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Nonlinear Control systems

مدرس المقرر

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Nonlinear Systems Analysis

- The objective of this part is to present various tools available for analyzing nonlinear control systems. The study of these nonlinear analysis techniques is important for a number of reasons.
- First, theoretical analysis is usually the least expensive way of exploring a system's characteristics.
- Second, simulation, though very important in nonlinear control, has to be guided by theory. Blind simulation of nonlinear systems is likely to produce few results or misleading results. This is especially true given the great richness of behavior that nonlinear systems can exhibit, depending on initial conditions and inputs.

Nonlinear Systems Analysis

- Third, the design of nonlinear controllers is always based on analysis techniques. Since design methods are usually based on analysis methods, it is almost impossible to master the design methods without first studying the analysis tools. Furthermore, analysis tools also allow us to assess control designs after they have been made, and, in case of inadequate performance, they may also suggest directions of modifying the control designs.

Nonlinear Systems Analysis

- It should not come as a surprise that no universal technique has been devised for the analysis of all nonlinear control systems. In linear control, one can analyze a system in the time domain or in the frequency domain. However, for nonlinear control systems, none of these standard approaches can be used, since direct solution of nonlinear differential equations is generally impossible, and frequency domain transformations do not apply

Nonlinear Systems Analysis
(**Phase plane**)

- Phase plane analysis is a graphical method of studying second-order nonlinear systems. Its basic idea is to solve a second order differential equation graphically, instead of seeking an analytical solution. The result is a family of system motion trajectories on a two-dimensional plane, called the phase plane, which allow us to visually observe the motion patterns of the system.
- While phase plane analysis has a number of important advantages, it has the fundamental disadvantage of being applicable only to systems which can be well approximated by a second-order dynamics. Because of its graphical nature, it is frequently used to provide intuitive insights about nonlinear effects.

Nonlinear Systems Analysis (**Lyapunov theory**)

- Basic Lyapunov theory comprises two methods introduced by Lyapunov, the indirect method and the direct method.
- The indirect method, or linearization method, states that the stability properties of a nonlinear system in the close vicinity of an equilibrium point are essentially the same as those of its linearized approximation. The method serves as the theoretical justification for using linear control for physical systems, which are always inherently nonlinear.

Nonlinear Systems Analysis (**Lyapunov theory**)

- The direct method is a powerful tool for nonlinear system analysis, and therefore the so-called Lyapunov analysis often actually refers to the direct method. The direct method is a generalization of the energy concepts associated with a mechanical system: the motion of a mechanical system is stable if its total mechanical energy decreases all the time. In using the direct method to analyze the stability of a nonlinear system, the idea is to construct a scalar energy-like function (a Lyapunov function) for the system, and to see whether it decreases. The power of this method comes from its generality: it is applicable to all kinds of control systems, be they time-varying or time-invariant, finite dimensional or infinite dimensional. Conversely, the limitation of the method lies in the fact that it is often difficult to find a Lyapunov function for a given system.

Nonlinear Systems Analysis (**Lyapunov theory**)

- Although Lyapunov's direct method is originally a method of stability analysis, it can be used for other problems in nonlinear control. One important application is the design of nonlinear controllers. The idea is to somehow formulate a scalar positive function of the system states, and then choose a control law to make this function decrease. A nonlinear control system thus designed will be guaranteed to be stable. Such a design approach has been used to solve many complex design problems, e.g., in robotics and adaptive control. The direct method can also be used to estimate the performance of a control system and study its robustness.

Nonlinear Systems Analysis (**Describing functions**)

- The describing function method is an approximate technique for studying nonlinear systems. The basic idea of the method is to approximate the nonlinear components in nonlinear control systems by linear "equivalents", and then use frequency domain techniques to analyze the resulting systems. Unlike the phase plane method, it is not restricted to second-order systems. Unlike Lyapunov methods, whose applicability to a specific system hinges on the success of a trial-and-error search for a Lyapunov function, its application is straightforward for nonlinear systems satisfying some easy-to-check conditions.

Nonlinear Systems Analysis (**Describing functions**)

- The method is mainly used to predict limit cycles in nonlinear systems. Other applications include the prediction of subharmonic generation and the determination of system response to sinusoidal excitation. The method has a number of advantages.
- First, it can deal with low order and high order systems with the same straightforward procedure.
- Second, because of its similarity to frequency-domain analysis of linear systems, it is conceptually simple and physically appealing, allowing users to exercise their physical and engineering insights about the control system.
- Third, it can deal with the "hard nonlinearities" frequently found in control systems without any difficulty.

Nonlinear Systems Analysis (**Describing functions**)

- As a result, it is an important tool for practical problems of nonlinear control analysis and design. The disadvantages of the method are linked to its approximate nature, and include the possibility of inaccurate predictions (false predictions may be made if certain conditions are not satisfied) and restrictions on the systems to which it applies (for example, it has difficulties in dealing with systems with multiple nonlinearities).

