MATHEMATICAL ANALAYSIS 1

Lecture 6

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DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

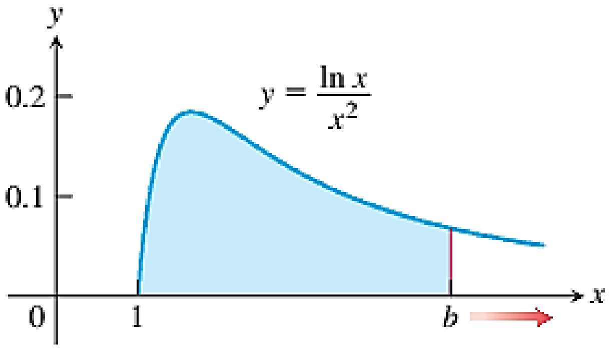
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

   where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

EXAMPLE 1 Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln b}{b} - \frac{1}{b} + 1.$$



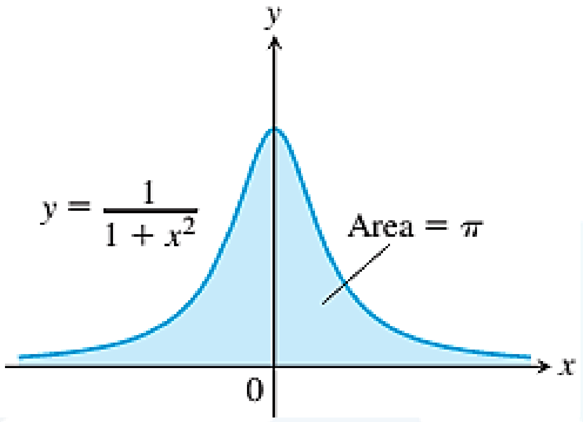
$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 \quad = 1.$$

EXAMPLE 2 Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_{0}^{\infty} \frac{dx}{1+x^2}$$



$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \Big]_0^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\frac{\pi}{2} = \pi$$

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

the improper integral converges if p >1 and diverges if $p \leq 1$

- If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big]_1^b = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

- If $p = 1$.

$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big]_1^b = \infty.$$

Integrands with Vertical Asymptotes

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

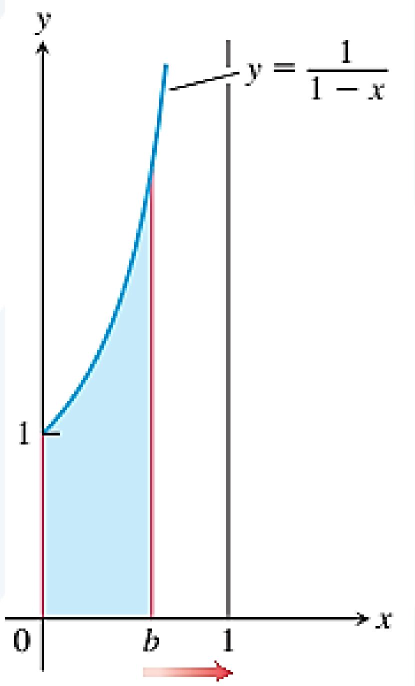
In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

EXAMPLE 4 Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx.$$

$$f(x) = \frac{1}{1-x}$$ continuous on [0, 1)

Discontinuous at x = 1



$$\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-}[-\ln(1-b)+0] = \infty.$$

The integral is divergent

EXAMPLE 5 Evaluate

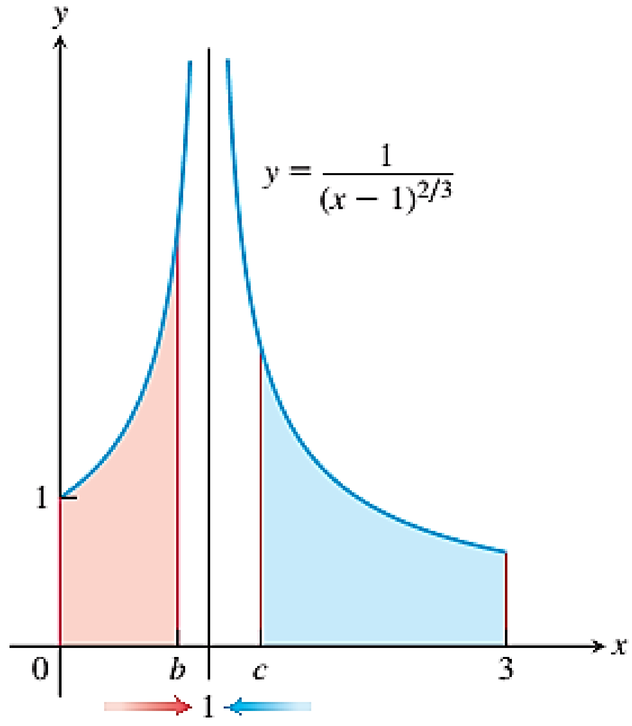
$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

$$f(x) = \frac{1}{(x-1)^{2/3}}$$ continuous on [0, 1) and (1,3]

