Problem sets 5 : Projection

CEDC102: Linear Algebra and Matrix Theory

Manara University

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Problem 1

Does there exist a . . . , as in each of (1), (2), (3) below? (If the answer is yes, show such a matrix and explain why it has the required property. If the answer is no, explain why such a matrix doesn't exist).

1. 2×3 matrix with column space spanned by $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and left nullspace spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

2. 3×2 matrix with column space spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and nullspace spanned by $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$?

3. 2×2 matrix with column space spanned by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and left nullspace spanned by $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$?

Problem 2

a. If $Ax = b$ has a solution x, then the closest vector to b in N(AT) is __________. (Try drawing a picture.)

b. If the rows of A (an $m \times n$ matrix) are independent, then the dimension $N(A^T A)$ is __________

c. If a matrix U has orthonormal rows, then $I =$ ________ and the projection matrix onto the row space of U is __________. (Your answers should be simplified expressions involving U and U^T only.)

Problem 3

In class, we saw that the orthogonal projection p of a vector b onto $C(A)$ is given by $p = A\hat{x}$ where $\hat{x}$ is a solution to the "normal equations" $A^T A\hat{x} = A^T b$. We showed in class that these equations are *always* solvable, because $A^T b \in C(AT) = C(A^T A)$.

a. The least-square solution $\hat{x}$ is unique if A is ___________, in which case $A^T A$ is ___________.

b. The least-square solution $\hat{x}$ is not unique if A is ___________, in which case $A^T A$ is ___________. However, the projection $p = A\hat{x}$ is still unique: if you have two solutions $\hat{x}$ and $\hat{\hat{x}}$ to the normal equations, $A\hat{x} - A\hat{\hat{x}}$ =_________ because ___________.

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Problem 4

Suppose we have data (e.g. from some experimental measurement) $b_1, b_2, b_3, \dots, b_{21}$ at the 21 equally spaced times $t = -10, -9, \dots, 9, 10$. All measurements are $b_k = 0$ except that $b_{11} = 1$ at the middle time $t = 0$

a. Using least squares, what are the best c and d to fit those 21 points by a straight line $c + dt$?

b. You are projecting the vector b onto what subspace? (Give a basis).

c. Find a nonzero vector perpendicular to that subspace.

Problem 5

Suppose $\hat{x}$ is the least squares solution to $Ax \approx b$ (i.e. it minimizes $\| Ax - b \|$) and $\hat{y}$ is the least squares solution to $Ay \approx c$, where A has full column rank. Does this tell you the least squares solution $\hat{z}$ to $Az \approx b + c$? If so, what is $\hat{z}$ and why?

Problem 6

(a) What matrix P projects every vector in $\mathbb{R}^3$ onto the line that passes through origin and $a = [3,4,5]$ (column vector)?

(b) What is the nullspace of that matrix P? (Give a basis.)

(c) What is the row space of P^7?

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Problem 7

Suppose that vectors $q_1, q_2, \dots, q_n$ in $\mathbb{R}^m$ are orthonormal.

a. Let $c_1, c_2, \dots, c_n$ be real numbers. What is $\|c_1 q_1 + c_2 q_2, + \dots + c_n q_n\|^2$?

b. Show that $q_1, q_2, \dots, q_n$ are linearly independent.

Problem 8

Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

a. Find Q and R such that $A = QR$ by Gram-Schmidt process on columns of A

b. Let $B = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, a matrix obtained by exchanging the first two columns of A

Find $\acute{Q}$ and $\acute{R}$ such that $B = \acute{Q}\acute{R}$ by Gram-Schmidt process on columns of B.