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قسم الروبوت و الأنظمة الذكية

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المحاضرة الرابعة

نظم التحكم اللاخطي
Nonlinear Control systems

مدرس المقرر

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Nonlinear System Behavior

- **Limit Cycles**

Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation. These oscillations are called limit cycles, or self-excited oscillations. This important phenomenon can be simply illustrated by a famous oscillator dynamics, first studied in the 1920's by the Dutch electrical engineer Balthasar Van der Pol.

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**

The second-order nonlinear differential equation

$$m\ddot{x} + 2c(x^2 - l)\,\dot{x} + kx = 0$$

where **m, c** and **k** are positive constants, is the famous Van der Pol equation. It can be regarded as describing a mass-spring-damper system with a position-dependent damping coefficient **$2c(x^2 - 1)$** (or, equivalently, an RLC electrical circuit with a nonlinear resistor).

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**
- For large values of **x**, the damping coefficient is positive and the damper removes energy from the system. This implies that the system motion has a convergent tendency. However, for small values of **x**, the damping coefficient is negative and the damper adds energy into the system. This suggests that the system motion has a divergent tendency.

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**
- Therefore, because the nonlinear damping varies with **x**, the system motion can neither grow unboundedly nor decay to zero. Instead, it displays a sustained oscillation independent of initial conditions, as illustrated in Figure. This so-called limit cycle is sustained by periodically releasing energy into and absorbing energy from the environment, through the damping term. This is in contrast with the case of a conservative mass spring system, which does not exchange energy with its environment during its vibration.