Phase Plane Analysis

- Phase plane analysis is a graphical method for studying second-order systems, which was introduced well before the turn of the century by mathematicians such as Henri Poincare. The basic idea of the method is to generate, in the state space of a second order dynamic system (a two-dimensional plane called the phase plane), motion trajectories corresponding to various initial conditions, and then to examine the qualitative features of the trajectories. In such a way, information concerning stability and other motion patterns of the system can be obtained.

Phase Plane Analysis

- Phase plane analysis has a number of useful properties.
- First, as a graphical method, it allows us to visualize what goes on in a nonlinear system starting from various initial conditions, without having to solve the nonlinear equations analytically.
- Second, it is not restricted to small or smooth nonlinearities, but applies equally well to strong nonlinearities and to "hard" nonlinearities.
- Finally, some practical control systems can indeed be adequately approximated as second-order systems, and the phase plane method can be used easily for their analysis.
- Conversely, of course, the fundamental disadvantage of the method is that it is restricted to second-order (or first order) systems, because the graphical study of higher-order systems is computationally and geometrically complex.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Portraits**
- The phase plane method is concerned with the graphical study of second-order autonomous systems described by

$$\dot{x}_1 = f_1(x_1, x_2) \qquad (1a)$$

$$\dot{x}_2 = f_2(x_1, x_2) \qquad (1b)$$

- where $\mathbf{x_1}$ and $\mathbf{x_2}$ are the states of the system, and $\boldsymbol{f_1}$, and $\boldsymbol{f_2}$ are nonlinear functions of the states. Geometrically, the state space of this system is a plane having $\mathbf{x_1}$ and $\mathbf{x_2}$ as coordinates. We will call this plane the phase plane.

Phase Plane Analysis
Concepts of Phase Plane Analysis

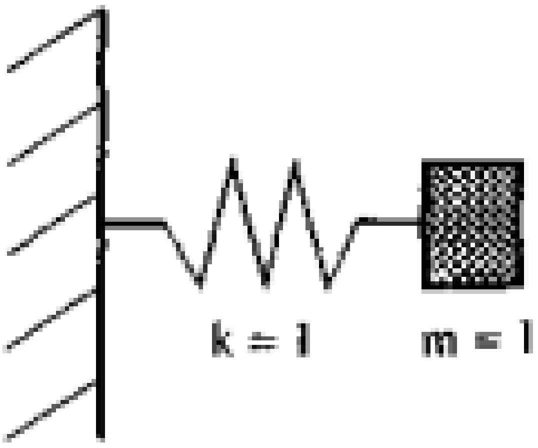
Phase Portraits

- Given a set of initial conditions $\mathbf{x(0)} = \mathbf{x_0}$, Equation ($\dot{x}_1=f_1(x_1, x_2)$, $\dot{x}_2=f_2(x_1, x_2)$) defines a solution $\mathbf{x(t)}$. With time $\mathbf{t}$ varied from **zero** to **infinity**, the solution $\mathbf{x(t)}$ can be represented geometrically as a curve in the phase plane. Such a curve is called a phase plane trajectory. A family of phase plane trajectories corresponding to various initial conditions is called a phase portrait of a system.
- To illustrate the concept of phase portrait, let us consider the following simple system.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Example 1**: **Phase portrait of a mass-spring system**
- The governing equation of the mass-spring system in Figure (a) is the familiar linear second order differential equation

$$\ddot{x} + x = 0 \qquad (2)$$



(a)

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Example 1**: **Phase portrait of a mass-spring system**
- Assume that the mass is initially at rest, at length $\mathbf{x_o}$. Then the solution of the equation is

$$x(t) = x_o \cos \boldsymbol{t}$$

$$\dot{x}(t) = - x_o \sin \boldsymbol{t}$$

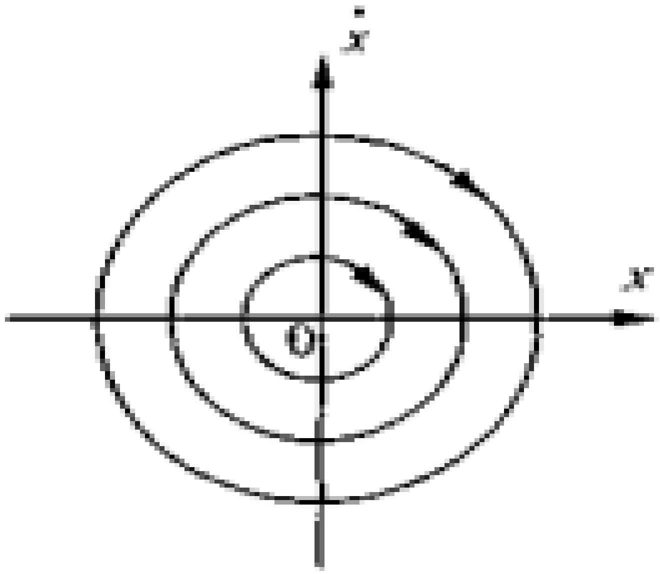
- Eliminating time ***t*** from the above equations, we obtain the equation of the trajectories