Discontinuous at x = 1

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

- $$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = 3$$
- $$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c \to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} = 3\sqrt[3]{2}$$



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$

The integral is convergent

Tests for Convergence and Divergence

THEOREM 2—Direct Comparison Test
Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

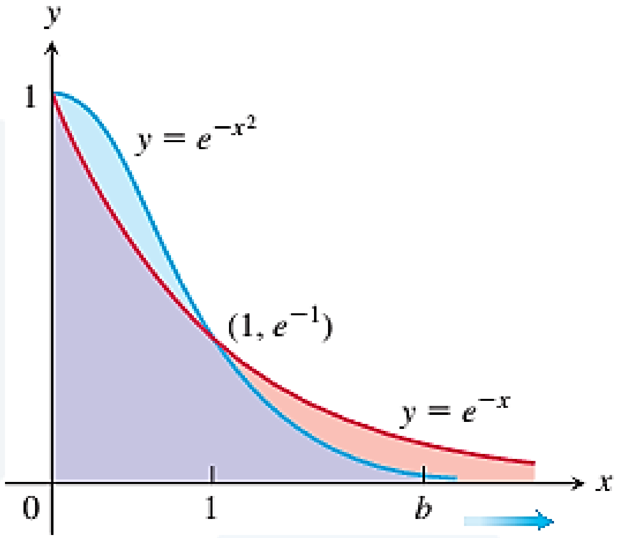
Tests for Convergence and Divergence

EXAMPLE 6 Does the integral $\int_1^{\infty} e^{-x^2}\,dx$ converge?

$$e^{-x^2} \le e^{-x} \quad ; x \ge 1$$

$$\int_1^b e^{-x^2}\,dx \le \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1}$$

$$\int_1^{\infty} e^{-x^2} dx \le \lim_{b\to\infty} \int_1^b e^{-x} dx = \lim_{b\to\infty}\left(e^{-1} - e^{-b}\right) = e^{-1}$$



The integral is convergent

EXAMPLE 7 These examples illustrate how we use Theorem 2.

(a) $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty)$$

$\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges.

(b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}}\,dx$

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty)$$

$\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]$$

$\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \sqrt{2\pi}$ converges.

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Tests for Convergence and Divergence

THEOREM 3—Limit Comparison Test
If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

either *both converge* or *both diverge*.

Example

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx$$

Comparing with $$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

$$\lim_{x\to\infty} \frac{\left(1/x^2\right)}{1/\left(x^2+1\right)} = 1 < \infty \quad \Longrightarrow \quad \int_1^{\infty} \frac{1}{1+x^2}\,dx \text{ converges.}$$

Example

$$\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$$

$$g(x) = \frac{1}{x},\ f(x) = \frac{1-e^{-x}}{x} \quad \Longrightarrow \quad \lim_{x\to\infty} \frac{\left(1-e^{-x}\right)/x}{1/x} = 1 < \infty$$

$$\int_1^{\infty} \frac{1}{x}\,dx \text{ diverges} \quad \Longrightarrow \quad \int_1^{\infty} \frac{1-e^{-x}}{x}\,dx \text{ diverges}$$

Absolute and conditional convergence

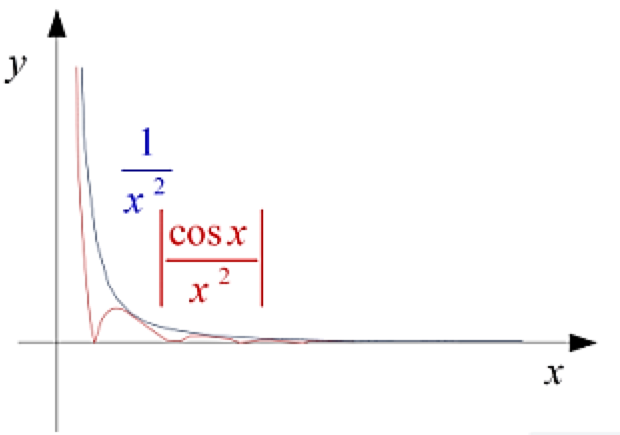
Absolute and conditional convergence. $\int_a^\infty f(x)dx$ is called *absolutely convergent* if $\int_a^\infty |f(x)|\,dx$ converges. If $\int_a^\infty f(x)\,dx$ converges but $\int_a^\infty |f(x)|\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ is called *conditionally convergent*.