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**

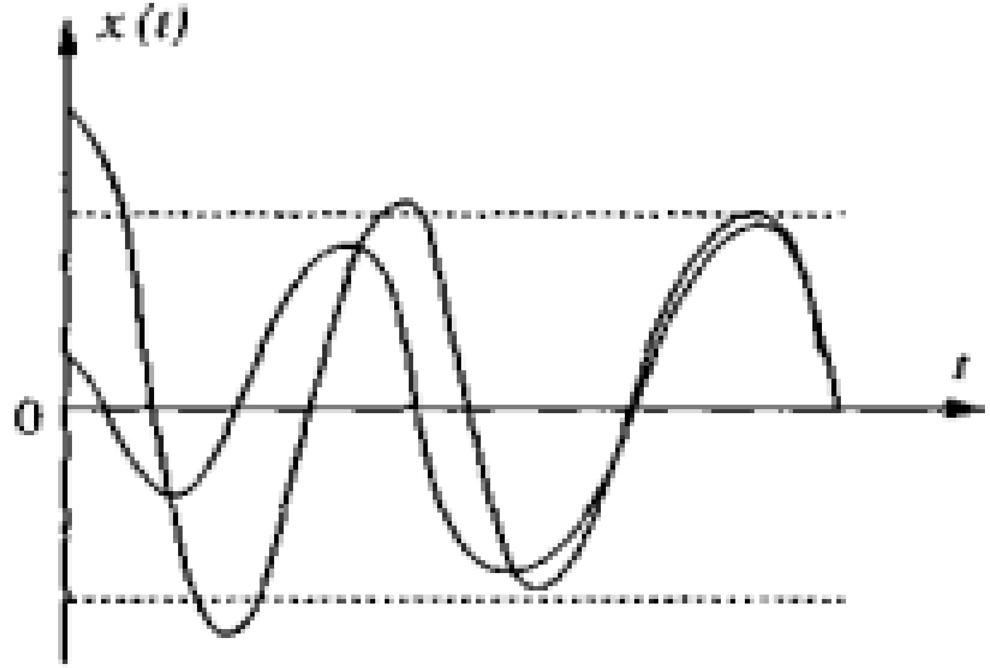


Figure 1.4 : Responses of the Van der Pol oscillator

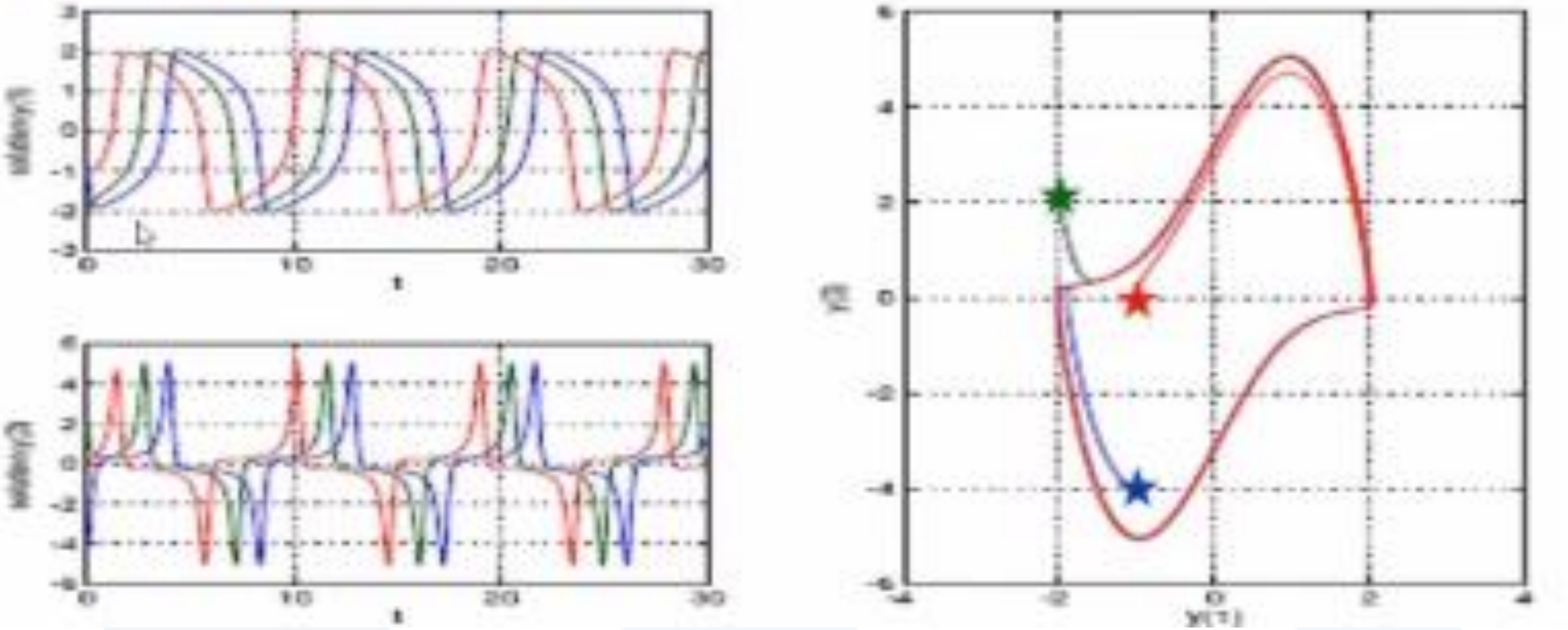
Nonlinear System Behavior

- **Limit Cycles**
- Of course, sustained oscillations can also be found in linear systems, in the case of marginally stable linear systems (such as a mass-spring system without damping) or in the response to sinusoidal inputs. However, limit cycles in nonlinear systems are different from linear oscillations in a number of fundamental aspects. First, the amplitude of the self-sustained excitation is independent of the initial condition, as seen in Figure, while the oscillation of a marginally stable linear system has its amplitude determined by its initial conditions. Second, marginally stable linear systems are very sensitive to changes in system parameters (with a slight change capable of leading either to stable convergence or to instability), while limit cycles are not easily affected by parameter changes.

Nonlinear System Behavior

- **Limit Cycles**

- Limit Cycle
  - ✓ Stable oscillations whose amplitude does not depend on the initial conditions.



Nonlinear System Behavior

- **Limit Cycles**
- Limit cycles represent an important phenomenon in nonlinear systems. They can be found in many areas of engineering and nature. Aircraft wing fluttering, a limit cycle caused by the interaction of aerodynamic forces and structural vibrations, is frequently encountered and is sometimes dangerous. The hopping motion of a legged robot is another instance of a limit cycle. Limit cycles also occur in electrical circuits, e.g., in laboratory electronic oscillators. As one can see from these examples, limit cycles can be undesirable in some cases, but desirable in other cases. An engineer has to know how to eliminate them when they are undesirable, and conversely how to generate or amplify them when they are desirable. To do this, however, requires an understanding of the properties of limit cycles and a familiarity with the tools for manipulating them.

