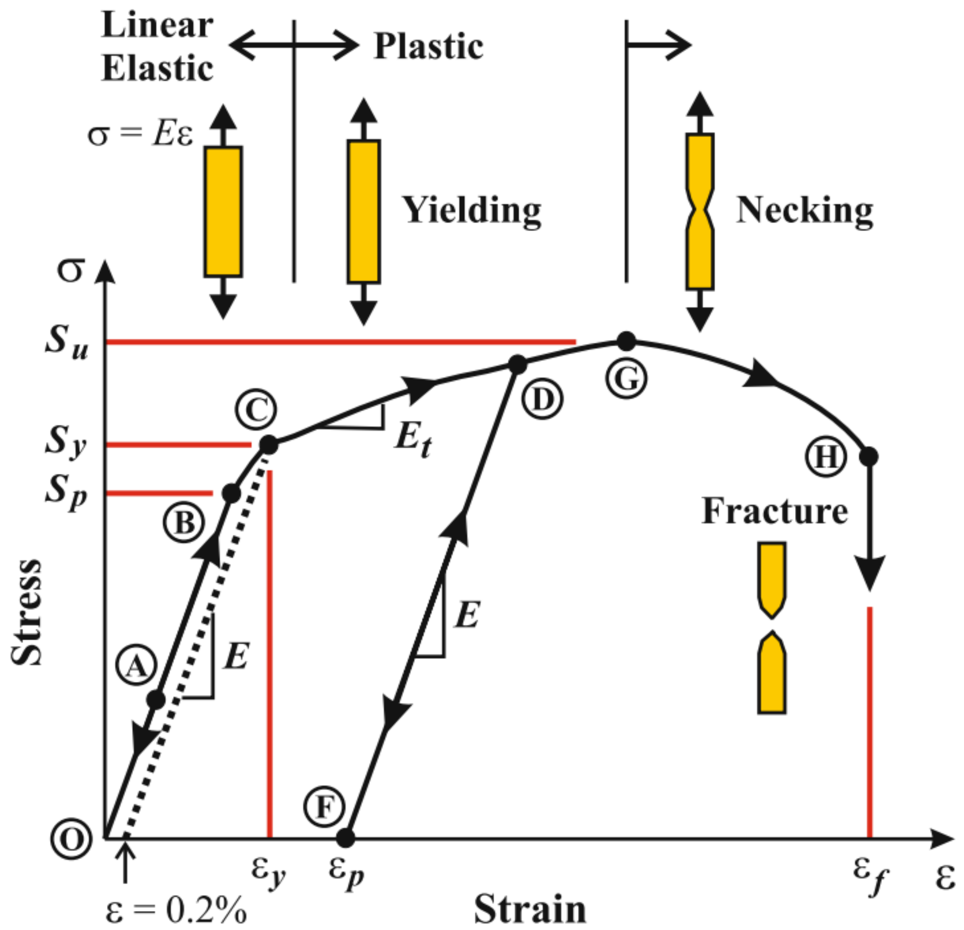
Unloading After Plastic Deformation and Reloading

By mechanical processing a material yield strength can be increased. However, a bar that has been strain-hardened is less ductile, the failure strain is reduced from ε_f to $\varepsilon_f - \varepsilon_p$ (although in most cases, there is still sufficient strain to failure).



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Material	E (Gpa)	S_y (Gpa)	ε_y(%)	ε_f(%)
Steels	207	250-1900	0.12-0.95	25+
Titanium alloys	115	200-1300	0.17-1.2	20
Aluminum alloys	70	100-600	0.14-0.86	15
Nickel Alloys	215	200-1600	0.1-0.74	30
Cast irons	180	220-1000	0.12-0.55	0(gray) 15(ductile)
Douglas fir (parallel to grain)	12.4	100 (S_u)	N/A	-
Glass	70	N/A	Very small	~0
Rubbers	0.01-0.1	30 (S_u)	>10	500+
Polymers	0.1-5	20-30	0.5-2	-
Engineering ceramics	300-450	N/A	Very small	~0
Carbon fiber/polymer matrix composite	70-200	1800 (S_u) In fiber direction	N/A	N/A

Values of yield strength S_y in metals depend on chemical composition, mechanical processing, thermal processing, etc.

Ceramics & glasses are brittle and exhibit little, if any, plastic strain.

General Comments

In $\sigma - \varepsilon$ experiments, it is standard practice to apply a displacement/strain and measure the required force/stress. The *displacement-controlled* or *strain-controlled* is test just discussed.

The alternative is to apply a force/stress and measure the resulting strain (a *force-controlled test*).

Data collection in the force-controlled test is difficult because beyond the proportional limit, the slope of the stress–strain curve decreases; small increments of stress cause large changes in strain.

Better results during yielding are achieved using the strain-controlled test; small increments of strain require very small changes in stress. Additionally, since force continuously increases in the force-controlled test, the decrease in stress at necking is not captured.

Unlike the modulus, the yield strength of a metal can be significantly increased by the addition of atoms of another element (*alloying*), by mechanical processing, or by heat treatment. Metals can, therefore, have a wide range of yield strengths, as shown in the previous Table.

By understanding processing techniques (metallurgy course), a metal alloy can be engineered to have a specific yield strength S_y .

Increasing the yield strength does not generally influence the value of Young's modulus E .

Of course, the more complex the processing route, the more expensive the material.

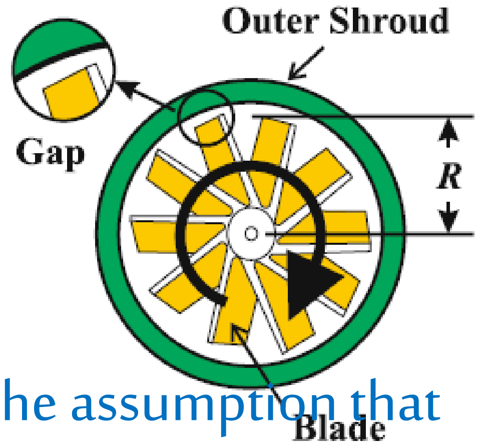
High strength metals are expensive, so only used in special applications where the cost is justified.

Yield strength S_y of steel used in buildings & bridges is down the range with a value of 250 MPa.

By contrast, modern pressure vessel is possible using high-strength steels with $S_y \approx 1900$ MPa .

High-strength, high-temperature nickel-based alloys are used in jet engines. Where the acting forces are large and a design with compact dimensions is only possible using strong nickel alloys. If only yield strength were considered in the design, the system could be made strong enough. However, larger allowable stress levels mean larger elastic deflections. An elastic extension is not necessarily negligible. Ex, a strain of ε =0.6% is within the elastic region for high-strength nickel alloys ($\varepsilon_y = 0.75\%, Sy = 1600\ MPa, E = 215\ GPa$).

If the radius of the engine's compressor disc R is1.0 m and the dynamic loading causes a strain of $\varepsilon = \Delta R\ /R = 0.006$, then the increase in radius is $\Delta R = 6$ mm.



The gap between the blades and the outer shroud (which itself may deform) must be large enough to accommodate this expansion (*Fig.*). Elastic deformation must be considered.

The majority of materials used in practice are ductile. Design methods are often based on the assumption that the stress in a material is limited by its yield strength S_y so that the material remains elastic.

The ability of a metal to yield before it breaks is a useful property, since plastic deformation provides a visible warning of impending failure.

2. Stress–Strain Curves for Brittle Materials

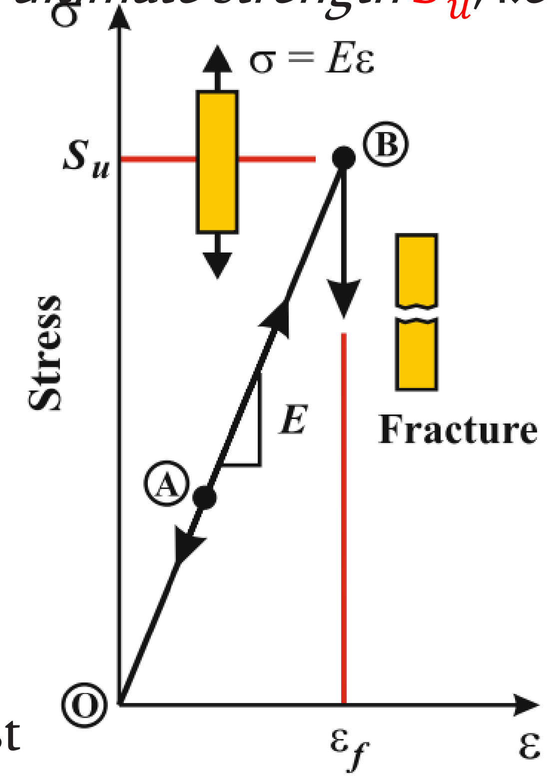
The stress–strain curve of a brittle material such as ceramic and glass is essentially limited to the elastic region, as shown in *Figure*. The strength is usually defined by the *ultimate strength* S_u, i.e., the stress at failure (fracture into two pieces).