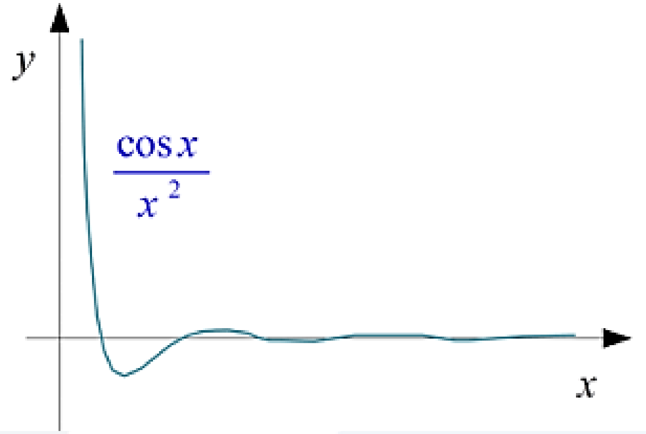
Theorem 2 If $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)\,dx$ converges. In words, an absolutely convergent integral converges.

Example

$$\int_1^{\infty} \frac{\cos x}{x^2}\,dx$$

$$\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$$





$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$ converges. $\Rightarrow$ $$\int_1^{\infty} \frac{\cos x}{x^2}\,dx$$ absolutely converges.

Exercises

Evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$$

0

$$\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$$

$$\frac{4+\pi}{2}$$

$$\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$$

π

Testing for Convergence

use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

$$\int_0^1 \frac{\ln x}{x^2}\,dx$$

diverges. $-\infty$

$$\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$$

converges,

$$\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$$

converges

$$\int_1^{\infty} \frac{1}{e^x-2^x}\,dx$$

converges

First-Order Differential Equations

General First-Order Differential Equations and Solutions

$$\frac{dy}{dx} = f(x, y)$$

first order

Unknown function

$$y' = y + x$$

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx} y(x) = f(x, y(x))$$

The general solution to a first order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant,

EXAMPLE 1 Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

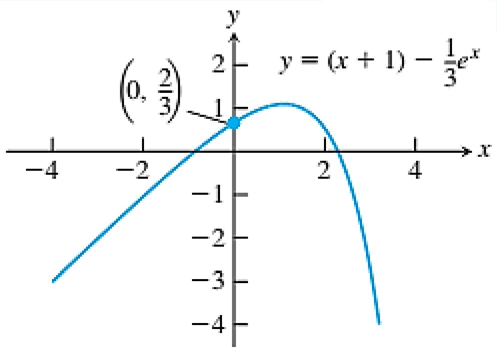
A first order initial value problem $y' = f(x, y)\ ,\ y(x_0) = y_0$

EXAMPLE 2 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$



First-Order Linear Equations

$$\frac{dy}{dx} + P(x)y = Q(x),$$

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

$$y(x) = \frac{1}{v(x)}\left(\int Q(x)v(x)dx + C\right)$$

$v(x)$ integrating factor

$Q(x) = 0$ → $$\frac{dy}{dx} + P(x)y = 0$$

Separating the variables → $$\frac{dy}{y} = -P(x)\,dx$$

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

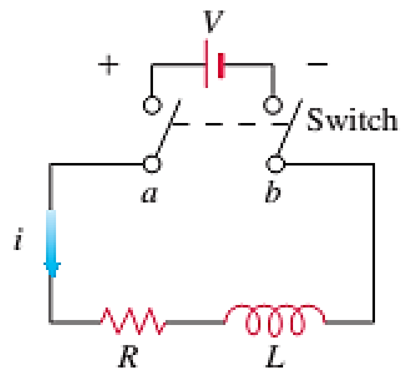
$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \longrightarrow y = -(\ln x + 4) + Cx^{1/3}.$$

$$y(1) = -2. \longrightarrow C = 2. \longrightarrow \boxed{y = 2x^{1/3} - \ln x - 4.}$$

RL Circuits

Ohm's Law, $V = RI$

$$L\frac{di}{dt} + Ri = V,$$



EXAMPLE 4 The switch in the RL circuit in Figure 9.9 is closed at time $t = 0$. How will the current flow as a function of time?