Nonlinear System Behavior

- **Bifurcations**
- As the parameters of nonlinear dynamic systems are changed, the stability of the equilibrium point can change (as it does in linear systems) and so can the number of equilibrium points. Values of these parameters at which the qualitative nature of the system's motion changes are known as critical or bifurcation values. The phenomenon of bifurcation, i.e., quantitative change of parameters leading to qualitative change of system properties, is the topic of bifurcation theory.

Nonlinear System Behavior

- **Bifurcations**
- For instance, the smoke rising from an incense stick (smokestacks and cigarettes are old-fashioned) first accelerates upwards (because it is lighter than the ambient air), but beyond some critical velocity breaks into swirls. More prosaically, let us consider the system described by the so-called undamped Duffing equation

$$\ddot{x} + \alpha x + x^3 = 0$$

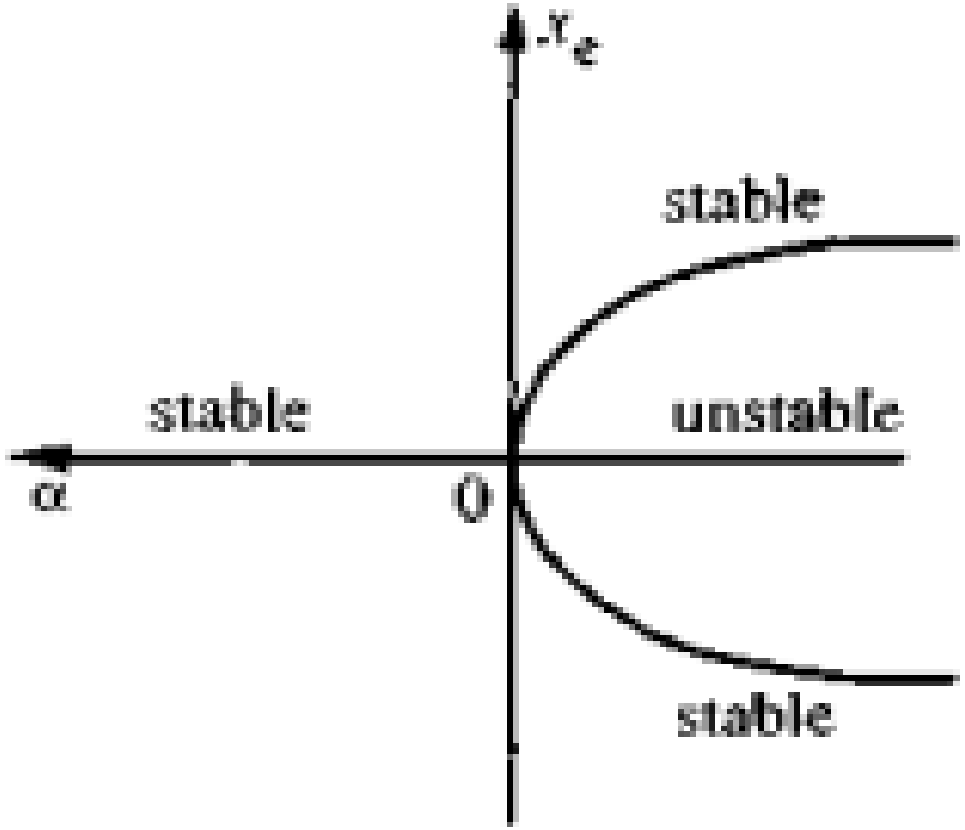
(the damped Duffing equation is $\ddot{x} + c\dot{x} + \alpha x + \beta x^3 = 0$, which may represent a mass-damper-spring system with a hardening spring).

Nonlinear System Behavior

- **Bifurcations**
- We can plot the equilibrium points as a function of the parameter **α**. As **α** varies from positive to negative, one equilibrium point splits into three points ($\mathbf{x_e = 0}, \sqrt{\alpha}, -\sqrt{\alpha}$), as shown in Figure (a). This represents a qualitative change in the dynamics and thus **α = 0** is a critical bifurcation value. This kind for bifurcation is known as a pitchfork, due to the shape of the equilibrium point plot in Figure (a).