$$x^2 + \dot{x}^2 = x_o^{\ 2}$$

- This represents a circle in the phase plane. Corresponding to different initial conditions, circles of different radii can be obtained. Plotting these circles on the phase plane, we obtain a phase portrait for the mass-spring system (Figure b).

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Example 1**: **Phase portrait of a mass-spring system**



(b)

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase portrait**
- The power of the phase portrait lies in the fact that once the phase portrait of a system is obtained, the nature of the system response corresponding to various initial conditions is directly displayed on the phase plane. In the above example, we easily see that the system trajectories neither converge to the origin nor diverge to infinity. They simply circle around the origin, indicating the marginal nature of the system's stability.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase portrait**
- A major class of second-order systems can be described by differential equations of the form

$$\ddot{x} + f(x, \dot{x}) = 0 \qquad (3)$$

- In state space form, this dynamics can be represented as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -f(x_1, x_2)$$

- with $\mathbf{x_1} = \mathbf{x}$ and $\mathbf{x_2} = \dot{\mathbf{x}}$. Most second-order systems in practice, such as mass-damper spring systems in mechanics, or resistor-coil-capacitor systems in electrical engineering, can be represented in or transformed into this form. For these systems, the states are **x** and its derivative $\dot{\mathbf{x}}$. Traditionally, the phase plane method is developed for the dynamics (3), and the phase plane is defined as the plane having **x** and $\dot{\mathbf{x}}$ as coordinates. But it causes no difficulty to extend the method to more general dynamics of the form ($\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$), with the ($\mathbf{x_1}$, $\mathbf{x_2}$) plane as the phase plane.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- An important concept in phase plane analysis is that of a singular point. A singular point is an equilibrium point in the phase plane. Since an equilibrium point is defined as a point where the system states can stay forever, this implies that $\dot{\mathbf{x}} = \mathbf{0}$, and using ($\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$).

$$f_1(x_1, x_2) = 0 \qquad\qquad f_2(x_1, x_2) = 0 \qquad (4)$$

- The values of the equilibrium states can be solved from (4).
- For a linear system, there is usually only one singular point (although in some cases there can be a continuous set of singular points, as in the system $\ddot{\mathbf{x}} + \dot{\mathbf{x}} = \mathbf{0}$, for which all points on the real axis are singular points). However, a nonlinear system often has more than one isolated singular point, as the following example shows.