It is generally difficult to specify a single value of S_u for a brittle material because there is so much scatter in tests.

The measured strength depends on the specimen's size and on the random distribution of pre-existing flaws or cracks in it.

S_u for brittle materials are often not tabulated, and given with conservative values, or are given with a broad range.

Gray cast iron is used extensively in castings of engine blocks for automobiles and diesel engines. When tested in tension, gray cast iron breaks into two with little warning.



Key to plot:

O. Origin.

A. Point on Linear-Elastic Curve.

B. Ultimate Strength, Failure Strain, Fracture.

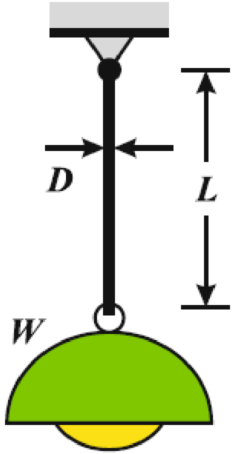
E: Young's Modulus.

S_u: Ultimate Strength.

E_f: Failure Strain.

Gray cast iron exhibits some of the characteristics of ductile materials with a failure strain ε_f typically between $[2Sy/E]$ and $[5Sy/E]$. Nevertheless, because the failure strain is small compared with ductile materials, gray cast iron is sometimes described as *semi-brittle*, sending a signal to the designer to proceed with caution.

Ex. 1. Hanging Lamp



A lamp weighing $W = 50$ N hangs from the ceiling by a steel wire of diameter $D = 2.5$ mm.

The wire has a yield strength of $S_y = 400$ MPa. The factor of safety against yielding is to be 2.5 (just in case someone pulls down on it, etc).

Determine the allowable (design) load P_{allow}.

Ex. 2. Tower Crane

The tower crane shown in (*Figure a*) consists of tower ***DCE*** fixed at the ground, and two jibs ***AC*** and ***CB***. The jibs are supported by tie bars ***AD*** and ***DB***, and are assumed to be attached to the tower by pinned connections.

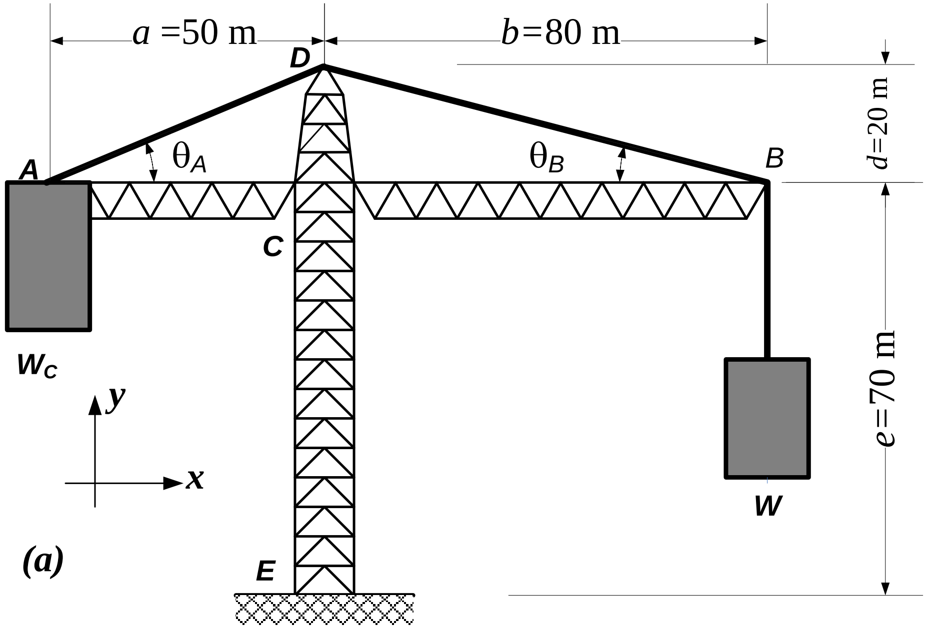
The counterweight W_C weighs 1750 kN and the crane has a lifting capacity of $W_{max} = 1200$ kN. Neglect the weight of the crane itself. Determine:

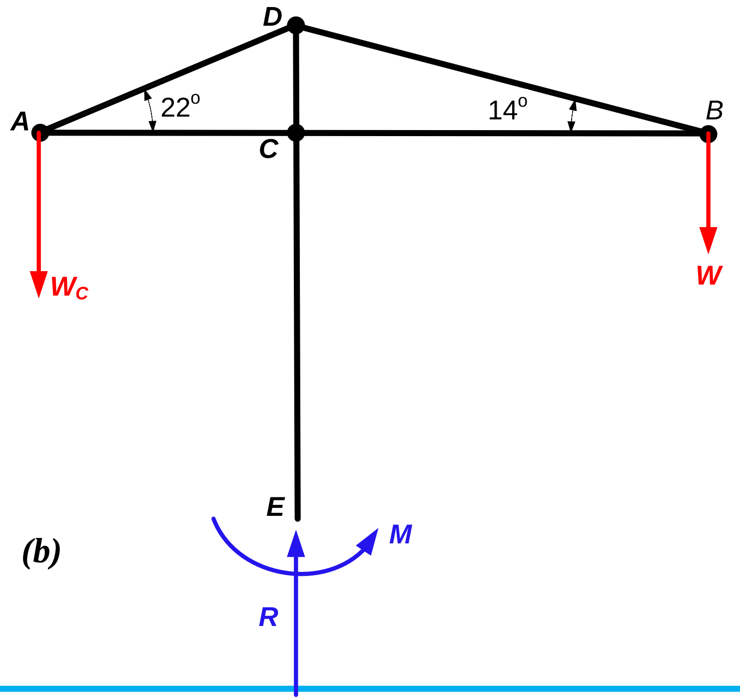
(a) the reactions at the base of the tower when the crane is lifting its capacity.

(b) the axial forces in tie bars ***AD*** and ***DB***, and jibs ***AC*** and ***CB***, and

(c) If the factor of safety against yielding is 2.0, determine the minimum cross-sectional area of tie bar *DB*.

(d) Using the area calculated in Part (c), determine the change in length Δ of *DB*.



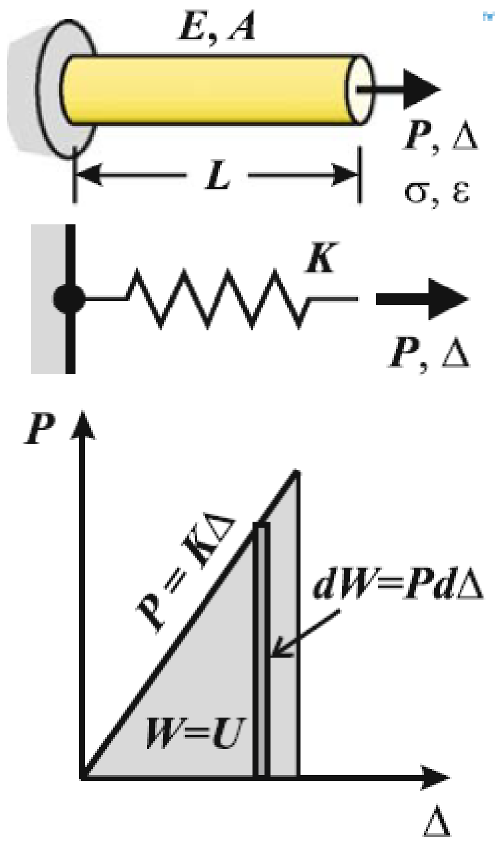


Elastic Strain Energy of a Tensile Bar

The relationship between axial force P and elongation Δ for an elastic bar is: $P = (EA/L)\Delta = K\Delta$

The bar behaves like a spring of stiffness K. The increment of work dW done by force P in deflecting the spring by an additional increment of displacement $d\Delta$, is:

$$dW = Pd\Delta = (K\Delta)d\Delta$$



The total work done to elastically deform the bar is determined by performing the integral:

$$W = \int_0^{\Delta} Pd\Delta = \int_0^{\Delta}(K\Delta)d\Delta = \frac{1}{2}K\Delta^2 = \frac{1}{2}\frac{EA}{L}\Delta^2$$

Using $P = K\Delta$, the total work done can also be: $$W = \frac{1}{2}P\Delta = \frac{1}{2}\frac{P^2}{K} = \frac{1}{2}\frac{L}{EA}P^2$$

The work done W is stored internally as *elastic strain energy* U. So $W = U$.