Nonlinear System Behavior

- **Bifurcations**



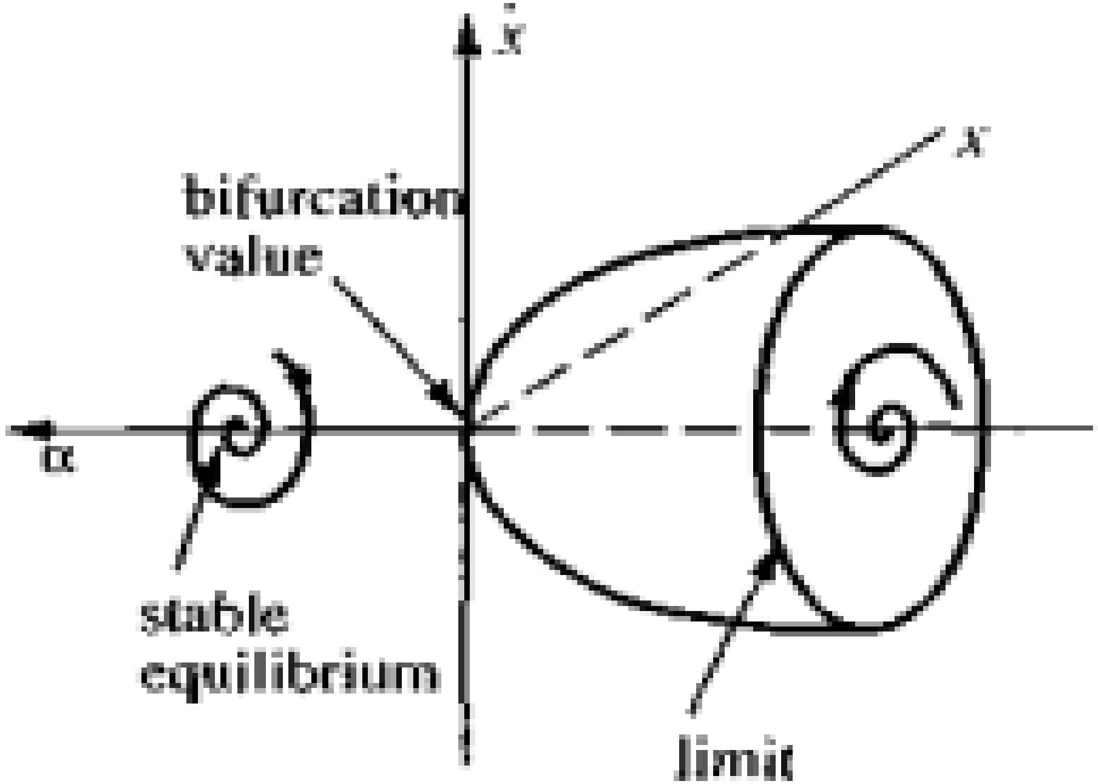
(a)

Nonlinear System Behavior

- **Bifurcations**
- Another kind of bifurcation involves the emergence of limit cycles as parameters are changed. In this case, a pair of complex conjugate eigenvalues $\mathbf{P_1 = \Upsilon + j\omega}$, $\mathbf{P_2 = \Upsilon - j\omega}$ cross from the left-half plane into the right-half plane, and the response of the unstable system diverges to a limit cycle. Figure (b) depicts the change of typical system state trajectories (states are $\mathbf{x}$ and $\dot{\mathbf{x}}$) as the parameter $\boldsymbol{\alpha}$ is varied. This type of bifurcation is called a Hopf bifurcation.

Nonlinear System Behavior

- **Bifurcations**



(b)

Nonlinear System Behavior

- **Chaos**
- For stable linear systems, small differences in initial conditions can only cause small differences in output. Nonlinear systems, however, can display a phenomenon called chaos, by which we mean that the system output is extremely sensitive to initial conditions. The essential feature of chaos is the unpredictability of the system output. Even if we have an exact model of a nonlinear system and an extremely accurate computer, the system's response in the long-run still cannot be well predicted.

Nonlinear System Behavior

- **Chaos**
- Chaos must be distinguished from random motion. In random motion, the system model or input contain uncertainty and, as a result, the time variation of the output cannot be predicted exactly (only statistical measures are available). In chaotic motion, on the other hand, the involved problem is deterministic, and there is little uncertainty in system model, input, or initial conditions.

Nonlinear System Behavior

- **Chaos**
- As an example of chaotic behavior, let us consider the simple nonlinear system

$$\ddot{x} + 0.1\dot{x} + x^5 = 6 \sin t$$

- which may represent a lightly-damped, sinusoidally forced mechanical structure undergoing large elastic deflections. Figure shows the responses of the system corresponding to two almost identical initial conditions, namely $\mathbf{x(0) = 2, \dot{x}(0) = 3}$ (thick line) and $\mathbf{x(0) = 2.01, \dot{x}(0) = 3.01}$ (thin line). Due to the presence of the strong nonlinearity in $\mathbf{x^5}$, the two responses are radically different after some time.

