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Planar Kinematics of a Rigid Body

**Motion Analysis : Acceleration** 



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**Relative-Motion Analysis : Acceleration** 

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# **Relative-Motion Analysis : Acceleration**

An equation that relates the accelerations of two points on a bar (rigid body) subjected to general plane motion may be determined by differentiating  $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$  with respect to time. This yields

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

The terms  $dv_B/dt = a_B$  and  $dv_A/dt = a_A$  represent the *absolute* accelerations of points B and A.

The last term represents the acceleration of B with respect to A as measured by an observer fixed to translating x', y' axes which have their origin at the base point A. It was shown that to this observer point Bappears to move along a *circular arc* that has a radius of curvature



 $r_{B/A}$ . Consequently,  $\mathbf{a}_{B/A}$  can be expressed in terms of its tangential and normal components; i.e.,  $\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$ , where  $(a_{B/A})_t = \alpha r_{B/A}$  and  $(\hat{a}_{B/A})_n = \omega^2 r_{B/A}$ . Hence, the relative-acceleration equation can be written in the form

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

where

- $\mathbf{a}_B = \operatorname{acceleration} \operatorname{of} \operatorname{point} B$
- $\mathbf{a}_A = \operatorname{acceleration} \operatorname{of} \operatorname{point} A$
- $(\mathbf{a}_{B/A})_t = \text{tangential acceleration component of } B \text{ with respect}$ to A. The magnitude is  $(a_{B/A})_t = \alpha r_{B/A}$ , and the *direction* is perpendicular to  $\mathbf{r}_{B/A}$ .
- $(a_{B/A})_n$  = normal acceleration component of *B* with respect to *A*. The *magnitude* is  $(a_{B/A})_n = \omega^2 r_{B/A}$ , and the *direction* is always from *B* toward *A*.



It is seen that at a given instant the acceleration of **B** is determined by considering the bar to translate with an acceleration  $\mathbf{a}_A$ , and simultaneously rotate about the base point A with an instantaneous angular velocity **w** and angular acceleration  $\mathbf{\alpha}$ .

Vector addition of these two effects, applied to *B*, *yields* **a**<sup>B</sup>. It should be noted that since points *A* and *B* move along curved paths, the accelerations of these points will have both tangential and normal components.





Since the relative-acceleration components represent the effect of *circular motion* observed from translating axes having their origin at the base point *A*, these terms can be expressed as  $(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A}$  and  $(\mathbf{a}_{B/A})_n = -\omega^2 \mathbf{r}_{B/A}$ 

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

where

 $\mathbf{a}_{B} = \text{acceleration of point } B$  $\mathbf{a}_{A} = \text{acceleration of the base point } A$  $\boldsymbol{\alpha} = \text{angular acceleration of the body}$  $\boldsymbol{\omega} = \text{angular velocity of the body}$  $\mathbf{r}_{B/A} = \text{position vector directed from } A \text{ to } B$ 



If equations are applied in a practical manner to study the accelerated motion of a rigid body which is *pin connected to two other* bodies, it should be realized that points which are *coincident at the pin* move with the same acceleration, since the path of motion over which they travel is the same. For example, point B lying on either rod BA or BC of the crank mechanism shown has the same acceleration, since the rods are pin connected at *B. Here the motion of B* is along a *circular path, so that* **a**<sup>B</sup> can be expressed in terms of its tangential and normal components. At the other end of rod BC point C moves along a straight-lined path, which is defined by the piston. Hence, **a**<sub>c</sub> is horizontal







Finally, consider a disk that rolls without slipping as shown As a result,  $v_A = 0$  and so from the kinematic diagram, the velocity of the mass center G is

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \mathbf{0} + (-\boldsymbol{\omega}\mathbf{k}) \times (r\mathbf{j})$$

So that

$$v_G = \omega r$$

This same result can also be determined using the IC method where point A is the IC.

Since G moves along a *straight line*, its acceleration in this case can be determined from the time derivative of its velocity.

$$\frac{dv_G}{dt} = \frac{d\omega}{dt}r$$
$$a_G = \alpha r$$





#### **Procedure for Analysis**

The relative acceleration equation can be applied between any two points *A* and *B* on a body either by using a Cartesian vector analysis, or by writing the *x* and *y* scalar component equations directly.

## Velocity Analysis.

 Determine the angular velocity ω of the body by using a velocity analysis.