Phase Plane Analysis
Concepts of Phase Plane Analysis

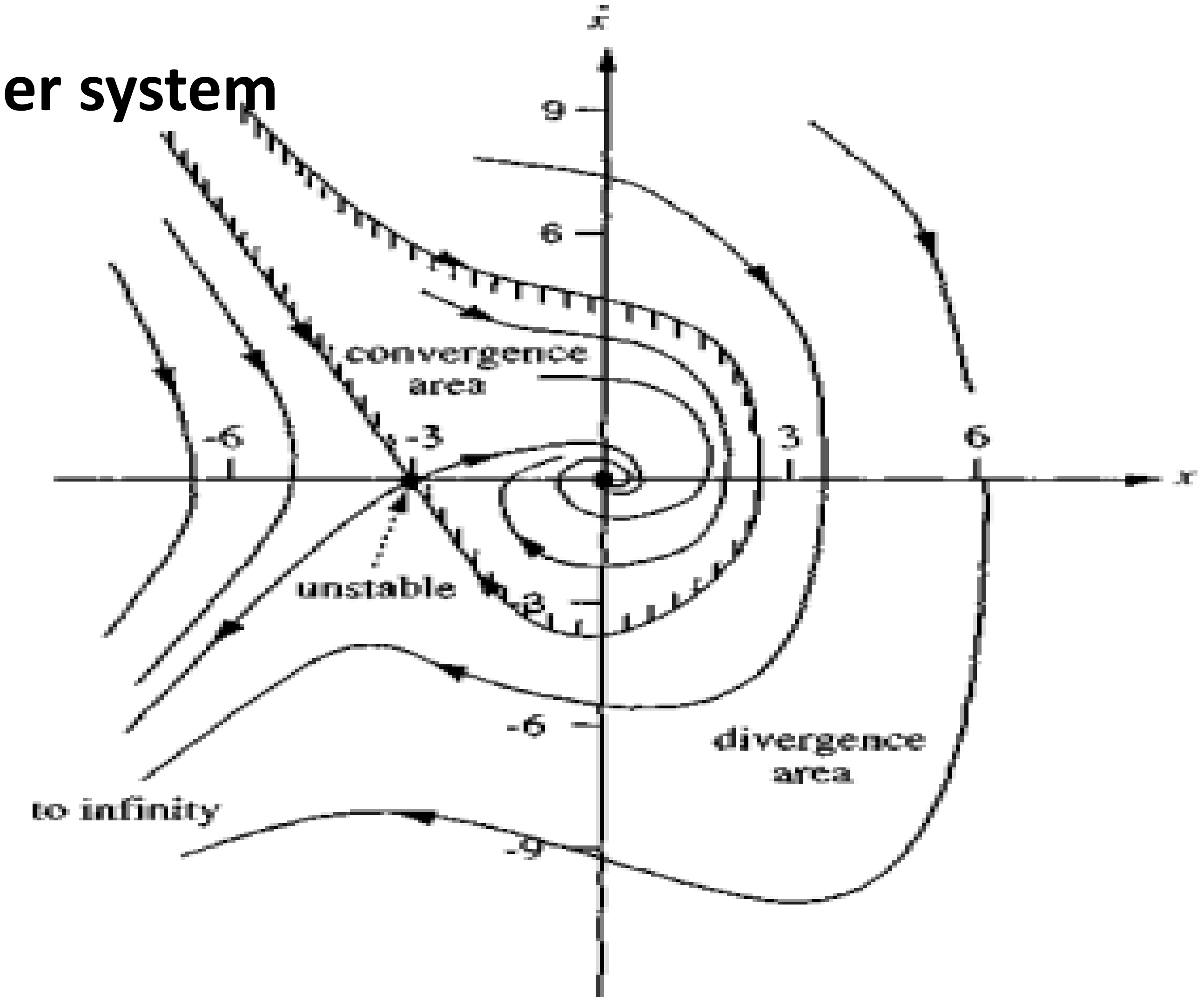
- **Singular Points**
- **Example.2: A nonlinear second-order system**
- Consider the system

$$\ddot{x} + 0.6\,\dot{x} + 3\,x + x^2 = 0$$

- whose phase portrait is plotted in Figure 2. The system has two singular points, one at (0, 0) and the other at (-3, 0). The motion patterns of the system trajectories in the vicinity of the two singular points have different natures. The trajectories move towards the point $x = 0$ while moving away from the point $x = -3$.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- **Example.2: A nonlinear second-order system**



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- One may wonder why an equilibrium point of a second-order system is called a singular point. To answer this, let us examine the slope of the phase trajectories. From ($\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$), the slope of the phase trajectory passing through a point ($\mathbf{x_1}, \mathbf{x_2}$) is determined by.

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2}$$

- With the functions f_1 and f_2 assumed to be single valued, there is usually a definite value for this slope at any given point in phase plane. This implies that the phase trajectories will not intersect.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- At singular points, however, the value of the slope is **0/0**, i.e., the slope is indeterminate. Many trajectories may intersect at such points, as seen from Figure. This indeterminacy of the slope accounts for the adjective "singular"**.**
- Singular points are very important features in the phase plane. Examination of the singular points can reveal a great deal of information about the properties of a system.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- In fact, the stability of linear systems is uniquely characterized by the nature of their singular points. For nonlinear systems, besides singular points, there may be more complex features, such as limit cycles.
- Note that, although the phase plane method is developed primarily for second order systems, it can also be applied to the analysis of first-order systems of the form

$$\dot{x} + f(x) = 0$$

- The idea is still to plot $\dot{x}$ with respect to x in the phase plane. The difference now is that the phase portrait is composed of a single trajectory.