The work expression can be transformed to show the *internal energy* expression as:

$$U = W = \frac{1}{2}\frac{L}{EA}P^2 = \frac{1}{2}\frac{L}{EA}(\sigma^2A^2) = \frac{1}{2}\frac{LA}{E}\sigma^2 = \frac{1}{2}LA\frac{\sigma}{E}\sigma = V\left(\frac{1}{2}\sigma\varepsilon\right) = VU_D$$

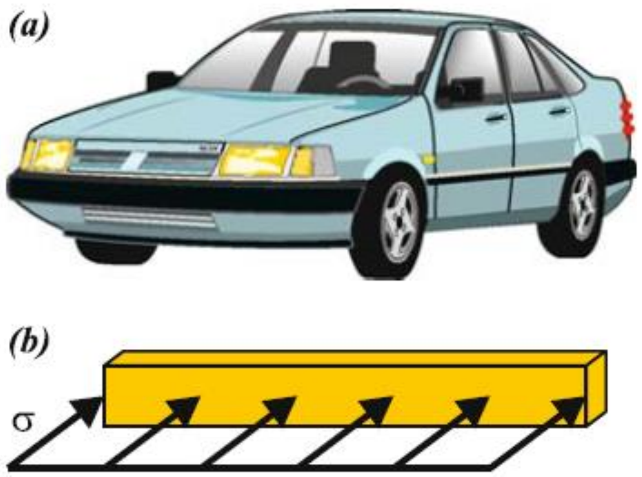
U_D, is the *elastic strain energy density.* $$U_D = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}\frac{\sigma^2}{E}$$

The maximum value of the *elastic strain energy density* is when the stress reaches the *yield strength*. The maximum elastic strain energy density is known as the *modulus of resilience* U_R : (معامل الممانعة الميكانيكية)

$$U_D = \frac{1}{2}\frac{\sigma^2}{E} \Rightarrow U_R = \frac{1}{2}\frac{S_y^2}{E}$$

The *resilience* (الممانعة الميكانيكية) is the maximum energy per unit volume that can be absorbed by the material without plastic deformation occurring.

Ex: Car Bumper Design



Car bumpers are often protected with a strip of rubber approximately 2.0 m long, 5 mm thick, and 100 mm high. The practical purpose of the strip is to absorb energy in low-speed accidental crashes such as in parking lots or when parallel parking. Assume the strip supports the entire load uniformly.

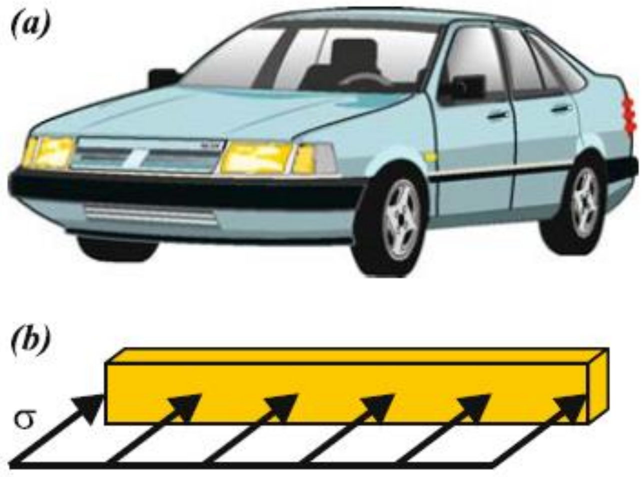
(a) From an elastic energy standpoint, why might rubber be a good choice of material compared to steel and aluminum?

(b) (b) If a 1100 kg car traveling at 2.2 m/s in a parking lot hits a wall, and comes to a complete stop, can the rubber pad absorb the energy without exceeding the elastic limit?

Solution:

(a) From tables giving S_y and E, $U_R = \frac{1}{2}\frac{S_y^2}{E}$ is computed to allow comparison

Material	S_y (MPa)	E (GPa)	U_R (MN·m/m^3)
Steel	250	200	
Aluminum	240	70	
Rubber	20	0.05	

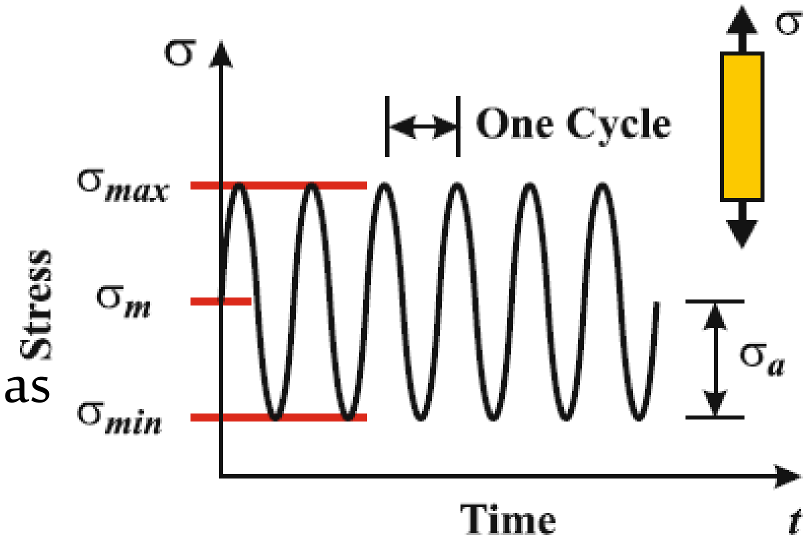


(b) Kinematic Energy of the car…

Cyclic Loading

Engineering systems are often subjected to *cyclic loading*.
Automobiles & machines (millions of cycles).
Electronic components (fluctuating temperature).

Material degradation with time under *cyclic loading* is known as ***fatigue* (التعب)**.



Typical $\sigma - t$ graph for cyclic loading.

Fatigue can cause materials to fail at stress levels well below their uniaxial strength. In many steels, the *fatigue strength* is less than half of the *ultimate strength*.

A few definitions are useful to describe the nature of cyclic loading.

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$$ Mean Stress الإجهاد المتوسط

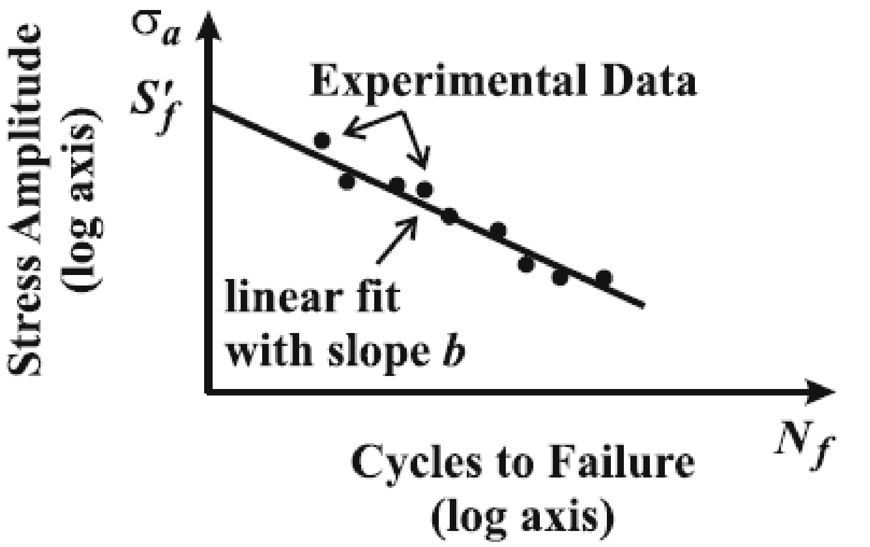
$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$$ Stress amplitude مطال الإجهاد

$$R = \frac{R_{min}}{R_{max}}$$ R-ratio

$\sigma - t$ graph for standard fatigue test:
$\sigma_a = \sigma_{\max} = |\sigma_{\min}|, \sigma_m = 0, R = -1.0.$

When the stress amplitude is plotted against the number of cycles to failure on a log–log set of axes (or on a semi-log set of axes), the graph is known as the ***S–N curve***

The ***fatigue strength*** is the stress amplitude $\sigma_a = \sigma_{\max}$ corresponding to a specified number of cycles to failure.