Nonlinear System Behavior

- **Chaos**

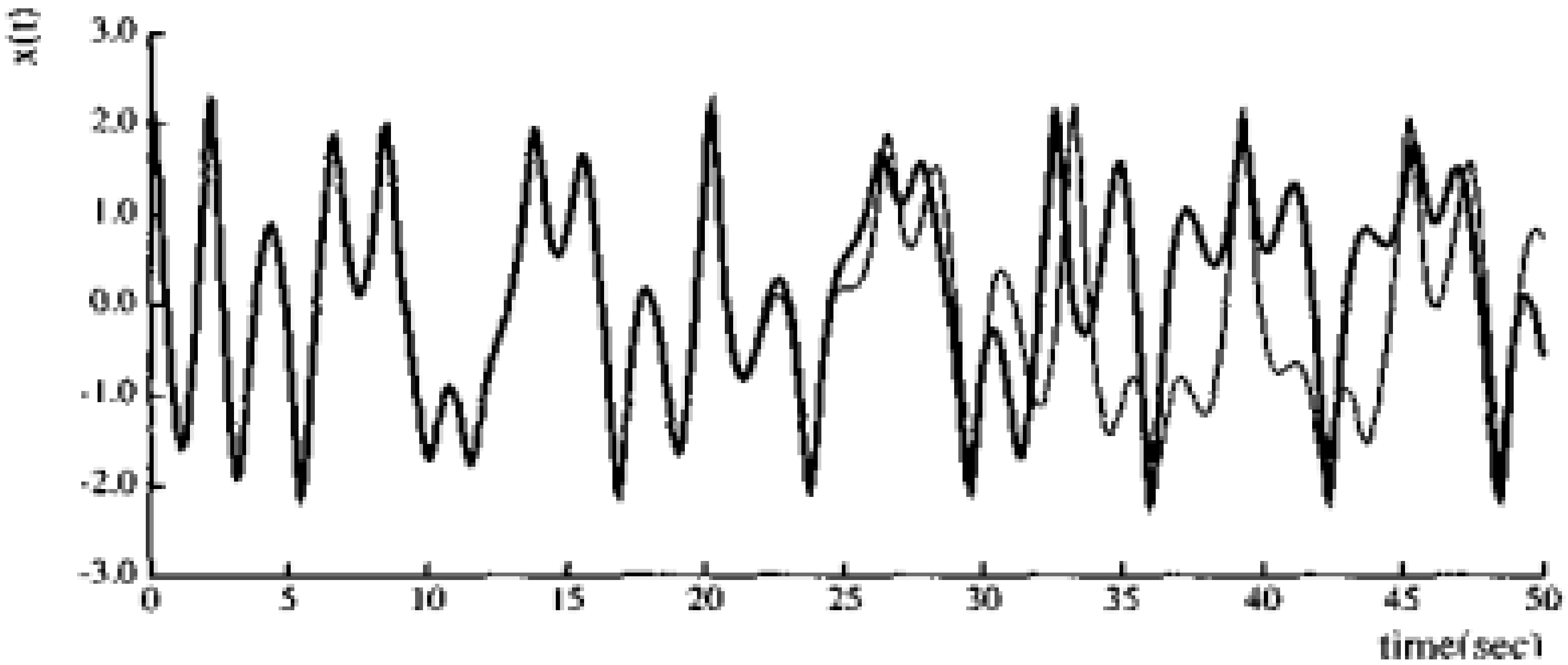


Figure 1.6 : Chaotic behavior of a nonlinear system

Nonlinear System Behavior

- **Chaos**
- Chaotic phenomena can be observed in many physical systems. The most commonly seen physical problem is turbulence in fluid mechanics (such as the swirls of our incense stick). Atmospheric dynamics also display clear chaotic behavior, thus making long-term weather prediction impossible. Some mechanical and electrical systems known to exhibit chaotic vibrations include buckled elastic structures, mechanical systems with play or backlash, systems with aeroelastic dynamics, wheelrail dynamics in railway systems, and, of course, feedback control devices.

