



MATHEMATICAL ANALYSIS 1

Lecture

3

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Trigonometric Integrals

Products of Powers of Sines and Cosines

$$\int \sin^m x \cos^n x dx$$

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x dx$ equal to $-d(\cos x)$.

Case 2 If n is odd in $\int \sin^m x \cos^n x dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Case 1

m is odd

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x)(\cos^2 x)(-\sin x) \, dx$$

$$u = \cos x$$

$$= \int (1 - u^2)(u^2)(-du) = \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Case 2

$m=0$ is even and
 $n=5$ is odd

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 d(\sin x)$$

$$u = \sin x$$

$$= \int (1 - u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Case 3

m and n both even

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 \, dx$$

$$= \frac{1}{8} \int (1 - \cos 2x)(1 + 2 \cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx$$

$$= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right]$$

$$\int (\cos^2 2x + \cos^3 2x) dx$$

$$\int \cos^3 2x dx = \int (1 - \sin^2 2x) \cos 2x dx$$

$$u = \sin 2x$$

$$= \frac{1}{2} \int (1 - u^2) du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right).$$

$$\int \sin^2 x \cos^4 x dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C.$$

Integrals of Powers of $\tan x$ and $\sec x$

$$\tan^2 x = \sec^2 x - 1$$

Integration by part when necessary

EXAMPLE 6 Evaluate

$$\int \sec^3 x \, dx.$$

$$u = \sec x, \quad dv = \sec^2 x \, dx, \quad v = \tan x, \quad du = \sec x \tan x \, dx.$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x + \int \sec x \, dx - \underline{\int \sec^3 x \, dx}.$$

$$\xrightarrow{2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx}$$

$$\xrightarrow{\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.}$$

Products of Sines and Cosines

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx$$

$$\sin mx \sin nx = \frac{1}{2} [\cos(m - n)x - \cos(m + n)x],$$

$$\sin mx \cos nx = \frac{1}{2} [\sin(m - n)x + \sin(m + n)x],$$

$$\cos mx \cos nx = \frac{1}{2} [\cos(m - n)x + \cos(m + n)x].$$

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

$m = 3$ and $n = 5$,

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Exercises

Evaluate the integrals

- $\int_0^\pi \sin^5 \frac{x}{2} dx$

$$\frac{16}{15}$$

- $\int \sec^4 x \tan^2 x dx$

$$\frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$$

- $\int \sin^2 \theta \cos 3\theta d\theta$

$$\frac{1}{6} \sin 3\theta - \frac{1}{4} \sin \theta - \frac{1}{20} \sin 5\theta + C$$

- $\int \cos^3 2x \sin^5 2x dx$

$$\frac{1}{12} \sin^6 2x - \frac{1}{16} \sin^8 2x + C$$

- $\int \cot^6 2x dx$

$$-\frac{1}{10} \cot^5 2x + \frac{1}{6} \cot^3 2x - \frac{1}{2} \cot 2x - x + C$$