When the data are plotted, the relationship between stress amplitude σ_a and cycles to failure N_f can be expressed in the following form:

$$\sigma_a = S'_f (N_f)^b$$

Where S'_f & b depend on the material, & are determined from a best-fit line of the material's ***S–N curve*** since:

$$\log(\sigma_a) = \log(S'_f) + b\log(N_f)$$

Quantity S'_f corresponds to the curve fit's intercept at $N_f = 1$ cycle, b is the slope of the curve in the log–log axes. Values for aluminum & steel are given

Property	Al 6061-T6	Steel A36
S_y	240 MPa	250 MPa
S_u	314 MPa	540 MPa
$S_f{}'$	505 MPa	1035 MPa
b	–0.082	–0.11
E	70 GPa	200 GPa

Ex. 1 Fatigue Strength of Structural Aluminum & Steel

Given: A tensile bar is subjected to cyclic loading, with zero mean stress

Required: Determine the fatigue strength, $\sigma_a = \sigma_{\max}$, (a) for an aluminum bar (Al 6061-T6) at 10^7 cycles and (b) for a steel bar (A36) at 10^6 cycles.

Solution: For aluminum, the fatigue strength corresponds to 10^7 cycles. From *Table* $S'_f = 505\ MPa$ & $b = -0.082$.

$\sigma_{a,al} = S'_f(N_f)^b = (505\text{ MPa})(10^7)^{-0.082} = 135\text{ MPa}.$ (56% S_y, 43% S_u)

For steel, the fatigue strength corresponds to 10^6 cycles. From *Table* $S'_f = 1035\ MPa$ & $b = -0.11$.

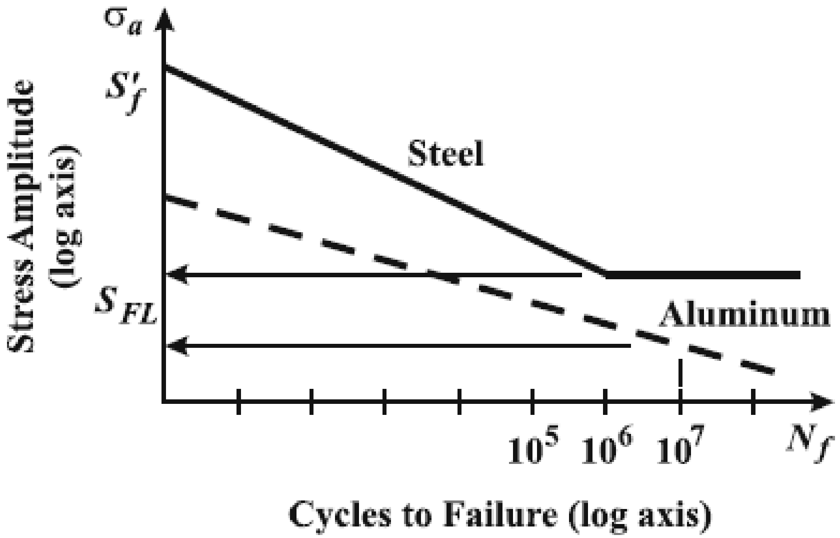
$\sigma_{a,st} = S'_f(N_f)^b = (1035\text{ MPa})(10^6)^{-0.11} = 226\text{ MPa}.$ (90% S_y, 42% S_u)

These values are also known as *fatigue limit* S_{FL} , as discussed below.

The fatigue limit for many steels is typically on the order of 35 - 50% of S_u.

Fatigue Limit

The ***S–N curve*** for most steels does't decrease to zero (*Fig*.). Below a certain value of σ_a, steel has essentially an infinite fatigue life. The stress amplitude below which fatigue failure does not occur is the *fatigue limit* S_{FL}, also known as the endurance limit حد التحمل.



Type equation here.For steels, the *fatigue limit* S_{FL} typically corresponds to a fatigue life of about 10^6 cycles. The S–N curve exhibits no further reduction and becomes horizontal.

Aluminum does not exhibit such limiting behavior. No matter how small the stress amplitude σ_a, aluminum will eventually fail by fatigue. Accordingly, when designing with aluminum, the fatigue limit S_{FL} is generally taken to be the stress amplitude for $N_f = 10^7$ cycles.

Effect of Mean Stress on Fatigue Strength

In general, cyclic stresses are applied with a non-zero mean stress σ_m.

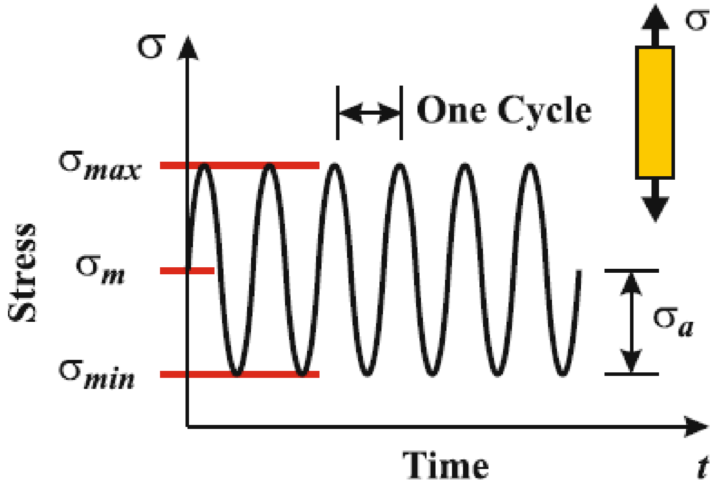
When this is the case, the fatigue strength for a given number of cycles to failure N_f is determined using the ***Goodman Diagram***

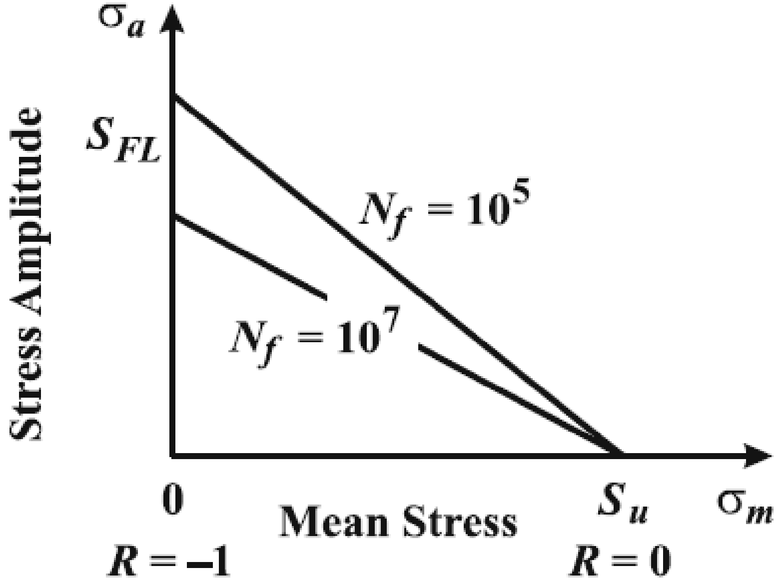
$$\frac{\sigma_a}{S_{FL}} + \frac{\sigma_m}{S_u} = 1$$

The new fatigue strength σ_a – the amplitude of the cyclic loading with mean stress σ_m for the specified cycles to failure N_f – is then:

$$\sigma_a = S_{FL}\left(1 - \frac{\sigma_m}{S_u}\right)$$

When $\sigma_m = S_u$, the materials breaks into two upon first loading; there can be no alternative stress σa ($\sigma_a = 0$). If the mean stress is zero, then the stress amplitude is the *fatigue limit* S_{FL} for the specified number of cycles to failure.





Ex. 2 Fatigue Strength With Non-Zero Mean Stress

Given: A tensile bar is subjected to cyclic loading with a mean stress of $\sigma_m = 100$ MPa.

Required: With the mean stress applied, determine the fatigue strength (a) for an Al 6061-T6 bar at 10^7 cycles and (b) for an A36 steel bar at 10^6 cycles.

Solution:

$$\sigma_{a,al} = S_{FL}\left(1 - \frac{\sigma_m}{S_u}\right) = 135\left(1 - \frac{100}{314}\right) = 91 \text{ MPa.}$$

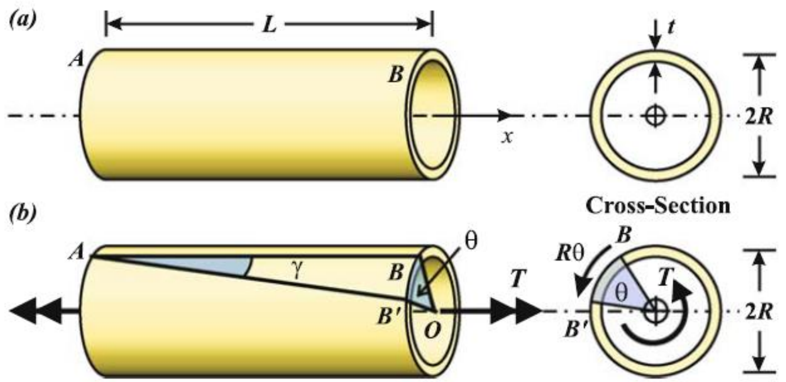
$$\sigma_{a,st} = S_{FL}\left(1 - \frac{\sigma_m}{S_u}\right) = 226\left(1 - \frac{100}{540}\right) = 184 \text{ MPa.}$$

3.2 The Torsion Test – Shear Properties

A thin-walled circular shaft of average radius R, thickness t & length L, is shown in Figure.

