Exercises 5: Vector Spaces

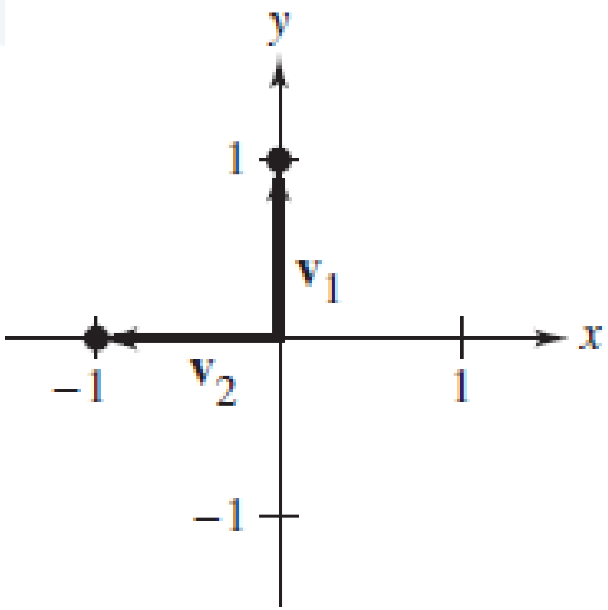
CEDC102 : Linear Algebra and Matrix Theory

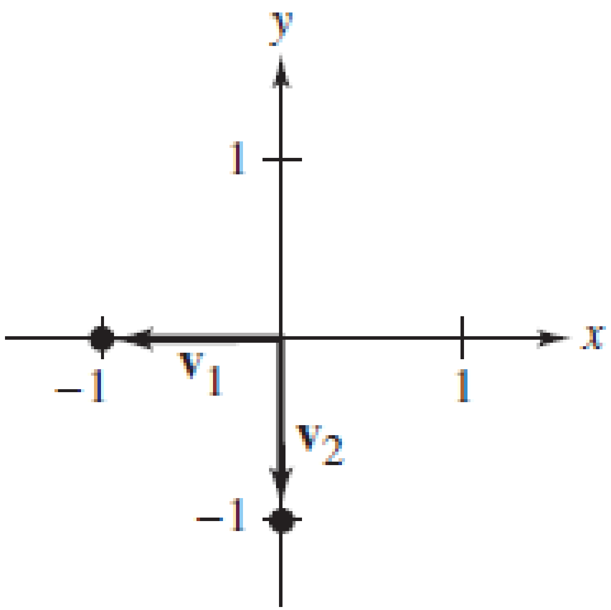
Manara University

2023-2024

Determine whether the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for $\boldsymbol{R}^2$