#### Vector Analysis Kinematic Diagram.

- Establish the directions of the fixed x, y coordinates and draw the kinematic diagram of the body. Indicate on it a<sub>A</sub>, a<sub>B</sub>, ω, α, and r<sub>B/A</sub>.
- If points A and B move along *curved paths*, then their accelerations should be indicated in terms of their tangential and normal components, i.e.,  $\mathbf{a}_A = (\mathbf{a}_A)_t + (\mathbf{a}_A)_n$  and  $\mathbf{a}_B = (\mathbf{a}_B)_t + (\mathbf{a}_B)_n$ .

Acceleration Equation.

- To apply a<sub>B</sub> = a<sub>A</sub> + α × r<sub>B/A</sub> ω<sup>2</sup>r<sub>B/A</sub>, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective i and j components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates that the sense of direction of the vector is opposite to that shown on the kinematic diagram.



#### Scalar Analysis Kinematic Diagram.

• If the acceleration equation is applied in scalar form, then the magnitudes and directions of the relative-acceleration components  $(\mathbf{a}_{B/A})_t$  and  $(\mathbf{a}_{B/A})_n$  must be established. To do this draw a kinematic diagram such as shown. Since the body is considered to be momentarily "pinned" at the base point *A*, the *magnitudes* of these components are  $(a_{B/A})_t = \alpha r_{B/A}$  and  $(a_{B/A})_n = \omega^2 r_{B/A}$ . Their *sense of direction* is established from the diagram such that  $(\mathbf{a}_{B/A})_t$  acts perpendicular to  $\mathbf{r}_{B/A}$ , in accordance with the rotational motion  $\boldsymbol{\alpha}$  of the body, and  $(\mathbf{a}_{B/A})_n$  is directed from *B* toward *A*.

Acceleration Equation.

• Represent the vectors in  $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$  graphically by showing their magnitudes and directions underneath each term. The scalar equations are determined from the x and y components of these vectors.



#### EXAMPLE

The rod AB shown is confined to move along the inclined planes at A and B. If point A has an acceleration of  $3 \text{ m/s}^2$ and a velocity of 2 m/s, both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.

## SOLUTION I (VECTOR ANALYSIS)

We will apply the acceleration equation to points A and B on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is  $\omega = 0.283 \text{ rad/s}$  using either the velocity equation or the method of instantaneous centers.





**Kinematic Diagram.** Since points *A* and *B* both move along straight-line paths, they have *no* components of acceleration normal to the paths. There are two unknowns, namely,  $a_B$  and  $\alpha$ .

#### Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

 $a_{B}$   $a_{B}$   $a_{B}$   $a_{B}$   $a_{A}$   $a_{B}$   $a_{B}$   $a_{B}$   $a_{A}$   $a_{B}$   $a_{B$ 

 $a_B \cos 45^\circ \mathbf{i} + a_B \sin 45^\circ \mathbf{j} = 3 \cos 45^\circ \mathbf{i} - 3 \sin 45^\circ \mathbf{j} + (\alpha \mathbf{k}) \times (10\mathbf{i}) - (0.283)^2 (10\mathbf{i})$ 

Carrying out the cross product and equating the **i** and **j** components yields



$$a_B \cos 45^\circ = 3 \cos 45^\circ - (0.283)^2 (10)$$
(1)  
$$a_B \sin 45^\circ = -3 \sin 45^\circ + \alpha (10)$$
(2)

Solving, we have

$$a_B = 1.87 \text{ m/s}^2 \measuredangle 45^\circ$$
$$\alpha = 0.344 \text{ rad/s}^2 \text{ S} \qquad Ans.$$

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#### SOLUTION II (SCALAR ANALYSIS)

From the kinematic diagram, showing the relative-acceleration components  $(\mathbf{a}_{B/A})_t$  and  $(\mathbf{a}_{B/A})_n$ , we have

$$\mathbf{a}_{B} = \mathbf{a}_{A} + (\mathbf{a}_{B/A})_{t} + (\mathbf{a}_{B/A})_{n}$$
$$\begin{bmatrix} a_{B} \\ \swarrow 45^{\circ} \end{bmatrix} = \begin{bmatrix} 3 \text{ m/s}^{2} \\ \lnot 545^{\circ} \end{bmatrix} + \begin{bmatrix} \alpha(10 \text{ m}) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.283 \text{ rad/s})^{2}(10 \text{ m}) \\ \leftarrow \end{bmatrix}$$

Equating the x and y components yields Eqs. 1 and 2, and the solution proceeds as before.





#### EXAMPLE

The disk rolls without slipping and has the angular motion shown Determine the acceleration of point *A* at this instant.