Phase Plane Analysis
Concepts of Phase Plane Analysis

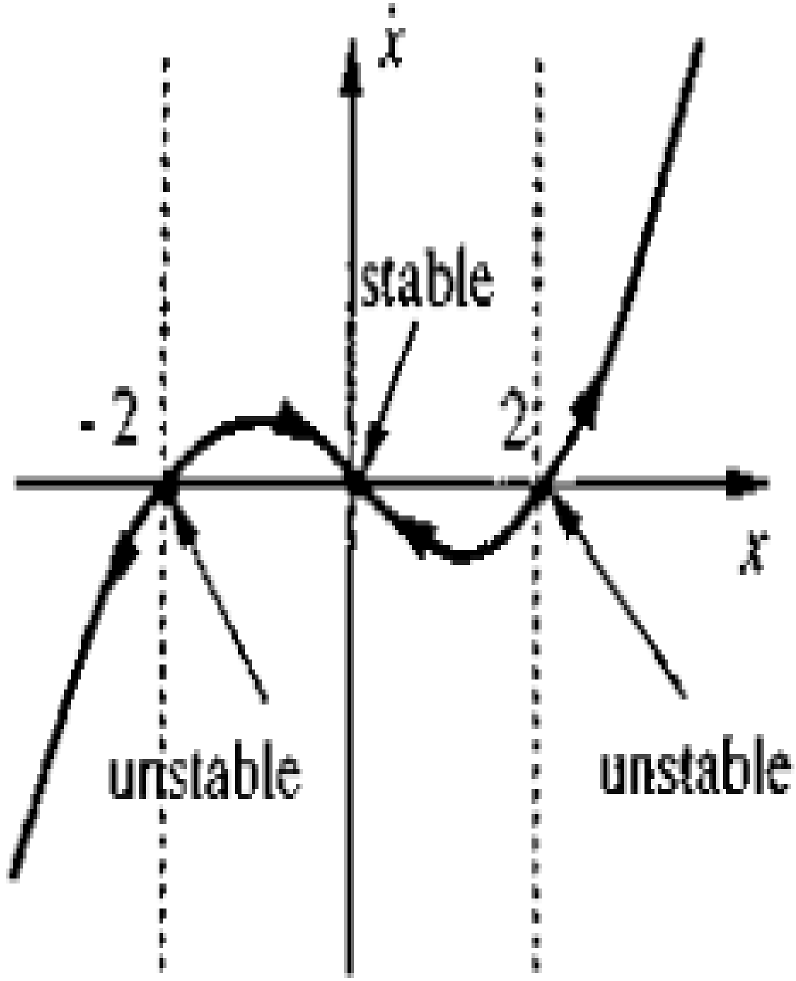
- **Singular Points**
- **Example 3: A first-order system**
- Consider the system

$$\dot{\mathbf{x}} = -4\,\mathbf{x} + \mathbf{x}^3$$

- There are three singular points, defined by $\mathbf{-\,4x + x^3 = 0}$, namely, $\mathbf{x = 0, -2}$, and 2. The phase portrait of the system consists of a single trajectory, and is shown in Figure. The arrows in the figure denote the direction of motion, and whether they point toward the left or the right at a particular point is determined by the sign of $\dot{\mathbf{x}}$ at that point. It is seen from the phase portrait of this system that the equilibrium point $\mathbf{x = 0}$ is stable, while the other two are unstable.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- **Example 3: A first-order system**



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Symmetry in Phase Plane Portraits**
- A phase portrait may have a priori known symmetry properties, which can simplify its generation and study. If a phase portrait is symmetric with respect to the $\mathbf{x_1}$ or the $\mathbf{x_2}$ axis, one only needs in practice to study half of it. If a phase portrait is symmetric with respect to both the $\mathbf{x_1}$ and $\mathbf{x_2}$ axes, only one quarter of it has to be explicitly considered.
- Before generating a phase portrait itself, we can determine its symmetry properties by examining the system equations. Let us consider the second-order dynamics ($\ddot{x} + f(x, \dot{x}) = 0$). The slope of trajectories in the phase plane is of the form

$$\frac{dx_2}{dx_1} = -\frac{f(x_1, x_2)}{x_2}$$

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Singular Points**
- Since symmetry of the phase portraits also implies symmetry of the slopes (equal in absolute value but opposite in sign), we can identify the following situations:
- **Symmetry about the x_1 axis**: The condition is

$$f(\mathbf{x_1}, \mathbf{x_2}) = f(\mathbf{x_1}, -\mathbf{x_2})$$

- This implies that the function $\boldsymbol{f}$ should be even in $\mathbf{x_2}$. The mass-spring system in Example **1** satisfies this condition. Its phase portrait is seen to be symmetric about $\mathbf{x_1}$ axis.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Symmetry about the x_2 axis**: Similarly

$$f(\mathbf{x_1}, \mathbf{x_2}) = -f(\mathbf{-x_1}, \mathbf{x_2})$$

- implies symmetry with respect to the $\mathbf{x_2}$ axis. The mass-spring system also satisfies this condition.
- **Symmetry about the origin**: When