1





2

1. $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2
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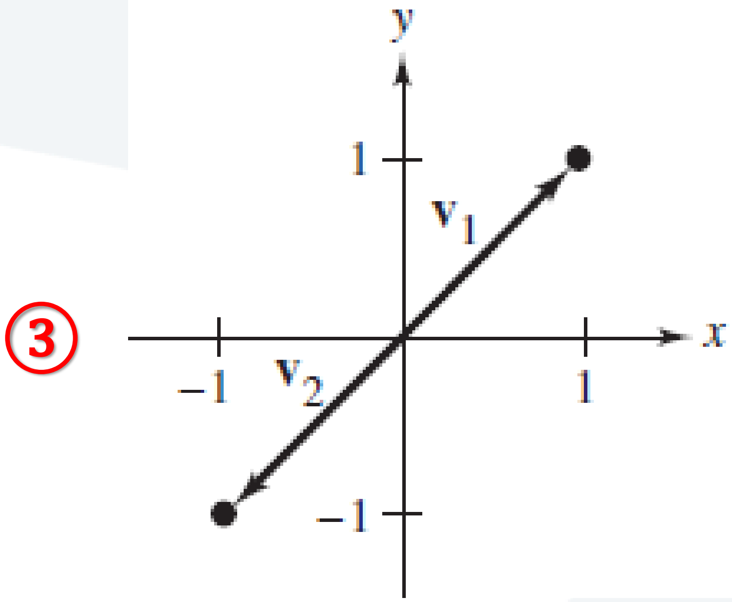
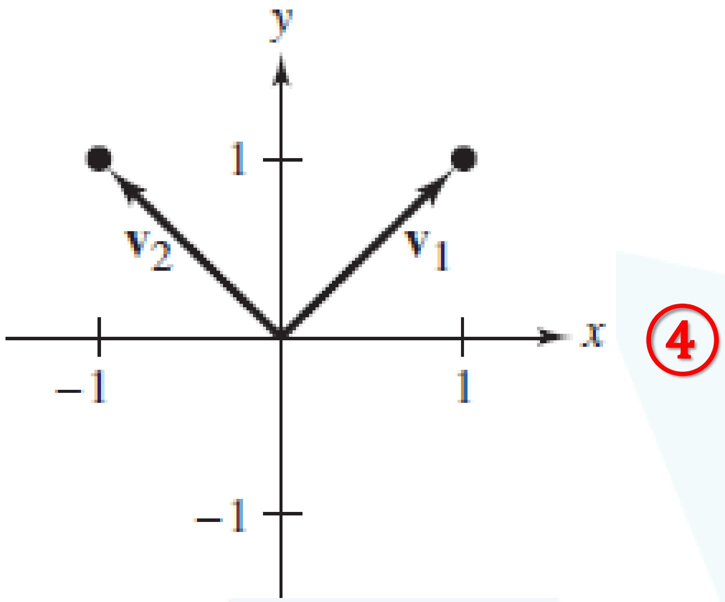
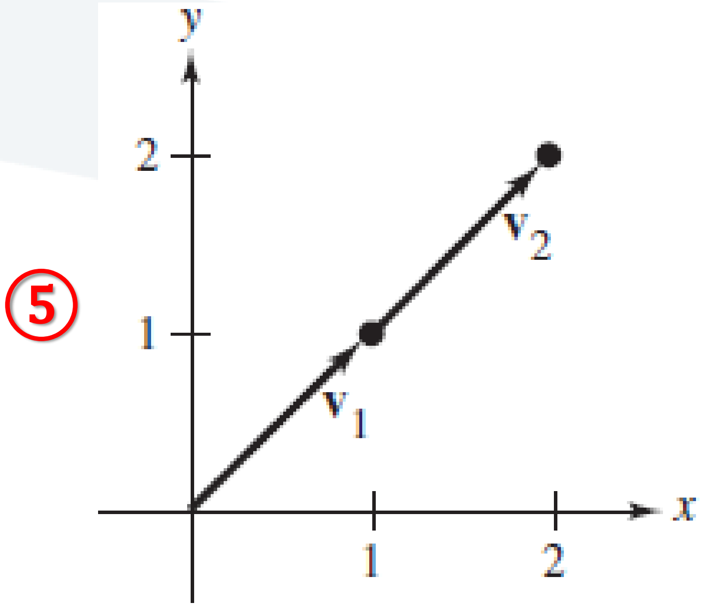
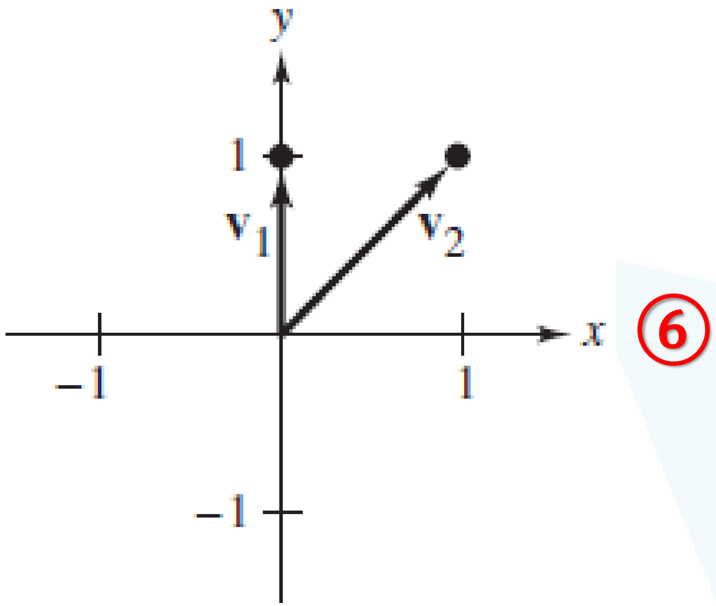




3. $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are multiples of each other, they do not form a basis for R^2