# SOLUTION I (VECTOR ANALYSIS)

Kinematic Diagram. Since no slipping occurs

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

# Acceleration Equation.

We will apply the acceleration equation to points G and A

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} - \boldsymbol{\omega}^2 \mathbf{r}_{A/G}$$
$$\mathbf{a}_A = -2\mathbf{i} + (4\mathbf{k}) \times (-0.5\mathbf{j}) - (6)^2(-0.5\mathbf{j})$$
$$= \{18\mathbf{j}\} \text{ ft/s}^2$$





#### SOLUTION II (SCALAR ANALYSIS)

Using the result for  $a_G = 2$  ft/s<sup>2</sup> determined above, and from the kinematic diagram, showing the relative motion  $\mathbf{a}_{A/G}$ , we have

$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_x + (\mathbf{a}_{A/G})_y$$

$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 2 \text{ ft/s}^2 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (4 \text{ rad/s}^2)(0.5 \text{ ft}) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6 \text{ rad/s})^2(0.5 \text{ ft}) \\ \uparrow \end{bmatrix}$$
  
$$\pm \qquad (a_A)_x = -2 + 2 = 0$$

$$+\uparrow$$
  $(a_A)_y = 18 \text{ ft/s}^2$ 

Therefore,

$$a_A = \sqrt{(0)^2 + (18 \text{ ft/s}^2)^2} = 18 \text{ ft/s}^2$$

Ans.





#### EXAMPLE

The spool shown unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s<sup>2</sup>. Determine the acceleration of point *B*.

## SOLUTION I (VECTOR ANALYSIS)

The spool "appears" to be rolling downward without slipping at point A. Therefore, we can determine the acceleration of point G

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

We will apply the acceleration equation to points G and B.





**Kinematic Diagram.** Point *B* moves along a *curved path* having an *unknown* radius of curvature. Its acceleration will be represented by its unknown *x* and *y* components as shown

## Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G} - \boldsymbol{\omega}^2 \mathbf{r}_{B/G}$$
$$(a_B)_x \mathbf{i} + (a_B)_y \mathbf{j} = -2\mathbf{j} + (-4\mathbf{k}) \times (0.75\mathbf{j}) - (3)^2 (0.75\mathbf{j})$$

Equating the i and j terms, the component equations are

$$(a_B)_x = 4(0.75) = 3 \text{ ft/s}^2 \rightarrow$$
 (1

$$(a_B)_y = -2 - 6.75 = -8.75 \text{ ft/s}^2 = 8.75 \text{ ft/s}^2 \downarrow$$
 (2)





The magnitude and direction of  $\mathbf{a}_B$  are therefore

$$a_B = \sqrt{(3)^2 + (8.75)^2} = 9.25 \, \text{ft/s}^2$$

$$\theta = \tan^{-1} \frac{8.75}{3} = 71.1^{\circ}$$

Ans.

## SOLUTION II (SCALAR ANALYSIS)

This problem may be solved by writing the scalar component equations directly. The kinematic diagram shows the relative-acceleration components  $(\mathbf{a}_{B/G})_t$  and  $(\mathbf{a}_{B/G})_n$ . Thus,

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_t$$

 $r_{B/G} = 0.75 \text{ ft}$  G  $\omega = 3 \text{ rad/s}$   $\alpha = 4 \text{ rad/s}^2$ 



$$\begin{bmatrix} (a_B)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_B)_y \\ \uparrow \end{bmatrix}$$
$$= \begin{bmatrix} 2 \text{ ft/s}^2 \\ \downarrow \end{bmatrix} + \begin{bmatrix} 4 \text{ rad/s}^2 (0.75 \text{ ft}) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (3 \text{ rad/s})^2 (0.75 \text{ ft}) \\ \downarrow \end{bmatrix}$$

The *x* and *y* components yield Eqs. 1 and 2 above.



## EXAMPLE

The collar *C* moves downward with an acceleration of  $1 \text{ m/s}^2$ . At the instant shown, it has a speed of 2 m/s which gives links *CB* and *AB* an angular velocity  $\omega_{AB} = \omega_{CB} = 10 \text{ rad/s}$ . Determine the angular accelerations of *CB* and *AB* at this instant.

## SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams of *both* links *AB* and *CB* are shown. To solve, we will apply the appropriate kinematic equation to each link.