Thin-walled ($t \leq 0.1R$): $t \ll R \Rightarrow R_{in} \cong R_{out} = R$ & material response is constant across t.

Torque T is applied to the shaft, with fixed left end, the right end rotates by angle θ and point B of the right end moves to point B'. The relative rotation between the two ends is the angle of twist θ of the shaft.

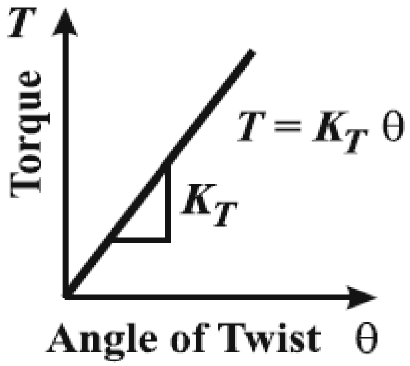


Experiments show that torque increases linearly with θ for small (elastic) rotations. The shaft behaves as a *torsional spring*, with K_T as its *stiffness*. Then $T = K_T\theta$.

The results of the torsion test are used to determine the *shear properties* of a material.

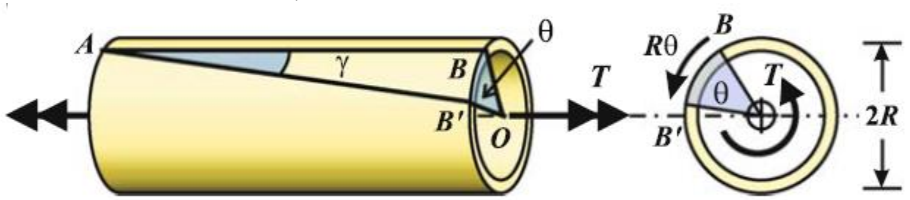
Then is determined the torsional strength & stiffness of any thin-walled circular shaft.

The torque-angle of twist response is investigated using *shear strain* & *shear stress*.



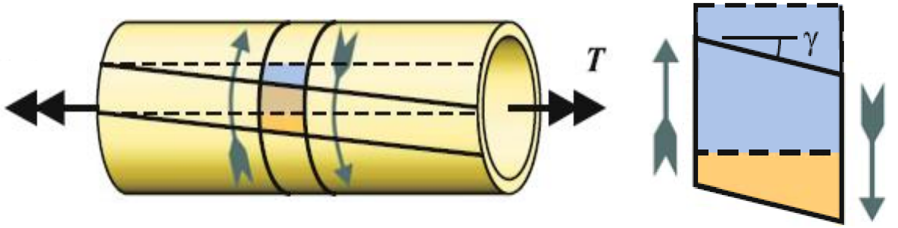
Shear Strain

The *shear strain* γ "gamma", (Fig.) is the ratio of the displacement (movement) of point B to the shaft length L, with left end taken as fixed. The distance point B moves is:



$BB' = R\theta = \gamma L$. (Both γ and θ are in radians) $\Rightarrow$ $\boxed{\gamma = R\theta/L}$ $\sim \varepsilon = \Delta/L$

Now consider a square material element on the surface of the shaft initially bound by two dashed axial lines and two solid circumferential lines Fig.



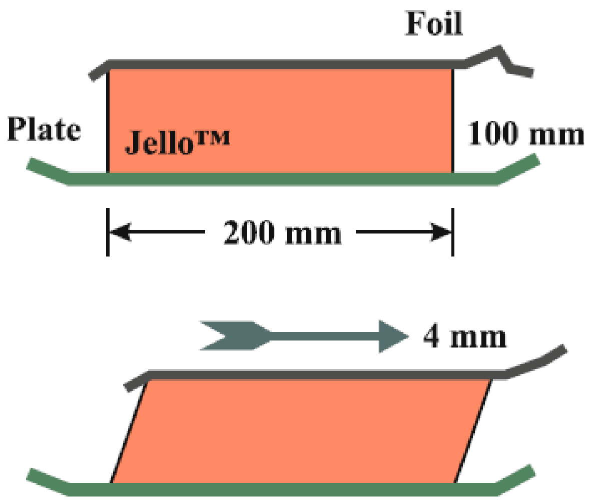
The torque deforms the dashed lines to the position of the solid angled lines. The square is transformed into a rhombus.

The *shear strain* γ is the change in right angle of the square element as it deforms into a rhombus, measured in radians.

Like normal strain ε, shear strain γ is generally small (~0.001). Thus $\sin\ \gamma \cong \tan\ \gamma \cong \gamma$ and $\cos\ \gamma \cong 1$.

In axial members, the Poisson effect changes volume. There is no change in volume due to shear. *Square's sides equal rhombus's sides*

Ex. 3.12 Shear Strain



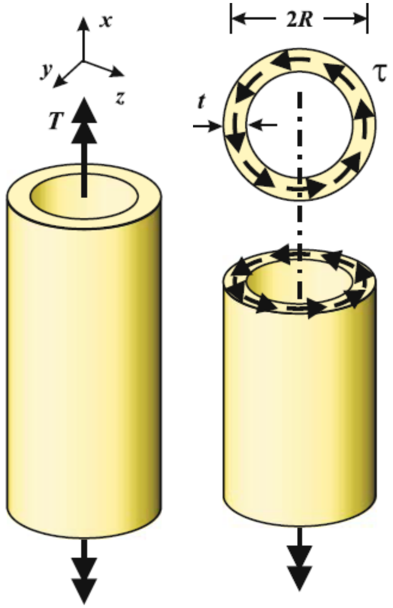
Given: A rectangular piece of Jello™, $h = 100$ mm tall by $b = 200$ mm wide, sits on a plate with a piece of aluminum foil on top (*Fig.*) The foil is moved to the right by $w = 4$ mm, pulling the top of the Jello™ along with it.

Required: Assume the small angle approximation holds. Determine the shear strain in the Jello™.

Shear Stress

Consider again the thin-walled shaft of thickness t and average radius R $(t << R)$, subjected to torque T.

A cut perpendicular to its axis, exposes an interior cross-section. The torque carried by this section is T.



The torque is not carried at a single point, but is supported uniformly over cross-sectional area A at distance R from the axis, by the *shear stress* τ ("tau").

The torque is thus $T = R\,\tau A$. A good approximation for the cross-sectional area of a thin-walled shaft is $A = 2\pi Rt$, then the *shear stress* τ can be determined in terms of T, R & t by:

$$\boxed{\tau = T/RA \text{ or } \tau = T/2\pi R^2 t.}$$

Shear stress has units of force per unit area (MPa,.. etc). Unlike the normal stress, the shear stress acts across, or parallel to, the interior material surface.

Note that the *shear stress* τ due to a torque constantly changes direction as it moves around the cross-sectional area making a zero resultant and a T resultant moment.

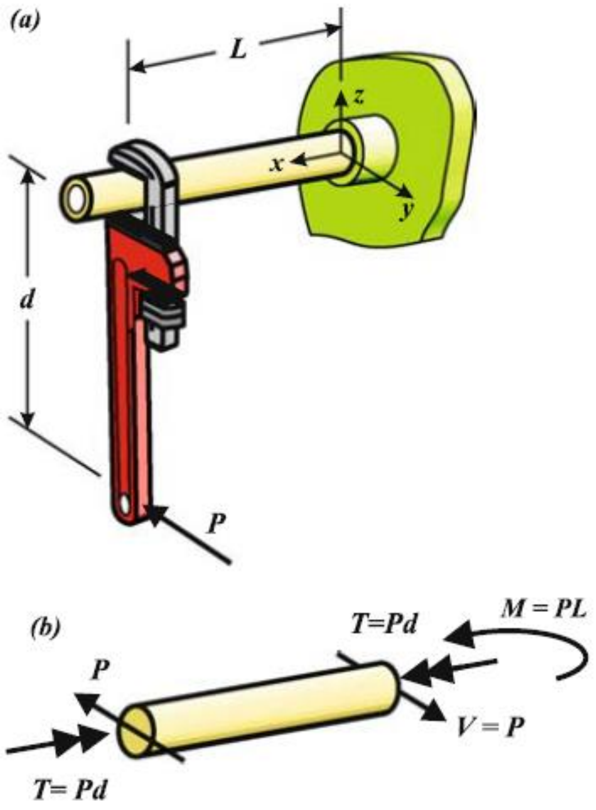
Also, the thin-wall assumption means that the thickness is so small that the shear stress can be considered constant through the wall thickness.