- $\int \frac{\sec^3 x}{\tan x} dx$

$$\sec x - \ln |\csc x + \cot x| + C$$

- $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} dx$

$$\sqrt{\frac{3}{2}} - \frac{2}{3}$$

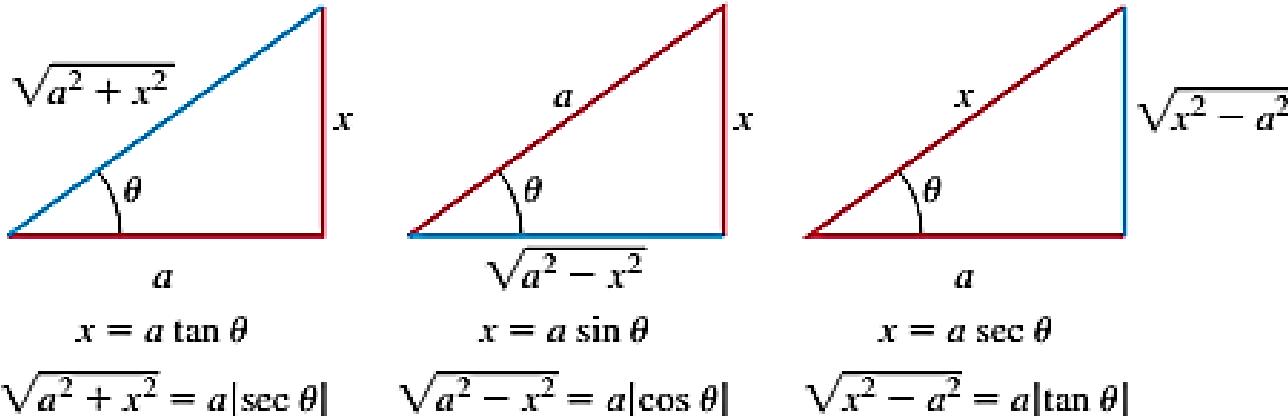
- $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx$

$$0$$

- $\int x \cos^3 x dx$

$$x \sin x - \frac{1}{3} x \sin^3 x + \frac{2}{3} \cos x + \frac{1}{9} \cos^3 x + C$$

Trigonometric Substitutions

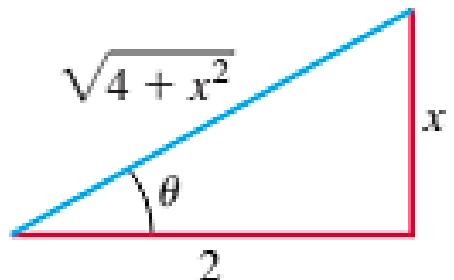


Procedure for a Trigonometric Substitution

1. Write down the substitution for x , calculate the differential dx , and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x .

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$



$$x = 2 \tan \theta, \quad dx = 2 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

$$\begin{aligned} \int \frac{dx}{\sqrt{4 + x^2}} &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C \end{aligned}$$

Exercises

- $\int \sqrt{25 - t^2} dt$

$$\frac{25}{2} \sin^{-1} \left(\frac{t}{5} \right) + \frac{t\sqrt{25-t^2}}{2} + C$$

- $\int \frac{\sqrt{x^2 + 4x + 3}}{x + 2} dx$

$$\sqrt{x^2 + 4x + 3} - \sec^{-1}(x + 2) + C$$

- $\int \frac{2 dx}{x^3 \sqrt{x^2 - 1}}, \quad x > 1$

$$\sec^{-1} x + \frac{\sqrt{x^2 - 1}}{x^2} + C$$

- $\int \frac{\sqrt{1 - (\ln x)^2}}{x \ln x} dx$

$$-\ln \left| \frac{1 + \sqrt{1 - (\ln x)^2}}{\ln x} \right| + \sqrt{1 - (\ln x)^2} + C$$

Rational Functions and Partial Fractions



$$f(x) = \frac{P(x)}{Q(x)}$$

numerator

denominator

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^n}$$

$$\deg(P(x)) \geq \deg(Q(x)) \xrightarrow{\text{Long Division}} f(x) = R(x) + \frac{K(x)}{H(x)}$$

$$\deg(H(x)) > \deg(K(x))$$

Case 1

$$Q(x) = (x - x_1)(x - x_2) \dots (x - x_m)$$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \dots + \frac{A_m}{x - x_m}$$

Find $\int \frac{1}{x(x-1)} dx$

Solution

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)}$$



$$\begin{aligned} A(x-1) + Bx &\equiv 1 \\ (A+B)x - A &\equiv 1 \end{aligned}$$

$$\left. \begin{aligned} x^0 : -A &= 1 \\ x^1 : A + B &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} A &= -1 \\ B &= 1 \end{aligned}$$

$$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$$



$$\int \frac{1}{x(x-1)} dx = \int \left(\frac{-1}{x} + \frac{1}{x-1} \right) dx = \ln \left| \frac{x-1}{x} \right| + C$$