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L},$$

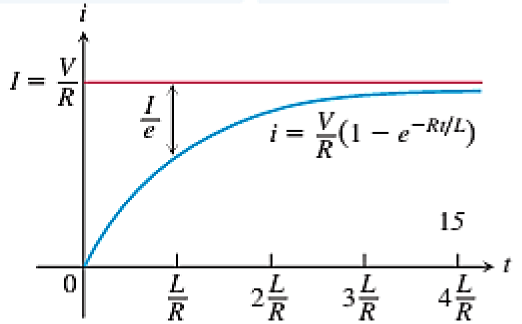
$i(0) = 0$

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t}.$$

$-\frac{V}{R}e^{-(R/L)t}.$ transient solution

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

steady-state value



Motion with Resistance Proportional to Velocity

Newton's second law of motion

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

If the resisting force is proportional to velocity

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

$$v(0) = v_0 \longrightarrow v = v_0 e^{-(k/m)t}.$$

Determining the position

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C. \xrightarrow{s(0)=0} s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)$$

Suppose that an object is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast?

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C. \xrightarrow{s(0)=0} s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right)$$

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) = \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.$$

$$\text{Distance coasted} = \frac{v_0 m}{k}.$$

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EXAMPLE 1 For a 192-lb ice skater, the k in Equation (1) is about $1/3$ slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

$$v = v_0 e^{-(k/m)t}. \tag{1}$$

$m = 6, v_0 = 11, v = 1$

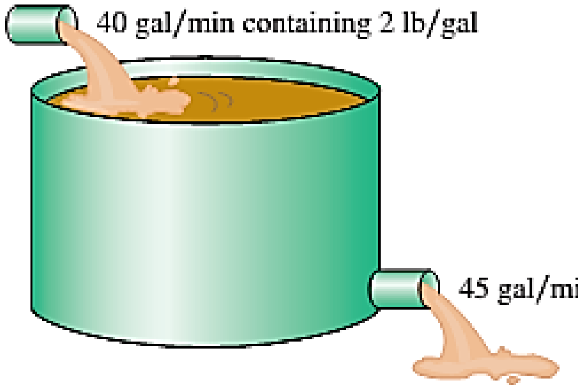
Pounds = slugs × 32,
gravitational acceleration 32 ft/sec^2.

$$11e^{-t/18} = 1$$

$$t = 18 \ln 11 \approx 43 \text{ sec.}$$

$$\text{Distance coasted} = \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3} = 198 \text{ ft.}$$

Mixture Problems

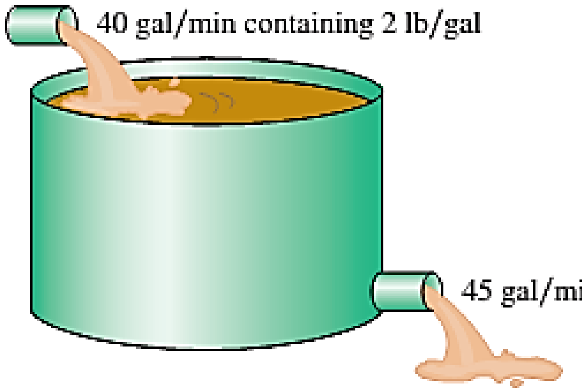


$$\text{Rate of change of amount in container} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix}.$$

$$\text{Departure rate} = \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix} \cdot (\text{outflow rate}). = \frac{y(t)}{V(t)} \cdot (\text{outflow rate})$$

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}).$$

EXAMPLE 3 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.15)?



$y(t)$ the amount (in pounds) of additive in the tank at time t

$y(0) = 100 \quad V(0) = 2000$

$$V(t) = 2000 \text{ gal} + \left(40\frac{\text{gal}}{\text{min}} - 45\frac{\text{gal}}{\text{min}}\right)(t \text{ min}) = (2000 - 5t) \text{ gal}.$$

$$\text{Rate out} = \frac{y(t)}{V(t)} \cdot \text{outflow rate} = \left(\frac{y}{2000-5t}\right)45 = \frac{45y}{2000-5t}\,\frac{\text{lb}}{\text{min}}.$$

$$\text{Rate in} = \left(2\,\frac{\text{lb}}{\text{gal}}\right)\left(40\,\frac{\text{gal}}{\text{min}}\right) = 80\,\frac{\text{lb}}{\text{min}}.$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t} \qquad \frac{dy}{dt} + \frac{45}{2000-5t}y = 80.$$