4. $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2





5. $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are multiplies of each other, they do not form a basis for R^2

6. $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

Determine whether S is a basis for the given vector space

1. $S = \{(4, -3), (5, 2)\}$ for R^2

   S consists of exactly two linearly independent vectors, it is a basis for R^2

2. $S = \{(1, 2), (1, -1), (-1, 2)\}$ for R^2

   S consists of more than two vectors, so S is linearly dependent, and it is not a basis for R^2

3. $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for R^3

   To determine if the vectors in S are linearly independent, find the solution to

   $c_1(1, 5, 3) + c_2\,(0, 1, 2) + c_3(0, 0, 6) = (0, 0, 0)$

   Which corresponds to the solution of

$$
\begin{aligned}
c_1 \qquad\qquad\qquad\qquad &= 0 \\
5c_1 + c_2 \qquad\qquad\quad &= 0 \\
3c_1 + 2c_2 + 6c_3 &= 0
\end{aligned}
$$

This system has only the trivial solution. So, S consists of exactly three linearly independent vectors, and is, therefore, a basis for R^3

4. $S = \{(2, 1, 0), (0, -1, 1)\}$ for R^3

S does not span R^3 (consists of less than three vectors), although it is linearly independent $\Rightarrow$ S is not a basis for R^3

5. $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for R^3

To determine if the vectors in S are linearly independent, find the solution to

$c_1(0, 3, -2) + c_2(4, 0, 3) + c_3(-8, 15, -16) = (0, 0, 0)$

which corresponds to the solution of

$$\begin{aligned} 4c_2 - 8c_3 &= 0 \\ 3c_1 + 15c_3 &= 0 \\ -2c_1 + 3c_2 - 16c_3 &= 0 \end{aligned}$$

This system has nontrivial solutions (for instance, $c_1 = -5$, $c_2 = 2$ and $c_3 = 1$), so the vectors are linearly dependent, and S is not a basis for R^3

6 $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for R^3

This set contains the zero vector, and is, therefore, linearly dependent. So, S is not a basis for R^3

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Determine whether the set, (a) is linearly independent, and (b) is a basis for R^3

1 $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$

(a) $c_1(1, -5, 4) + c_2(11, 6, -1) + c_3(2, 3, 5) = (0, 0, 0)$

$$\begin{aligned} c_1 + 11c_2 + 2c_3 &= 0 \\ -5c_1 + 6c_2 + 3c_3 &= 0 \\ 4c_1 - c_2 + 5c_3 &= 0 \end{aligned}$$

This system has only the trivial solution. So, S is linearly independent.

(b) S consists of exactly three linearly independent vectors, and is, therefore, a basis for R^3

2 $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$

(a) S is linearly dependent because the 4th vector is a linear combination of the first three $(-1, 2, -3) = -1(1, 0, 0) + 2(0, 1, 0) - 3(0, 0, 1)$

(b) S is not a basis because it is not linearly independent

Find the **rank** and nullity of the matrix $\boldsymbol{A}$

1 $A = \begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix}$

2 $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$,
$\text{nullity}(A) = 4 - 2 = 2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\operatorname{rank}(A) = 2$,
$\operatorname{nullity}(A) = 3 - 2 = 1$

Given the coordinate matrix of $\boldsymbol{x}$ relative to a (nonstandard) basis $\boldsymbol{B}$ for $\boldsymbol{R}^n$, find the coordinate matrix of $\boldsymbol{x}$ relative to the standard basis

1. $B = \{(1, 1), (-1, 1)\}$, $[\boldsymbol{x}]_B = [3 \;\; 5]^T$

   $\boldsymbol{x} = 3(1, 1) + 5(-1, 1) = (-2, 8)$

$(-2, 8) = -2(1, 0) + 8(0, 1)$, the coordinate vector of $\boldsymbol{x}$ relative to the standard basis is $[\boldsymbol{x}]_S = [-2 \;\; 8]^T$

② $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}$, $[\boldsymbol{x}]_B = [2 \;\; 0 \;\; -1]^T$

$$\boldsymbol{x} = 2(1, 0, 0) + 0(1, 1, 0) - 1(0, 1, 1) = (2, -1, -1)$$

$(-2, -1, -1) = 2(1, 0, 0) - 1(0, 1, 0) - 1(0, 0, 1)$, the coordinate vector of $\boldsymbol{x}$ relative to the standard basis is $[\boldsymbol{x}]_B = [2 \;\; -1 \;\; -1]^T$