Case 2

$$Q(x) = (x - x_1)^{r_1} (x - x_2)^{r_2} \dots (x - x_m)^{r_m}$$

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{A_1}{x - x_1} + \frac{A_2}{(x - x_1)^2} + \dots + \frac{A_{r_1}}{(x - x_1)^{r_1}} \\ &\quad + \frac{B_1}{x - x_2} + \frac{B_2}{(x - x_2)^2} + \dots + \frac{B_{r_2}}{(x - x_2)^{r_2}} \\ &\quad + \dots + \frac{C_1}{x - x_m} + \frac{C_2}{(x - x_m)^2} + \dots + \frac{C_{r_m}}{(x - x_m)^{r_m}}\end{aligned}$$

Find $\int \frac{1}{x^2(x+1)} dx$

Solution

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \Rightarrow \frac{1}{x^2(x+1)} = \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$Ax(x+1) + B(x+1) + Cx^2 \equiv 1$$

$$(A+C)x^2 + (A+B)x + B \equiv 1 \Rightarrow \left. \begin{array}{l} x^0 : B = 1 \\ x^1 : A + B = 0 \\ x^2 : A + C = 0 \end{array} \right\} \Rightarrow \begin{array}{l} A = -1 \\ B = 1 \\ C = 1 \end{array}$$

$$\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\int \frac{1}{x^2(x+1)} dx = \int \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx = \ln \left| \frac{x+1}{x} \right| - \frac{1}{x} + C$$

Case 3

$$Q(x) = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2) \dots (a_mx^2 + b_mx + c_m)$$

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \dots + \frac{A_mx + B_m}{a_mx^2 + b_mx + c_m}$$

Case 4

$$Q(x) = (a_1x^2 + b_1x + c_1)^{r_1} (a_2x^2 + b_2x + c_2)^{r_2} \dots (a_mx^2 + b_mx + c_m)^{r_m}$$

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{(a_1x^2 + b_1x + c_1)^2} + \dots + \frac{A_{r_1}x + B_{r_1}}{(a_1x^2 + b_1x + c_1)^{r_1}}$$

$$+ \frac{C_1x + D_1}{a_2x^2 + b_2x + c_2} + \frac{C_2x + D_2}{(a_2x^2 + b_2x + c_2)^2} + \dots + \frac{C_{r_2}x + D_{r_2}}{(a_2x^2 + b_2x + c_2)^{r_2}}$$

$$+ \dots + \frac{E_1x + F_1}{a_mx^2 + b_mx + c_m} + \frac{E_2x + F_2}{(a_mx^2 + b_mx + c_m)^2} + \dots + \frac{E_{r_m}x + F_{r_m}}{(a_mx^2 + b_mx + c_m)^{r_m}}$$

$$\int \frac{Ax + B}{ax^2 + bx + c} dx$$

Case 1

$$\int \frac{2ax + b}{ax^2 + bx + c} dx = \ln(ax^2 + bx + c) + c_1$$

Case 2

$$\int \frac{Ax + B}{ax^2 + bx + c} dx = \frac{1}{a} \int \frac{Ax + B}{x^2 + \frac{b}{a}x + \frac{c}{a}} dx = \frac{1}{a} \int \frac{Ax + B}{x^2 + px + q} dx$$

$$Ax + B = (2x + p)\frac{A}{2} + B - \frac{Ap}{2}$$

$$\frac{Ax + B}{x^2 + px + q} = \frac{A}{2} \cdot \underbrace{\frac{2x + p}{x^2 + px + q}}_I + \underbrace{\frac{B - \frac{Ap}{2}}{x^2 + px + q}}_{II}$$

$$\frac{Ax + B}{x^2 + px + q} = \frac{A}{2} \cdot \underbrace{\frac{2x + p}{x^2 + px + q}}_{I} + \underbrace{\frac{B - \frac{Ap}{2}}{x^2 + px + q}}_{II}$$