$$f(\mathbf{x_1}, \mathbf{x_2}) =- f(\mathbf{-x_1}, \mathbf{- x_2})$$

- the phase portrait of the system is symmetric about the origin.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- Today, phase portraits are routinely computer-generated. In fact, it is largely the advent of the computer in the early 1960's, and the associated ease of quickly generating phase portraits, which spurred many advances in the study of complex nonlinear dynamic behaviors such as chaos. However, of course (as e.g., in the case of root locus for linear systems), it is still practically useful to learn how to roughly sketch phase portraits or quickly verify the plausibility of computer outputs.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- There are a number of methods for constructing phase plane trajectories for linear or nonlinear systems, such as the so-called analytical method, the method of isoclines, the delta method, Lienard's method, and Pell's method. We shall discuss two of them in this section, namely, the analytical method and the method of isoclines. These methods are chosen primarily because of their relative simplicity. The analytical method involves the analytical solution of the differential equations describing the systems. It is useful for some special nonlinear systems, particularly piece-wise linear systems, whose phase portraits can be constructed by piecing together the phase portraits of the related linear systems. The method of isoclines is a graphical method which can conveniently be applied to construct phase portraits for systems which cannot be solved analytically, which represent by far the most common case.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **ANALYTICAL METHOD**
- There are two techniques for generating phase plane portraits analytically. Both techniques lead to a functional relation between the two phase variables $\mathbf{x_1}$ and $\mathbf{x_2}$ in the form

$$g(\mathbf{x_1}, \mathbf{x_2}, \mathbf{c}) = 0 \qquad (6)$$

- where the constant c represents the effects of initial conditions (and, possibly, of external input signals). Plotting this relation in the phase plane for different initial conditions yields a phase portrait.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **ANALYTICAL METHOD**
- The first technique involves solving equations ($\dot{x}_1=f_1(x_1, x_2)$, $\dot{x}_2=f_2(x_1, x_2)$) for $\mathbf{x_1}$ and $\mathbf{x_2}$ as functions of time *t, i.e.*,

$$X_1(t)=g_1(\mathbf{t}) \qquad\qquad X_2(t)=g_2(\mathbf{t})$$

- and then eliminating time *t* from these equations, leading to a functional relation in the form of ($g(\mathbf{x_1}, \mathbf{x_2}, \mathbf{c}) = 0$). This technique was already illustrated in Example 1.
- The second technique, on the other hand, involves directly eliminating the time variable, by noting that

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2}$$

- and then solving this equation for a functional relation between $\mathbf{x_1}$ and $\mathbf{x_2}$. Let us use this technique to solve the mass-spring equation again.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **ANALYTICAL METHOD**
- **Example 4: Mass-spring system**
- By noting that $\boldsymbol{\ddot{x} = (d\dot{x}/dx)(dx/dt)}$, we can rewrite ($\ddot{x} + x = 0$) as

$$\dot{x}\frac{d\dot{x}}{dx} + x = 0$$

- Integration of this equation yields

$$\dot{x}^2 + x^2 = {x_0}^2$$

- One sees that the second technique is more straightforward in generating the equations for the phase plane trajectories.
- Most nonlinear systems cannot be easily solved by either of the above two techniques. However, for piece-wise linear systems, an important class of nonlinear systems, this method can be conveniently used, as the following example shows.

Phase Plane Analysis
Concepts of Phase Plane Analysis

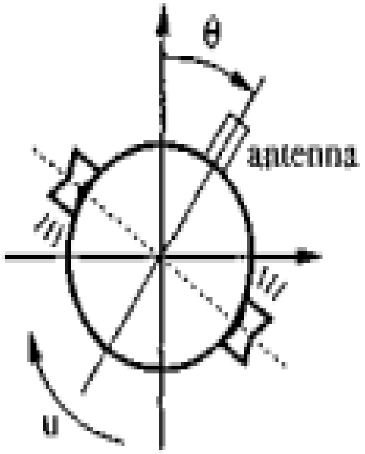
- **Constructing Phase Portraits**
- **ANALYTICAL METHOD**
- **Example 5: A satellite control system**
- Figure 4 shows the control system for a simple satellite model. The satellite, depicted in Figure 5(a), is simply a rotational unit inertia controlled by a pair of thrusters, which can provide either a positive constant torque ***U*** (positive firing) or a negative torque (negative firing). The purpose of the control system is to maintain the satellite antenna at a zero angle by appropriately firing the thrusters. The mathematical model of the satellite is