Ex. 3.13 Thin-walled Pipe Tightened with a Wrench

Given: A pipe of diameter D= 25.4 mm (1.0 in) and thickness t=2.54 mm (0.10 in) is twisted by a wrench with P=125 N and d=20 cm, as shown.

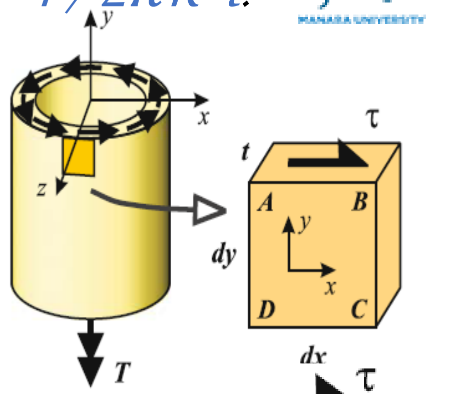
Required: Use the thin-wall formula to estimate

the shear stress in the thin-walled pipe due only to torque T.



Complementary Shear Stress

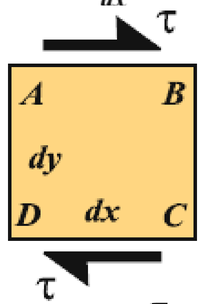
The shear stress on a cross-section of the thin-walled circular shaft in torsion was determined as: $\tau = T/2\pi R^2 t$.



This stress on the cross-section is not the only active shear stress.

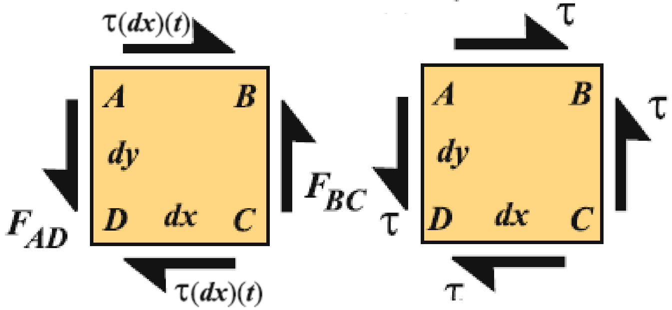
Consider a 3D element with sides dx, dy, & t removed from the shaft, showing only the shear stress on the exposed cross-section (Upper *Fig.*).

Then the element is drawn in 2D showing only the shear stresses acting on cross-sectional planes *AB* and *CD*. Equilibrium of forces requires that these stresses be equal and opposite.



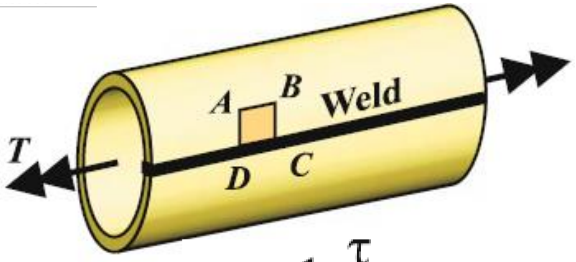
The horizontal stresses cause equal but not directly opposite forces $\tau(t\,dx)$ on surfaces *AB* and *CD*. These forces produce a clockwise couple: $M = [\tau(t\,dx)](dy)$

The only way of maintaining rotational equilibrium is to apply a counteracting couple in the form of shear forces on sides *AD* and *BC*, F_{AD} and F_{BC}, which are dx apart. So $|F_{AD}| = |F_{BC}| = M/dx = \tau(t\,dy)$.

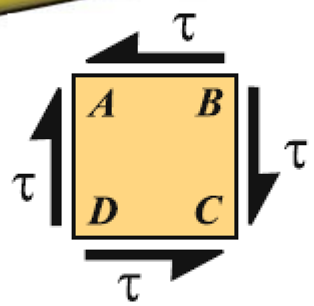


The shear stresses on *AD* & *BC* are equal and at right angles to the applied shear stress on *AB* & *CD*, so referred to as *complementary shear stresses*.

Ex. 3.14 Complementary Shear Stress – A Welded Pipe



Given: A thin-walled pipe (radius R, thickness $t \ll R$) is formed by rolling a plate into a cylinder and welding the edges together to form a continuous shaft. The shaft is subjected to a torque T.

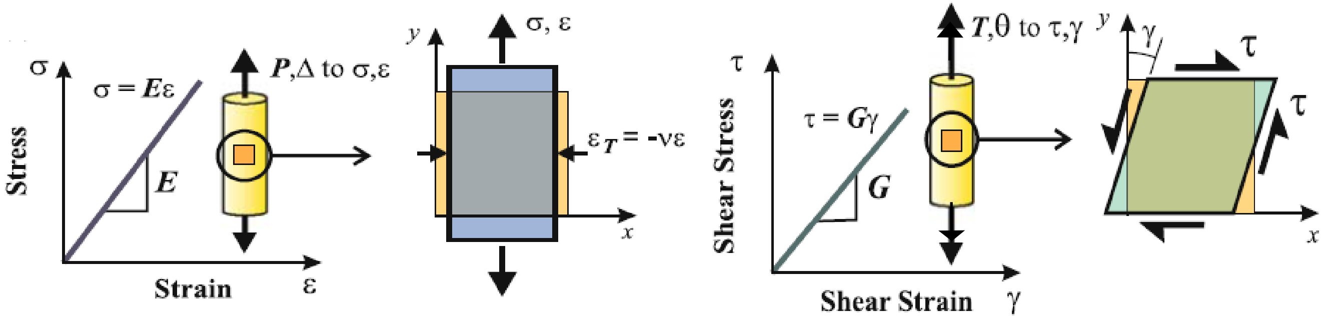


Required: Determine the shear stress on the weld, $\tau_W = \tau_{CD}$ (along line CD).

Elastic Shear Modulus

The relationship between normal stress and strain is given by Hooke's Law, $\sigma = E\,\varepsilon$.

A similar relationship exists between shear stress τ and shear strain γ: $\tau = G\,\gamma$.



where G is the *shear modulus*. Typical values for this modulus are found in next table

Material	*E* (GPa)	*G* (GPa)
Steels	207	80
Titanium alloys	115	43
Aluminum alloys	70	26
Nickel Alloys	215	83
Cast irons	180	70
Douglas fir (parallel to grain)	12.4	???
Glass	70	29
Rubbers	0.01-0.1	0.003–0.03
Polymers	0.1-5	0.03–1.7
Engineering ceramics	300-450	125–190
Carbon fiber/polymer matrix composite	70-200 ???	7–40 ???

The values of G are not always readily accessible. For *homogeneous* & *isotropic* materials: $G = E/2(1+\nu)$.

E, G, & ν *elastic properties* of a material.

Since $\nu \cong 1/3$ for many metals, then G can be approximated: $G = (3/8)E$.

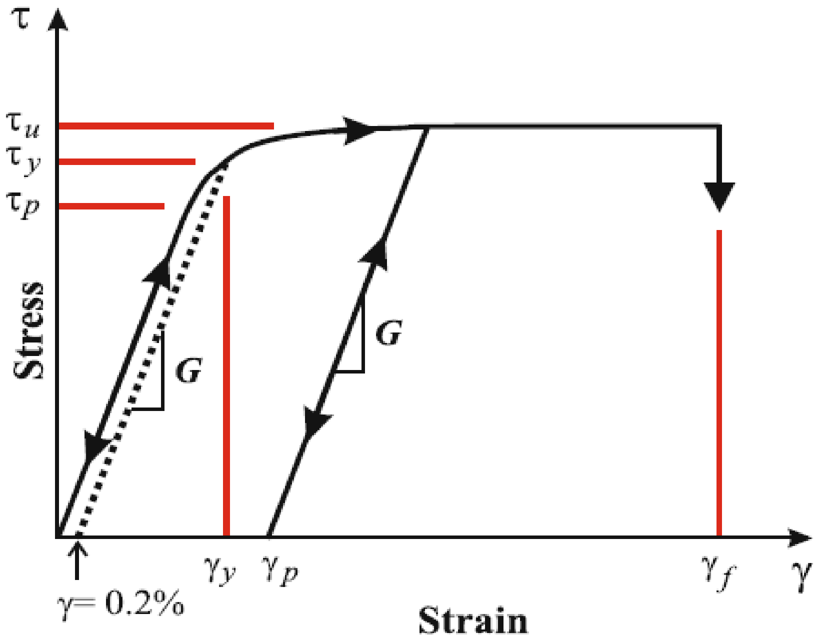
For aluminum, $E = 70$ GPa & $\nu = 0.33$, so $G = 26$ GPa, as in table.
For steel, $E = 207$ GPa & $\nu = 0.3$, so $G = 78$ GPa. as in table.