Completing the square

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}$$

Case 1

Using integral tables

Example

$$I = \int \frac{3x + 4}{x^2 + 7x + 14} dx$$

$$\frac{3x + 4}{x^2 + 7x + 14} = \frac{(2x + 7)\frac{3}{2} + 4 - \frac{21}{2}}{x^2 + 7x + 14} = \frac{(2x + 7)\frac{3}{2} - \frac{13}{2}}{x^2 + 7x + 14}$$

$$I = \frac{3}{2} \int \frac{2x + 7}{x^2 + 7x + 14} dx - \frac{13}{2} \int \frac{dx}{\left(x + \frac{7}{2}\right)^2 + \frac{7}{4}}$$

$$= \frac{3}{2} \ln(x^2 + 7x + 14) - \frac{13}{\sqrt{7}} \operatorname{arctg} \left(\frac{2x + 7}{\sqrt{7}} \right) + c$$

$$\int \frac{Ax + B}{(ax^2 + bx + c)^n} dx$$

$$\frac{Ax + B}{(x^2 + px + q)^n} = \underbrace{\frac{A}{2} \cdot \frac{2x + p}{(x^2 + px + q)^n}}_I + \left(B - \frac{Ap}{2} \right) \cdot \underbrace{\frac{1}{(x^2 + px + q)^n}}_{II}$$

$$\int \frac{du}{u^n} = -\frac{1}{1-n} \cdot \frac{1}{u^{n-1}} + C \quad (n \neq 1)$$

$$\frac{1}{(x^2 + px + q)^n} = \frac{1}{\left[\left(x + \frac{p}{2} \right)^2 + \left(q - \frac{p^2}{4} \right) \right]^n}$$

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

$$I_{n+1} = \frac{1}{2n \cdot a^2} \left[\frac{x}{(x^2 + a^2)^n} + (2n-1)I_n \right]$$

$$I = \int \frac{2x - 1}{(3x^2 + x + 7)^2} dx$$

$$= \frac{1}{9} \int \frac{2x + \frac{1}{3}}{\left(x^2 + \frac{x}{3} + \frac{7}{3}\right)^2} dx - \frac{4}{27} \int \frac{dx}{\left[\left(x + \frac{1}{6}\right)^2 + \frac{83}{36}\right]^2}$$

$$\frac{1}{9} \int \frac{2x + \frac{1}{3}}{\left(x^2 + \frac{x}{3} + \frac{7}{3}\right)^2} dx = -\frac{1}{9} \frac{1}{\left(x^2 + \frac{x}{3} + \frac{7}{3}\right)}$$

$$-\frac{4}{27} \int \frac{dx}{\left[\left(x + \frac{1}{6}\right)^2 + \frac{83}{36}\right]^2} = -\left(\frac{4}{27}\right) \left[\frac{1}{2(1)\left(\frac{83}{36}\right)} \right] \cdot \left[\frac{x + \frac{1}{6}}{\left(x + \frac{1}{6}\right)^2 + \frac{83}{36}} + I_1 \right]$$

$$-\frac{4}{27} \int \frac{dx}{\left[\left(x + \frac{1}{6}\right)^2 + \frac{83}{36}\right]^2} = -\left(\frac{4}{27}\right) \left[\frac{1}{2(1)\left(\frac{83}{36}\right)} \right] \cdot \left[\frac{x + \frac{1}{6}}{\left(x + \frac{1}{6}\right)^2 + \frac{83}{36}} + I_1 \right]$$

$$I = -\frac{1}{3(3x^2 + x + 7)} - \frac{8}{249} \left[\frac{x + \frac{1}{6}}{\left(x + \frac{1}{6}\right)^2 + \frac{83}{36}} + \frac{6}{\sqrt{83}} \operatorname{arctg} \left(\frac{6x + 1}{\sqrt{83}} \right) \right] + c$$

$$= -\frac{1}{3(3x^2 + x + 7)} - \frac{4}{3} \left[\frac{6x + 1}{83(3x^2 + x + 7)} + \frac{12}{83\sqrt{83}} \operatorname{arctg} \left(\frac{6x + 1}{\sqrt{83}} \right) \right] + c$$

