Exercises 4: Vector Spaces

CEDC102 : Linear Algebra and Matrix Theory

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Determine whether $\boldsymbol{W}$ is a subspace of the vector space $\boldsymbol{V}$

1. $W = \{(x, y) : x = 2y\}, \; V = R^2$

W is nonempty and $W \subset R^2$, W is closed under addition and scalar multiplication
$\Rightarrow$ W is a subspace of R^2

2. $W = \{(x, 2x, 3x) : x \text{ is a real number}\}, \; V = R^3$

W is nonempty and $W \subset R^3$, W is closed under addition and scalar multiplication
$\Rightarrow$ W is a subspace of R^3

3. $W = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 0\}$ $\qquad W = \{\mathbf{0}\}$, W is a subspace of R^3

4. $W = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$

W is not closed under addition or scalar multiplication, so it is not a subspace of R^3.
$(1, 0, 0) \in W$, and yet $2(1, 0, 0) = (2, 0, 0) \notin W$

Write each vector as a linear combination of the vectors in S (if possible)

1 $S = \{(2, -1, 3), (5, 0, 4)\}$

(a) $\boldsymbol{u} = (1, 1, -1)$

(b) $\boldsymbol{v} = (8, -1/4, 27/4)$

(a) $\boldsymbol{u} = (1, 1, -1) = c_1(2, -1, 3) + c_2(5, 0, 4)$

$$\begin{aligned} 2c_1 + 5c_2 &= 1 \\ -c_1 &= 1 \\ 3c_1 + 4c_2 &= -1 \end{aligned}$$

This system has no solution. So, $\boldsymbol{u}$ cannot be written as a linear combination of vectors in S

(b) $\boldsymbol{v} = (8, -1/4, 27/4) = c_1(2, -1, 3) + c_2(5, 0, 4)$

$$
\begin{aligned}
2c_1 + 5c_2 &= 8 \\
-c_1 \qquad\quad &= -1/4 \\
3c_1 + 4c_2 &= 27/4
\end{aligned}
$$

The solution to this system is $c_1 = 1/4$ and $c_2 = 3/2$. So, $\boldsymbol{v}$ can be written as a linear combination of vectors in S

2 $S = \{(2, 0, 7), (2, 4, 5), (2, -12, 13)\}$

(a) $\boldsymbol{u} = (-1, 5, -6)$

(b) $\boldsymbol{v} = (-3, 15, 18)$

(a) $\boldsymbol{u} = (-1, 5, -6) = c_1(2, 0, 7) + c_2(2, 4, 5) + c_3(2, -12, 13)$

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= -1 \\ 4c_2 - 12c_3 &= 5 \\ 7c_1 + 5c_2 + 13c_3 &= -6 \end{aligned}$$

One solution to this system is $c_1 = -7/4$, $c_2 = 5/4$ and $c_3 = 0$. So, $\boldsymbol{u}$ can be written as a linear combination of vectors in S

(b) $\boldsymbol{v} = (-3, 15, 18) = c_1(2, 0, 7) + c_2\,(2, 4, 5) + c_3\,(2, -12, 13)$

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= -3 \\ 4c_2 - 12c_3 &= 15 \\ 7c_1 + 5c_2 + 13c_3 &= 18 \end{aligned}$$

This system has no solution. So, $\boldsymbol{v}$ cannot be written as a linear combination of vectors in S

Determine whether the set S is linearly independent or linearly dependent

1. $S = \{(-2, 2), (3, 5)\}$

   $(-2, 2)$ is not a scalar multiple of $(3, 5)$. So the set S is linearly independent

2. $S = \{(0, 0), (1, -1)\}$

   $\mathbf{0} \in S \Rightarrow S$ is linearly dependent

3. $S = \{(3, -6), (-1, 2)\}$

   $(3, -6) = -3\,(-1, 2) \Rightarrow S$ is linearly dependent

4. $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$

   These vectors are multiples of each other, the set S is linearly dependent

5. $S = \{(-2, 1, 3), (2, 9, -3), (2, 3, -3)\}$

$$c_1(-2, 1, 3) + c_2\,(2, 9, -3) + c_3(2, 3, -3) = \mathbf{0} = (0, 0, 0)$$

$$\begin{aligned} -2c_1 + 2c_2 + 2c_3 &= 0 \\ c_1 + 9c_2 + 3c_3 &= 0 \\ 3c_1 - 3c_2 - 3c_3 &= 0 \end{aligned}$$