Shear Stress–Strain Curves

The *shear stress–strain curve* in strain-controlled tests on metal shafts is similar to the one of tension tests.

Initially, it is *linear–elastic*: $\tau = G\gamma$, but, when τ exceeds a critical value, additional small increments in plastic strain result in smaller additional increments of stress.

The *shear yield strength* τ_y is determined by the dashed line starting at γ=0.2% parallel to the initial linear-elastic line, then intersecting the experimental curve at (τ_y, γ_y).

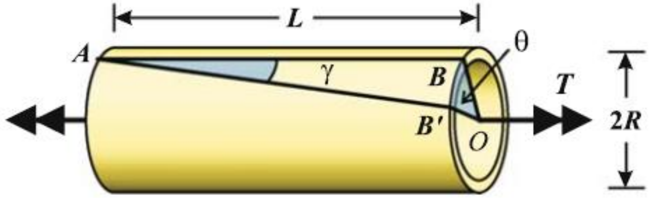


From experiments on ductile metals, it is observed that τ_y can be deduced from S_y as follows: $\tau_y = S_y/(3)^{1/2} = S_y/1.73$. For example, the tabulated values of yield strength for aluminum 6061-T6 are: $S_y = 240$ MPa & $\tau_y = 139$ MPa.

For advanced composites, non-isotropic materials and brittle materials, this relation is not generally valid.

Torsional Stiffness/flexibility of a Thin-Walled Circular Shaft

With G known, the torsional stiffness of any straight, thin-walled circular shaft of length L, *radius* R, & wall thickness t, subjected to torque T can be calculated from:

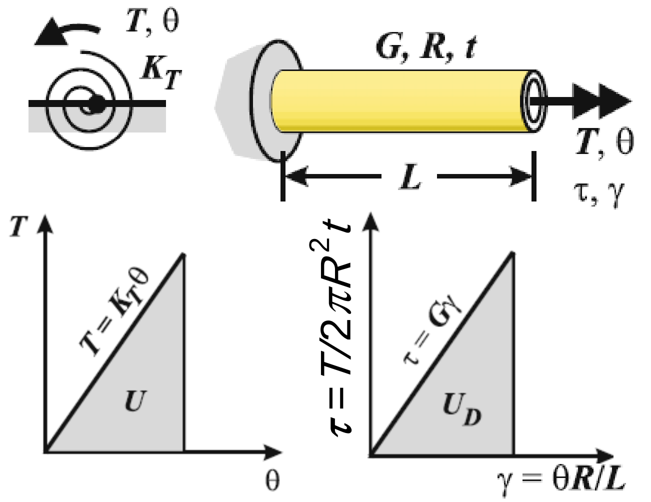


$$\tau = G\gamma \Rightarrow T/2\pi R^2 t = GR\theta/L \quad \Rightarrow T = (2\pi R^3 tG/L)\theta$$

$\Rightarrow$The *torsional stiffness* $K_T = T/\theta = 2\pi R^3 tG/L$ Compare with the axial stiffness $K = AE/L$

The *torsional flexibility* f_T of the shaft is: $f_T = \theta/T = 1/K_T = L/2\pi R^3 tG$

Elastic Strain Energy of a Thin-Walled Circular Shaft



The thin-walled circular shaft is like a torsional spring of stiffness K_T. The total work to twist the shaft by angle θ during elastic deformation is the area under the linear–elastic curve of: $W_T = (1/2)T\theta = (1/2)K_T\theta^2 = (1/2)f_T T^2$.

The work done on the shaft W_T is stored as *elastic shear strain energy* U. Hence:

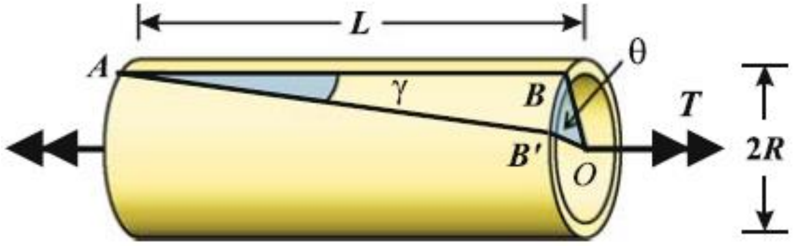
$U = W_T = (1/2)K_T\theta^2 = (1/2)(2\pi R^3 tG/L)(\gamma L/R)^2 = (1/2)(2\pi R\, tL)G\gamma^2 = (1/2)G\gamma^2$ [Volume].

Then *elastic shear strain energy density* $U_{D,\tau}$ is: $U_{D,\tau} = (1/2)G\gamma^2 = (1/2)\tau^2/G = = (1/2)\tau\gamma$.

The *maximum elastic shear strain energy density* is then: $U_{R,\tau} = (1/2)\tau_y^2/G \cong (4/9)\, S_y^2/E$.

Ex. 3.15 Thin-Walled Circular Shaft in Torsion

Given: A thin-walled circular shaft has average diameter $D = 2R = 150$mm, thickness $t = 10.0$ mm, and length $L = 2.0$m *Fig.* The material is a high-strength steel with shear modulus $G = 82$ GPa and tensile yield strength $S_y = 600$ MPa. The applied torque is $T = 42.0$ kN·m.

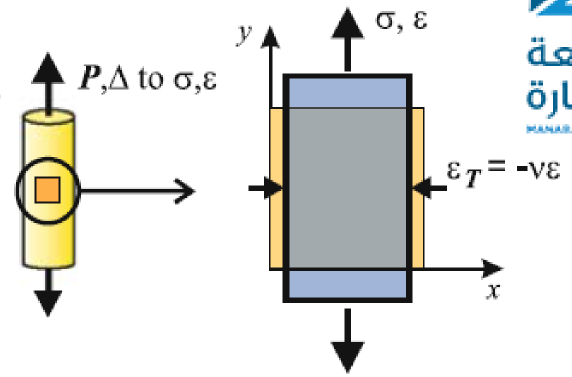


Required: Determine (a) the shear stress in the shaft, (b) the angle of twist between the ends of the shaft, (c) the torsional stiffness, (d) the factor of safety against yielding, and (e) the elastic shear strain energy density.

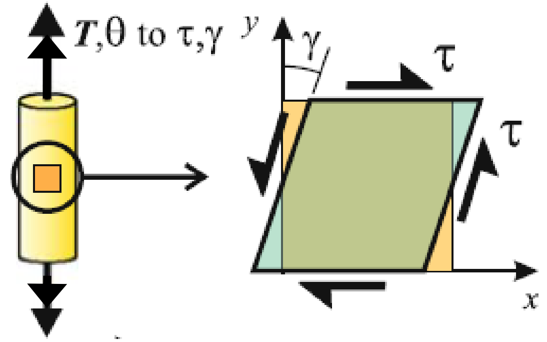
3.3 General Stress and Strain

An axial bar has uniform axial stress and strain; the thin-walled circular shaft also has uniform shear stress and shear strain.

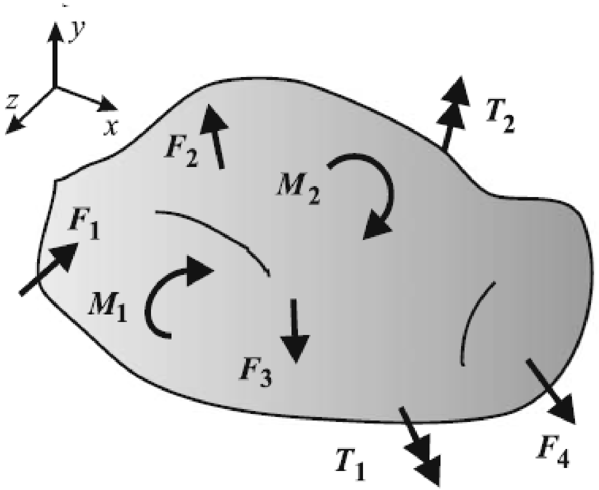
Uniform stress (strain) means that the stress (strain) is the same at every point in a body.



For the axial and torsion members studied in this chapter, the stress state at any point can be visualized on 2D (square) elements since the stresses act in a single plane.

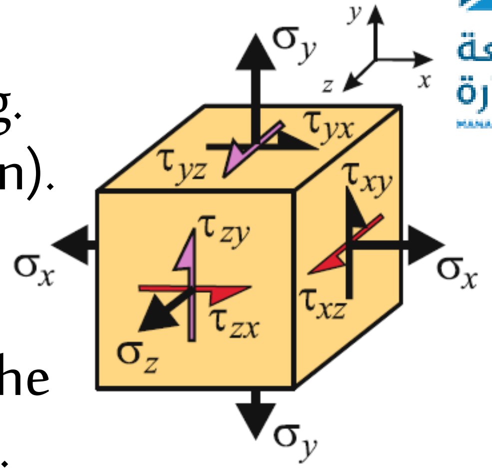


In general, however, stresses and strains vary from point-to-point in a body. From the next Fig., it is evident that every point in the arbitrary body is affected differently; each point has a different stress, and thus a different strain.