$$\ddot{\theta} = u$$

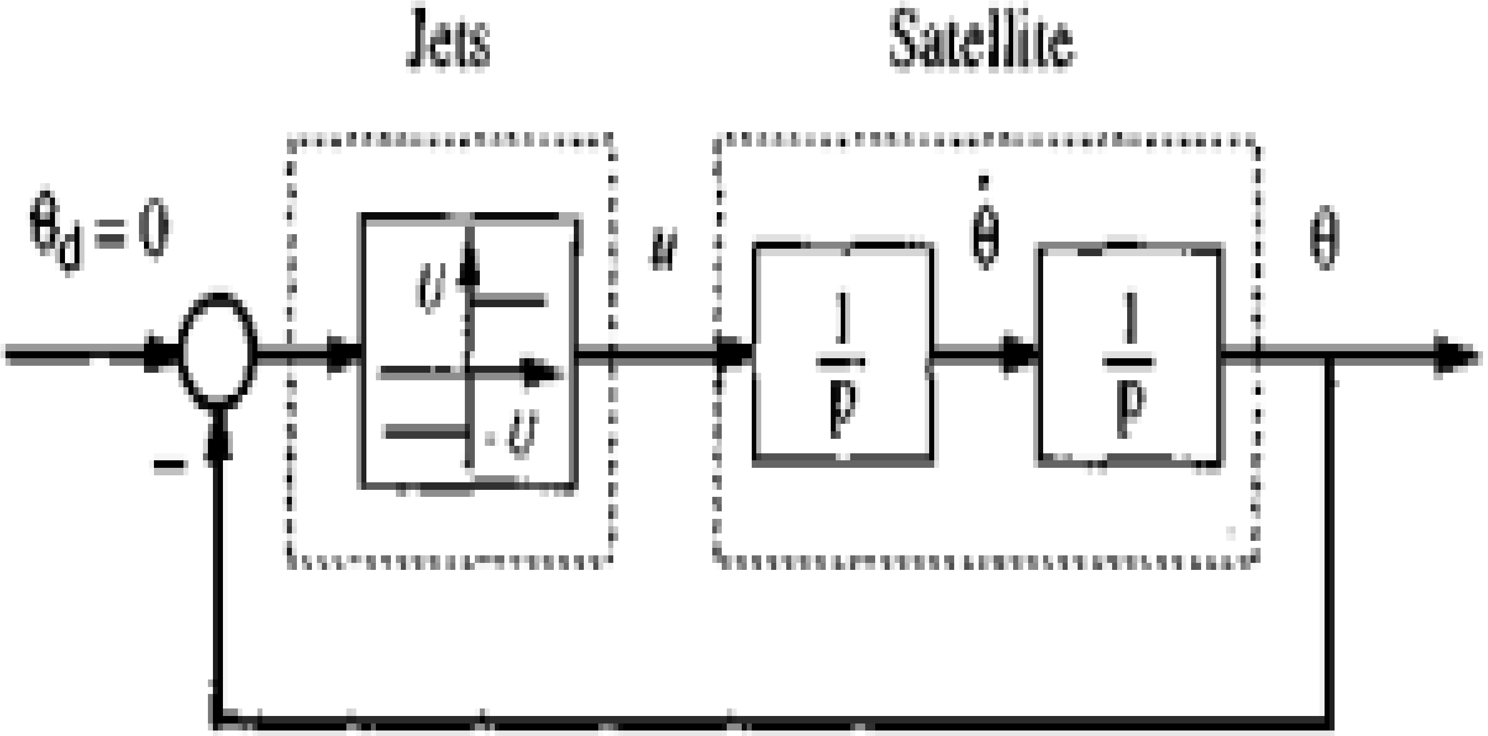
- where u is the torque provided by the thrusters and θ is the satellite angle.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- ANALYTICAL METHOD
- **Example 5: A satellite control system** **(4)**
-
- **(5a)**
-



(a)



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- ANALYTICAL METHOD
- **Example 5: A satellite control system**
- Let us examine on the phase plane the behavior of the control system when the thrusters are fired according to the control law

$$\boldsymbol{u(t)} = \begin{cases} -\boldsymbol{u},\ \boldsymbol{if}\ \ \boldsymbol{\theta > 0} \\ \ \ \boldsymbol{u},\ \boldsymbol{if}\ \ \boldsymbol{\theta < 0} \end{cases} \qquad (7)$$

- which means that the thrusters push in the counterclockwise direction if $\boldsymbol{\theta}$ is positive, and vice versa**.**
- As the first step of the phase portrait generation, let us consider the phase portrait when the thrusters provide a positive torque ***U***. The dynamics of the system is