The system has many solutions. One solution is $(3, -2, 5)$, so S is linearly dependent

6. $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$

$$c_1(-4, -3, 4) + c_2\,(1, -2, 3) + c_3(6, 0, 0) = \mathbf{0} = (0, 0, 0)$$

$$\begin{aligned} -4c_1 + c_2 + 6c_3 &= 0 \\ -3c_1 - 2c_2 &= 0 \\ 4c_1 + 3c_2 &= 0 \end{aligned}$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$. So S is linearly independent

7 $S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$

$(1, -5, 3) = (1, 0, 0) + 5/4(0, 4, 0) + 1/2(0, 0, -6)$

The fourth vector is a linear combination of the first three. So S is linearly dependent

Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be three linearly independent vectors in a vector space $\boldsymbol{V}$. Is the set $\{\boldsymbol{v}_1 - 2\boldsymbol{v}_2, 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3, 3\boldsymbol{v}_3 - \boldsymbol{v}_1\}$ linearly dependent or linearly independent? Explain

To see if the given set is linearly independent, solve the equation

$$c_1(\boldsymbol{v}_1 - 2\boldsymbol{v}_2) + c_2(2\boldsymbol{v}_2 - 3\boldsymbol{v}_3) + c_3(3\boldsymbol{v}_3 - \boldsymbol{v}_1) = \mathbf{0}$$

$$(c_1 - c_3)\boldsymbol{v}_1 + (-2c_1 + 2c_2)\boldsymbol{v}_2 + (-3c_2 + 3c_3)\boldsymbol{v}_3 = \mathbf{0}$$

$\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be three linearly independent vectors $\Rightarrow$

$$\begin{aligned} c_1 \qquad\qquad - c_3 &= 0 \\ -2c_1 + 2c_2 \qquad\quad &= 0 \\ -3c_2 + 3c_3 &= 0 \end{aligned}$$

This system has infinitely many solutions, so $\{\boldsymbol{v}_1 - 2\boldsymbol{v}_2, 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3, 3\boldsymbol{v}_3 - \boldsymbol{v}_1\}$ is linearly dependent

Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be three linearly independent vectors in a vector space V. Is the set $\{\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \boldsymbol{v}_3 + \boldsymbol{v}_1\}$ linearly dependent or linearly independent? Explain

To see if the given set is linearly independent, solve the equation

$$c_1(\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_2(\boldsymbol{v}_2 + \boldsymbol{v}_3) + c_3(\boldsymbol{v}_3 + \boldsymbol{v}_1) = \mathbf{0}$$
$$(c_1 + c_3)\boldsymbol{v}_1 + (c_1 + c_2)\boldsymbol{v}_2 + (c_2 + c_3)\boldsymbol{v}_3 = \mathbf{0}$$

$\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be three linearly independent vectors $\Rightarrow$

$$\begin{aligned} c_1 \quad\quad\; + c_3 &= 0 \\ c_1 + c_2 \quad\quad\; &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

This system has only the trivial solution, so
$\{\boldsymbol{v}_1 - 2\boldsymbol{v}_2, 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3, 3\boldsymbol{v}_3 - \boldsymbol{v}_1\}$ is linearly dependent

Determine whether W is a subspace of the vector space V

1. $W = \{(x, y): x - y = 1\}, \ V = R^2$

2. Which of the subsets of R^3 is a subspace of R^3?

(a) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 0\}$

(b) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 1\}$

Write v as a linear combination of u_1, u_2, and u_3, if possible

1. $\boldsymbol{v} = (3, 0, -6),\ \boldsymbol{u}_1 = (1, -1, 2),\ \boldsymbol{u}_2 = (2, 4, -2),\ \boldsymbol{u}_3 = (1, 2, -4)$

2. $\boldsymbol{v} = (4, 4, 5),\ \boldsymbol{u}_1 = (1, 2, 3),\ \boldsymbol{u}_2 = (-2, 0, 1),\ \boldsymbol{u}_3 = (1, 0, 0)$

Determine whether the set S is linearly independent or linearly dependent

1. $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$

2. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$

3. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$

4. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$

5. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$