Stresses are, in general, 3D, but in many cases can be reduced to 2D.

A material point may be visualized as a very small cube or element.
The stresses at that point are the average stresses acting on that cube, as shown in Fig.
A cube is a natural stress element as it provides a built in coordinate system (cartesian).
Any object can be thought of as being made of many infinitesimally small cubes.



Each face of the stress element has three stresses acting on it: *a normal stress* σ, which is tensile (positive) or compressive (negative) and acts perpendicular to the face; & two *shear stresses* τ, which act parallel to the face in the other two directions.

All of the stresses acting at a point are shown on the cube in *Fig.*; they are drawn in their positive senses as defined below.

The values of the stresses specify a *state of stress*. Equilibrium of forces & moments on the cube are used to show that there are six unique stresses: σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}.

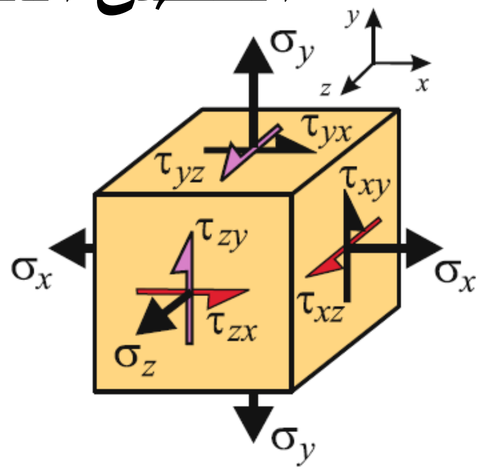
Note that $\tau_{xy}=\tau_{yx}$, $\tau_{yz}=\tau_{zy}$, & $\tau_{zx}=\tau_{xz}$, since at any point, shear stresses on perpendicular planes are equal (*complementary shear stress*).

Stress Subscripts: The subscripts on the stress symbols represent:

1. the face on which the stress acts, & 2. the direction in which the stress acts.

Thus, τ_{xy} is a shear stress on the *x*-face acting in the *y*-direction. Likewise, τ_{yz} is a shear stress on the *y*- face in the *z*- direction.

Positive Sense/Sign Convention



A *positive stress* physically acts on a positive face in a positive direction, or on a negative face in a negative direction.

Two such stresses are in equilibrium with each other.

A *negative stress* physically acts on a positive face in a negative direction, or on a negative face in a positive direction.

Internal forces (stress resultants), torques, and moments are defined positive or negative by *special convention* respecting as possible the same manner as the stresses.

General Stress-Strain Relationship

For a *homogeneous* (same at every point) & *isotropic* (same in every direction) material (e.g., metals, ceramics, etc.), the strains caused by the stresses are defined by 3D Hooke's Law. The normal (axial) strains are given as:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

The normal strain in any direction is caused by the stress in that direction – the *direct stress* – and by the normal stresses in the two perpendicular directions due to the *Poisson effect*.

For the same types of materials and as there is no change of volume due to shear, (no *Poisson effect* with shear) the shear strains are related to shear stresses by:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}; \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}; \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

These six equations defining the stress–strain relationship can be written in matrix form as follows:

$$[\boldsymbol{\varepsilon}]_{6\times1} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = [\boldsymbol{\sigma}]_{6\times1}$$

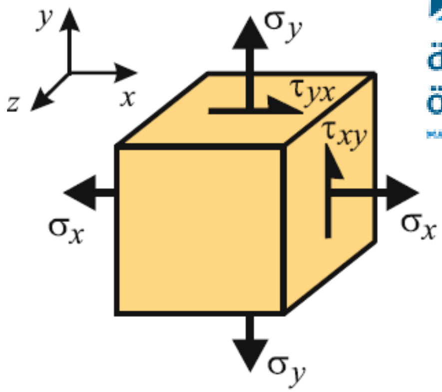
In short matrix notation: $[\boldsymbol{\varepsilon}]_{6\times1} = [\boldsymbol{F}]_{6\times6}[\boldsymbol{\sigma}]_{6\times1}$

Column matrices $[\boldsymbol{e}]_{6\times1}$ & $[\boldsymbol{s}]_{6\times1}$ are respectively the *strain vector* & the *stress vector*.

The square matrix $[\boldsymbol{F}]_{6\times6}$ is the *flexibility matrix* of the material.

The inverse of the *flexibility matrix* is the *stiffness matrix* $[\boldsymbol{K}] = [\boldsymbol{F}]^{-1}$.

In short matrix notation: $[\boldsymbol{\sigma}]_{6\times1} = [\boldsymbol{K}]_{6\times6}[\boldsymbol{\varepsilon}]_{6\times1}$

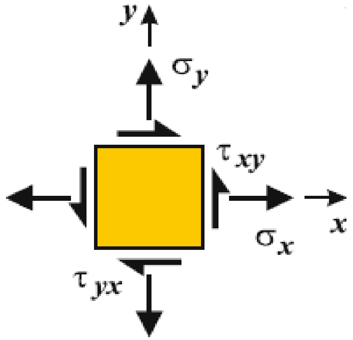


Special Stress and Strain States **:** Plane Stress

In many cases, loads applied to a system act in a single plane. When the stresses on an element act only in one plane (e.g., the *x–y plane in Fig.*), the state of stress is called ***plane stress***.

Plane stress generally exists when the loaded element is relatively thin in the out-of-plane direction.

The plane stress element is usually drawn in 2D (e.g., in the *x–y* plane).



Taking the *z-axis* as the out-of-plane direction, then the non-zero stresses are: σ_x, σ_y & τ_{xy}. The out-of-plane stresses (with *z* in their subscripts) are all zero: $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. *The non-zero strains are:*

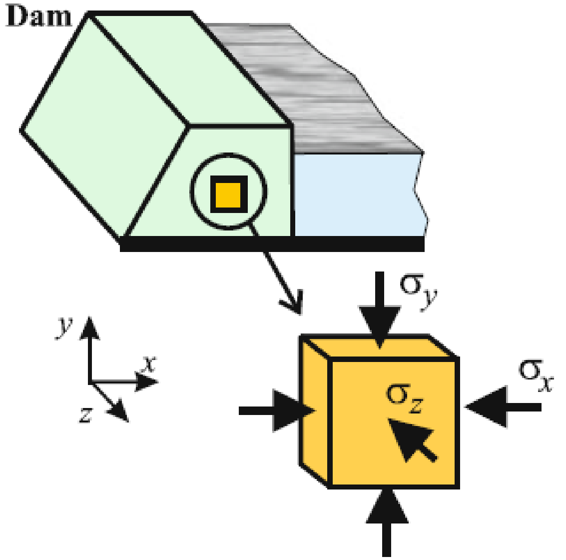
$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \varepsilon_z = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right) \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

Note that there is a non-zero normal strain in the *z-direction* ε_z due to the *Poisson effect*. However, the shear strains with components in the *z-direction* are zero ($\gamma_{yz} = \gamma_{zx} = 0$) since there are no out-of-plane shear stresses.

Special Stress and Strain States : Plane Strain

For *plane strain problems*, no deformation is allowed in the out-of-plane direction.

Plane strain generally occurs when the out-of-plane thickness is comparable to, or larger than, the in-plane dimensions (*Fig.*), or where a part is constrained between rigid objects (objects made of a much stiffer material).



With the *z-axis* as the out-of-plane direction: $\varepsilon_z = g_{yz} = g_{zx} = 0$. In order for there to be no out-of-plane normal strain:

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) = 0$$

an out-of-plane normal stress must be applied $\quad \sigma_z = \nu(\sigma_x + \sigma_y)$

The non-zero strains are:

$$\left\{\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{1}{E}\left\{\sigma_x - \nu\left[\sigma_y + \nu(\sigma_x + \sigma_y)\right]\right\} = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) \\
\varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] = \frac{1}{E}\left\{\sigma_y - \nu\left[\nu(\sigma_x + \sigma_y) + \sigma_x\right]\right\} = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}\right.$$

Special Stress and Strain States : Other Stress States

The following are special stress states of interest

(a) *Uniaxial stress:* Normal stress on an element that acts in only one direction; no shear stress acts.

(b) *Biaxial stress:* Normal stresses in 2 directions; no shear stress.

(c) *Triaxial stress:* Normal stresses in all 3 directions; no shear stress.

(special case) *Hydrostatic stress: Equal normal* stresses in all 3 directions; no shear stress.

(d) *Pure shear:* Only shear stresses act on an element (usually 2D).