## Acceleration Equation.

Link AB (rotation about a fixed axis):

$$\mathbf{a}_{B} = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B} - \boldsymbol{\omega}_{AB}^{2} \mathbf{r}_{B}$$
$$\mathbf{a}_{B} = (\boldsymbol{\alpha}_{AB} \mathbf{k}) \times (-0.2\mathbf{j}) - (10)^{2}(-0.2\mathbf{j})$$
$$\mathbf{a}_{B} = 0.2\boldsymbol{\alpha}_{AB}\mathbf{i} + 20\mathbf{j}$$





Note that  $\mathbf{a}_B$  has *n* and *t* components since it moves along a *circular path*. Link *BC* (general plane motion):

$$\mathbf{a}_{B} = \mathbf{a}_{C} + \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{B/C} - \boldsymbol{\omega}_{CB}^{2} \mathbf{r}_{B/C}$$
$$0.2\boldsymbol{\alpha}_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + (\boldsymbol{\alpha}_{CB}\mathbf{k}) \times (0.2\mathbf{i} - 0.2\mathbf{j}) - (10)^{2}(0.2\mathbf{i} - 0.2\mathbf{j})$$
$$0.2\boldsymbol{\alpha}_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + 0.2\boldsymbol{\alpha}_{CB}\mathbf{j} + 0.2\boldsymbol{\alpha}_{CB}\mathbf{i} - 20\mathbf{i} + 20\mathbf{j}$$

Thus,

$$0.2\alpha_{AB} = 0.2\alpha_{CB} - 20$$
$$20 = -1 + 0.2\alpha_{CB} + 20$$

Solving,

$$\alpha_{CB} = 5 \text{ rad/s}^2 5 \qquad Ans.$$
  
$$\alpha_{AB} = -95 \text{ rad/s}^2 = 95 \text{ rad/s}^2 2 \qquad Ans.$$



#### EXAMPLE

The crankshaft *AB* turns with a clockwise angular acceleration of 20 rad/s<sup>2</sup>. Determine the acceleration of the piston at the instant *AB* is in the position shown. At this instant  $\omega_{AB} = 10$  rad/s and  $\omega_{BC} = 2.43$  rad/s.

# SOLUTION (VECTOR ANALYSIS)

**Kinematic Diagram.** The kinematic diagrams for both AB and BC are shown. Here  $a_C$  is vertical since C moves along a straight-line path.

Acceleration Equation. Expressing each of the position vectors in Cartesian vector form

$$\mathbf{r}_{B} = \{-0.25 \sin 45^{\circ} \mathbf{i} + 0.25 \cos 45^{\circ} \mathbf{j}\} \text{ ft} = \{-0.177 \mathbf{i} + 0.177 \mathbf{j}\} \text{ ft}$$
$$\mathbf{r}_{C/B} = \{0.75 \sin 13.6^{\circ} \mathbf{i} + 0.75 \cos 13.6^{\circ} \mathbf{j}\} \text{ ft} = \{0.177 \mathbf{i} + 0.729 \mathbf{j}\} \text{ ft}$$





Crankshaft AB (rotation about a fixed axis):

$$\mathbf{a}_{B} = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B} - \omega_{AB}^{2} \mathbf{r}_{B}$$
  
= (-20k) × (-0.177i + 0.177j) - (10)<sup>2</sup>(-0.177i + 0.177j)  
= {21.21i - 14.14j} ft/s<sup>2</sup>

Connecting Rod BC (general plane motion): Using the result for  $a_B$  and noting that  $a_C$  is in the vertical direction, we have

 $\mathbf{a}_{C} = \mathbf{a}_{B} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^{2} \mathbf{r}_{C/B}$   $a_{C}\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + (\alpha_{BC}\mathbf{k}) \times (0.177\mathbf{i} + 0.729\mathbf{j}) - (2.43)^{2}(0.177\mathbf{i} + 0.729\mathbf{j})$   $a_{C}\mathbf{j} = 21.21\mathbf{i} - 14.14\mathbf{j} + 0.177\alpha_{BC}\mathbf{j} - 0.729\alpha_{BC}\mathbf{i} - 1.04\mathbf{i} - 4.30\mathbf{j}$   $0 = 20.17 - 0.729\alpha_{BC}$   $a_{C} = 0.177\alpha_{BC} - 18.45$ 



Solving yields

$$\alpha_{BC} = 27.7 \text{ rad/s}^2 \text{ }$$

$$a_C = -13.5 \text{ ft/s}^2 \qquad Ans.$$

**NOTE:** Since the piston is moving upward, the negative sign for  $a_C$  indicates that the piston is decelerating, i.e.,  $\mathbf{a}_C = \{-13.5\mathbf{j}\}$  ft/s<sup>2</sup>. This causes the speed of the piston to decrease until *AB* becomes vertical, at which time the piston is momentarily at rest.



# انتهت المحاضرة

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