Nonlinear System Behavior

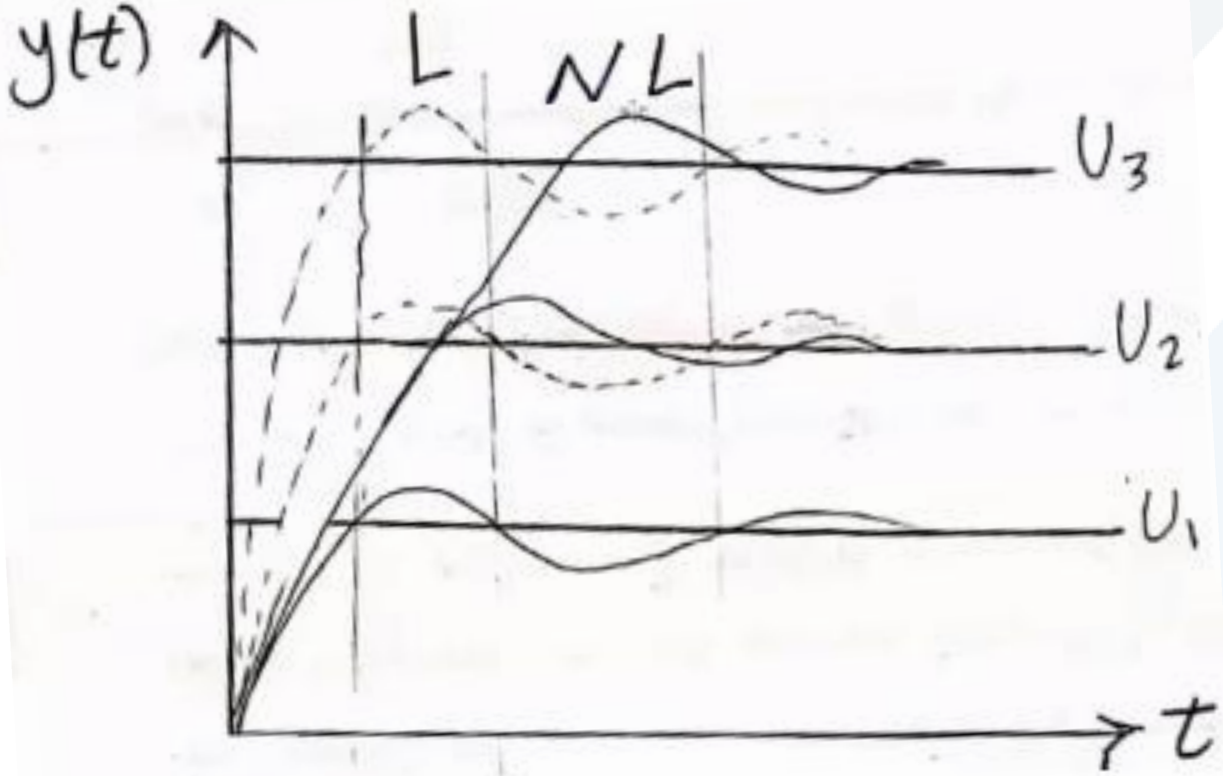
- **Chaos**
- Chaos occurs mostly in strongly nonlinear systems. This implies that, for a given system, if the initial condition or the external input cause the system to operate in a highly nonlinear region, it increases the possibility of generating chaos. Chaos cannot occur in linear systems. Corresponding to a sinusoidal input of arbitrary magnitude, the linear system response is always a sinusoid of the same frequency. By contrast, the output of a given nonlinear system may display sinusoidal, periodic, or chaotic behaviors, depending on the initial condition and the input magnitude.

Nonlinear System Behavior

- **Chaos**
- In the context of feedback control, it is of course of interest to know when a nonlinear system will get into a chaotic mode (so as to avoid it) and, in case it does, how to recover from it. Such problems are the object of active research.