Find the coordinate matrix of $\boldsymbol{x}$ in $\boldsymbol{R}^n$ relative to the basis $\boldsymbol{B'}$

① $B' = \{(5, 0), (0, -8)\}$, $\boldsymbol{x} = (2, 2)$

$$c_1(5, 0) + c_2(0, -8) = (2, 2)$$

The resulting system of linear equations is

$$\begin{aligned} 5c_1 \qquad\qquad &= 2 \\ -8c_2 \quad &= 2 \end{aligned}$$

So $c_1 = 2/5$, $c_2 = -1/4 \Rightarrow [\boldsymbol{x}]_{B'} = [2/5 \;\; -1/4]^T$

② $B' = \{(1, 2, 3), (1, 2, 0), (0, -6, 2)\}$, $\boldsymbol{x} = (3, -3, 0)$

$$c_1(1, 2, 3) + c_2(1, 2, 0) + c_3(0, -6, 2) = (3, -3, 0)$$

$$\begin{aligned} c_1 + c_2 \qquad\qquad &= 3 \\ 2c_1 + 2c_2 - 6c_3 &= -3 \\ 3c_1 \qquad + 2c_3 &= 0 \end{aligned}$$

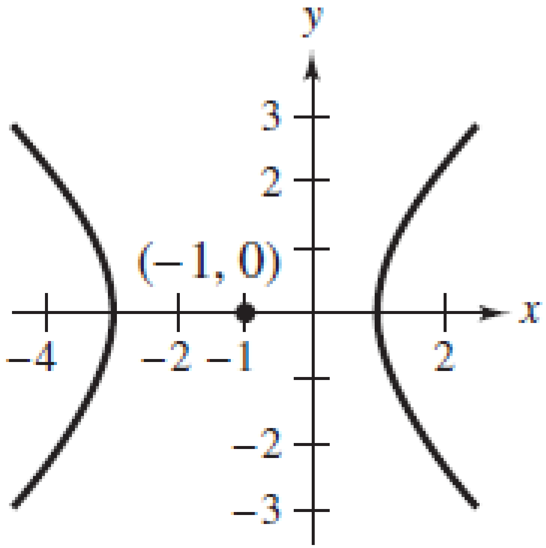
The solution is $c_1 = -1$, $c_2 = 4$ and $c_3 = 3/2 \Rightarrow [\boldsymbol{x}]_{B'} = [-1 \;\; 4 \;\; 3/2]^T$

Identify and sketch the graph of the conic section

① $x^2 - y^2 + 2x - 3 = 0$

$$(x+1)^2 - 1 - y^2 - 3 = 0$$

$$(x+1)^2 - y^2 = 4 \quad \Rightarrow \quad \frac{(x+1)^2}{2^2} - \frac{y^2}{2^2} = 1$$



② $16x^2 + 25y^2 - 32x - 50y + 16 = 0$

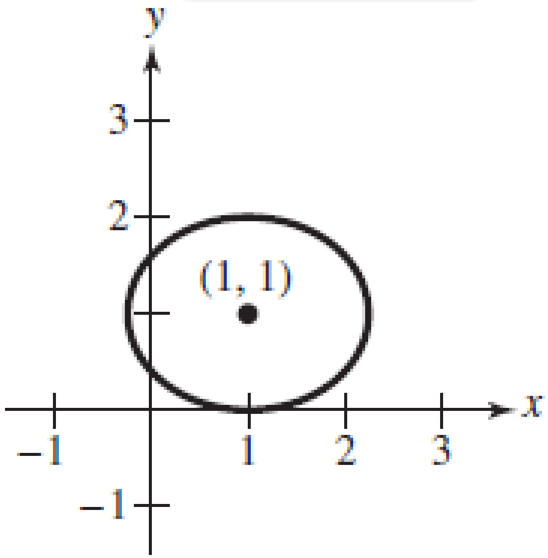
$$16(x^2 - 2x) + 25(y^2 - 2y) + 16 = 0$$

$$16(x-1)^2 - 16 + 25(y-1)^2 - 25 + 16 = 0$$

$$16(x-1)^2 + 25(y-1)^2 = 25$$

$$\frac{(x-1)^2}{25/16} - (y-1)^2 = 1$$

This is the equation of an ellipse centered at (1, 1) and a = 5/4, b = 1



Perform a rotation of axes to eliminate the xy-term, and sketch the graph of the conic

$$7x^2 + 6\sqrt{3}xy + 13y^2 - 16 = 0$$

$$\cot 2\theta = \frac{a-c}{b} = \frac{7-13}{6\sqrt{3}} = \frac{-1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{6}$$

$$\Rightarrow \sin\theta = -\frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}$$