Exercises

1)
$$\int \frac{7x - 8}{x^2 + 5x + 17} dx$$

$$\frac{7}{5} \ln(x^2 + 5x + 17) - \frac{51}{\sqrt{43}} \arctan\left(\frac{1}{\sqrt{43}}(2x + 5)\right) + C$$

2)
$$\int \frac{dz}{(1+z^2)^3}$$

$$\frac{1}{4} \frac{z}{(1+z^2)^2} + \frac{3}{8} \frac{z}{1+z^2} + \frac{3}{8} \arctan z + C$$

3)
$$\int \frac{3x + 5}{(x^2 + 2x + 5)^2} dx$$

$$\frac{1}{16} \frac{4x - 20}{x^2 + 2x + 5} + \frac{1}{8} \arctan\left(\frac{x+1}{2}\right) + C$$

4)
$$\int \frac{dx}{x^2 + 4x + 14}$$

$$\frac{1}{\sqrt{10}} \arctan\left(\frac{1}{\sqrt{10}}(x+2)\right) + C$$

5)
$$\int \frac{x^3 + x + 2}{(x-3)(x-4)} dx$$

$$\frac{1}{2}x^2 - x + \frac{66}{7} \ln(x+4) + \frac{32}{7} \ln(x-3) + C$$

6)
$$\int \frac{dx}{1-x^4}$$

$$\frac{1}{2} \arctan(x) + \frac{1}{2} \operatorname{arctanh}(x) + C$$

Integral of Rational Functions Containing sine and cosine

$$I = \int R(\sin x, \cos x) dx$$



$$\tan \frac{x}{2} = t$$

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt$$

$$I = \int \frac{P(t)}{Q(t)} dt$$

Example

$$I = \int \frac{5 + 6 \sin x}{\sin x (4 + 3 \cos x)} dx$$



$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt$$

$$\int \frac{5t^2 + 12t + 5}{t(7+t^2)} dt$$

Special Cases

Case 1

$R(\sin x, \cos x)$ is **odd** with respect to $\sin x$

$$I = \int R(\sin x, \cos x) dx$$



$$\cos x = t$$

$$I = \int \frac{P(t)}{Q(t)} dt$$

Case 2

$R(\sin x, \cos x)$ is **odd** with respect to $\cos x$

$$I = \int R(\sin x, \cos x) dx$$



$$\sin x = t$$

$$I = \int \frac{P(t)}{Q(t)} dt$$

Special Cases

Case 3

$R(\sin x, \cos x)$ is even with respect to $\sin x$ and $\cos x$

$$I = \int R(\sin x, \cos x) dx$$



$$\operatorname{tg} x = t$$

$$I = \int \frac{P(t)}{Q(t)} dt$$

Examples

1 $I = \int \frac{\cos^5 x}{\sin^4 x} dx$

Case 2

$\sin x = t$

$\int \left(\frac{1}{t^4} - \frac{2}{t^2} + 1 \right) dt$

2 $I = \int \frac{dx}{\sin^3 x \cdot \cos^2 x}$

Case 1

$\cos x = t$

$-\int \frac{dt}{(1-t^2)^2 \cdot t^2}$

3 $I = \int \frac{dx}{\sin^4 x \cdot \cos^2 x}$

Case 3

$\operatorname{tg} x = t$

$\int \frac{1+2t^2+t^4}{t^4} dt$