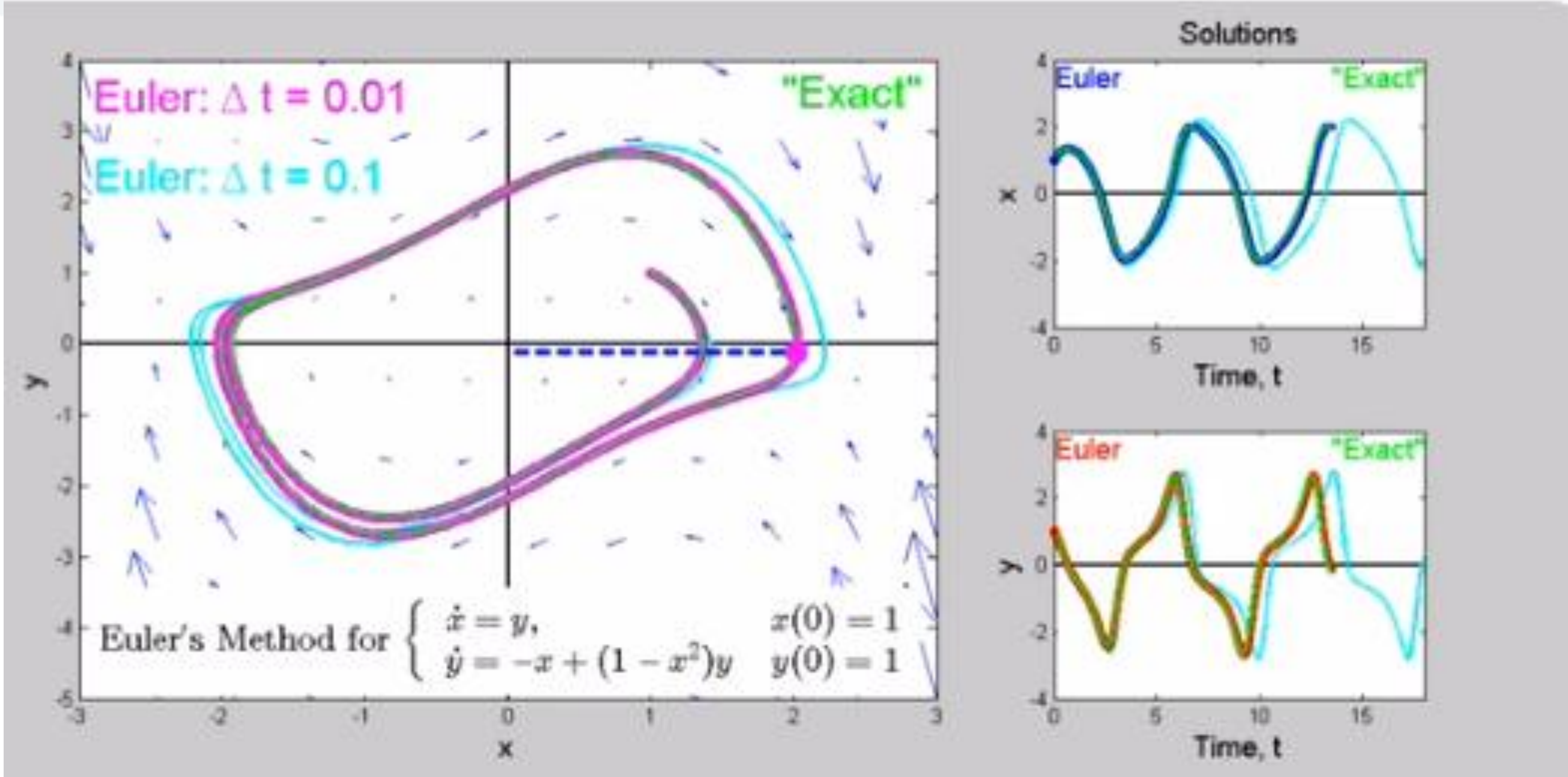
Nonlinear System Behavior

- **Other behaviors**
- Other interesting types of behavior, such as jump resonance, subharmonic generationand frequency-amplitude dependence of free vibrations, can also occur and become important in some system studies. However, the above description should provide ample evidence that nonlinear systems can have considerably richer and more complex behavior than linear systems.


y(t)
L
NL
U3
U2
U1
t

Stability & Output of systems

- Stability depends on the system's parameter (linear)
- Stability depends on the initial conditions, input signals as well as the system parameters (nonlinear).
- Output of a linear system has the same frequency as the input although its amplitude and phase may differ.
- Output of a nonlinear system usually contains additional frequency components and may, in fact, not contain the input frequency.


Euler: Δ t = 0.01
Euler: Δ t = 0.1
"Exact"
Euler's Method for $\begin{cases} \dot{x} = y, & x(0) = 1 \\ \dot{y} = -x + (1 - x^2)y & y(0) = 1 \end{cases}$
x
y
Solutions
Euler
"Exact"
x
Time, t
Euler
"Exact"
y
Time, t