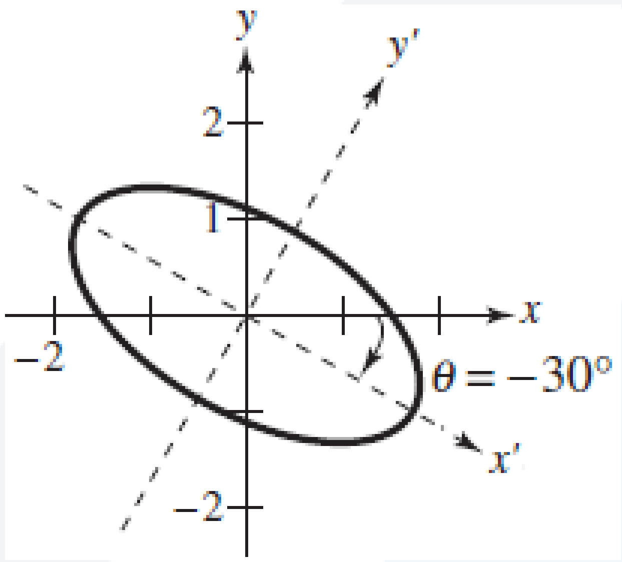
$$x = x'\cos\theta - y'\sin\theta = \frac{1}{2}(\sqrt{3}x' + y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{2}(\sqrt{3}y' - x')$$

$$7x^2 + 6\sqrt{3}xy + 13y^2 - 16 = 0 \Rightarrow 4(x')^2 + 16(y')^2 = 16$$

$$\frac{(x')^2}{4}+(y')^2=1$$

Ellipse centered at (0, 0) with the major axis along the x'-axis and $a=2,\ b=1$



Determine whether the set, (a) is linearly independent, and (b) is a basis for $\boldsymbol{R}^3$

1. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$

2. $S = \{(-1/2, 3/4, -1), (5, 2, 3), (-4, 6, -8)\}$

3. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$

4. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$

Given the coordinate matrix of $\boldsymbol{x}$ relative to a (nonstandard) basis $\boldsymbol{B}$ for $\boldsymbol{R}^n$, find the coordinate matrix of $\boldsymbol{x}$ relative to the standard basis

1. $B = \{(2, 4), (-1, 1)\}, [\boldsymbol{x}]_B = [4 \quad -7]^T$

2. $B = \{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}, [\boldsymbol{x}]_B = [4 \quad 0 \quad 2]^T$

Find the **rank** and nullity of the matrix $\boldsymbol{A}$

1 $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 4 & -2 & 4 & -2 \\ -2 & 0 & 1 & 3 \end{bmatrix}$

2 $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 4 & 0 & 3 \\ -2 & 3 & 0 & 2 \\ 1 & 2 & 6 & 1 \end{bmatrix}$

Find the coordinate matrix of $\boldsymbol{x}$ in $\boldsymbol{R}^n$ relative to the basis $\boldsymbol{B'}$

1. $B' = \{(2, 2), (0, -1)\}$, $\boldsymbol{x} = (-1, 2)$

2. $B' = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$, $\boldsymbol{x} = (4, -2, 9)$

Identify and sketch the graph of the conic section

1. $x^2 + y^2 + 4x - 2y - 11 = 0$

2. $9x^2 + 9y^2 + 18x - 18y + 14 = 0$

3. $2x^2 - 20x - y + 46 = 0$

4. $4x^2 + y^2 + 32x + 4y + 63 = 0$

Perform a rotation of axes to eliminate the ***xy***-term, and sketch the graph of the conic

1. $xy = 3$

2. $9x^2 + 4xy + 9y^2 - 20 = 0$

3. $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y = 0$