$$\ddot{\boldsymbol{\theta}}=\boldsymbol{U}$$

Phase Plane Analysis
Concepts of Phase Plane Analysis

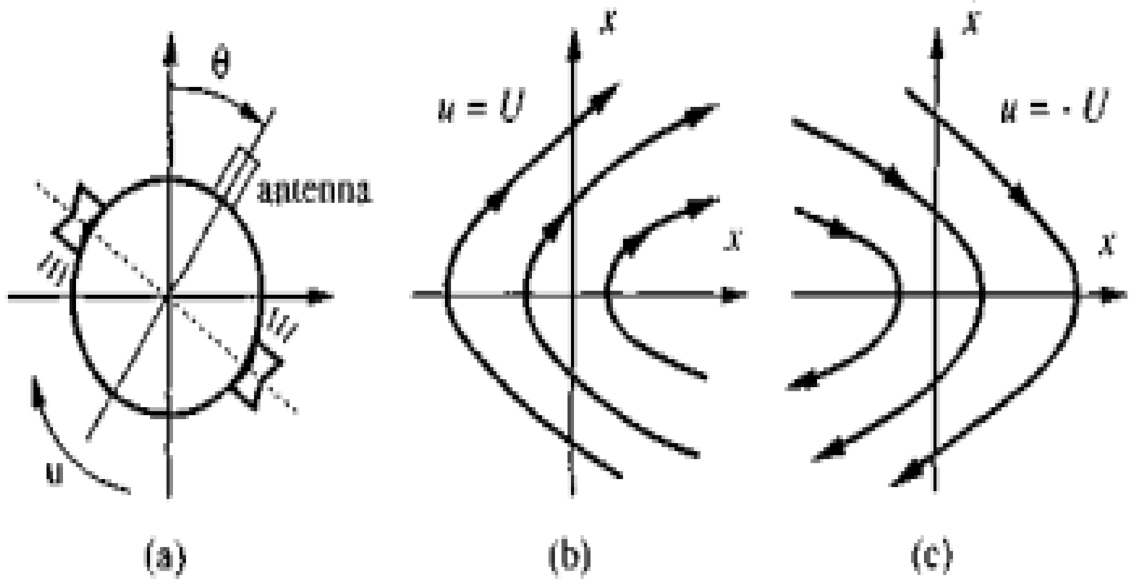
- **Constructing Phase Portraits**
- **ANALYTICAL METHOD**
- **Example 5: A satellite control system**
- which implies that $\dot{\theta} d\dot{\theta} = U d\theta$. Therefore, the phase trajectories are a family of parabolas defined by

$$\dot{\theta}^2 = 2U\theta + c_1$$

- where c_1 is a constant. The corresponding phase portrait of the system is shown in Figure 5(b).
- When the thrusters provide a negative torque $-U$, the phase trajectories are similarly found to be

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- ANALYTICAL METHOD
- **Example 5: A satellite control system**



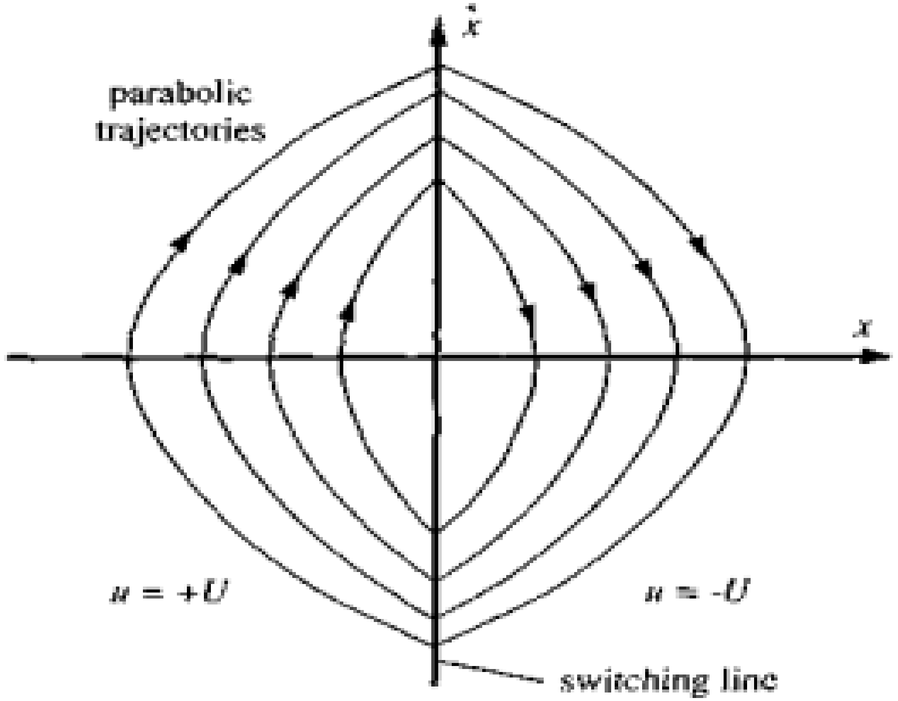
- (5)

$$\dot{\theta}^2 = -2Ux + c_1$$

- with the corresponding phase portrait shown in Figure 5(c)

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- ANALYTICAL METHOD (6)
- **Example 5: A satellite control system**



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **ANALYTICAL METHOD**
- **Example 5: A satellite control system**
- The complete phase portrait of the closed-loop control system can be obtained simply by connecting the trajectories on the left half of the phase plane in 5(b) with those on the right half of the phase plane in 5(c), as shown in Figure 6. The vertical axis represents a switching line, because the control input and thus the phase trajectories are switched on that line. It is interesting to see that, starting from a nonzero initial angle, the satellite will oscillate in periodic motions under the action of the jets. One concludes from this phase portrait that the system is marginally stable, similarly to the mass-spring system in Example 1. Convergence of the system to the zero angle can be obtained by adding rate feedback .