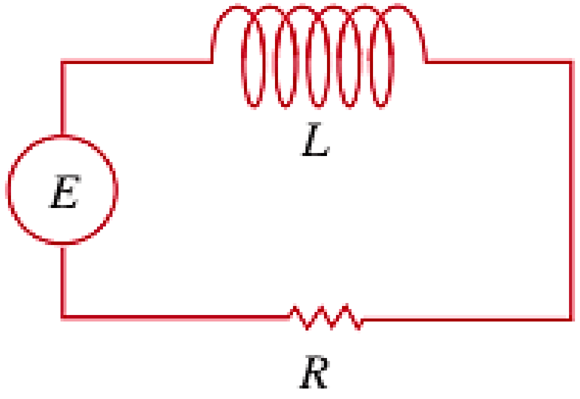
$$P(t) = 45/(2000-5t) \longrightarrow v(t) = e^{\int P\,dt} = e^{\int \frac{45}{2000-5t}dt} = (2000-5t)^{-9}.$$

$$y(t) = \frac{1}{(2000-5t)^{-9}}\left(\int 80(2000-5t)^{-9}\,dt + C\right) = 2(2000-5t) + C(2000-5t)^{9}$$

$$y(0) = 100 \longrightarrow y = 2(2000-5t) - \frac{3900}{(2000)^9}(2000-5t)^9.$$

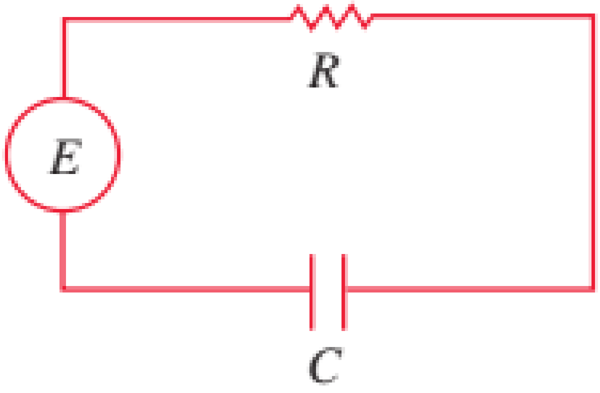
$$y(20) = 2[2000-5(20)] - \frac{3900}{(2000)^9}[2000-5(20)]^9 \approx 1342 \text{ lb}.$$

LR-series circuit



$$L\frac{di}{dt} + Ri = E(t),$$

RC-series circuit



$$Ri + \frac{1}{C}q = E(t).$$

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

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EXAMPLE

A 12-volt battery is connected to an LR-series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

Solution

$$L = 1/2,\ R = 10,\ E = 12,\ i(0) = 0$$

$$L\frac{di}{dt} + Ri = E(t), \quad \Rightarrow \quad \frac{1}{2}\frac{di}{dt} + 10i = 12 \qquad i(0) = 0.$$

$$i(t) = e^{-\int 20dt}\left[24\int e^{\int 20dt} + C\right] = e^{-20t}\left[\frac{24}{20}e^{20t} + C\right] = \frac{6}{5} + Ce^{-20t}$$

$$i(0) = 0$$

$$\Rightarrow \quad 0 = i(0) = \frac{6}{5} + C \quad \Rightarrow \quad C = -\frac{6}{5} \quad \Rightarrow \quad i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}$$

EXAMPLE

A 100-volt electromotive force is applied to an RC-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current $i(t)$.

Solution

$$C = 10^{-4}, \; R = 200, E = 100$$

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

$$\Rightarrow 200\frac{dq}{dt} + 10000q = 100 \;,\; q(0) = 0 \Rightarrow \frac{dq}{dt} + 50q = 0.5 \;,\; q(0) = 0$$

$$q(t) = e^{-\int 50dt}\left[0.5\int e^{\int 50dt} + C\right] = e^{-50t}\left[0.01e^{50t} + C\right] = 0.01 + Ce^{-50t}$$

$q(0) = 0$

$$\Rightarrow 0 = q(0) = 0.01 + C \Rightarrow C = -0.01 \Rightarrow q(t) = 0.01 - 0.01e^{-50t}$$

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}\left(0.01 - 0.01e^{-50t}\right) = 0.5e^{-50t}$$

Exercises

Solve the differential equations

$$y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$$

$$\sin x \cos x + C\cos x$$

$$(t-1)^3\frac{ds}{dt} + 4(t-1)^2 s = t + 1, \quad t > 1$$

$$\frac{t^3}{3(t-1)^4} - \frac{t}{(t-1)^4} + \frac{C}{(t-1)^4}$$

Current in a closed *RL* circuit How many seconds after the switch in an *RL* circuit is closed will it take the current i to reach half of its steady-state value? Notice that the time depends on R and L and not on how much voltage is applied.

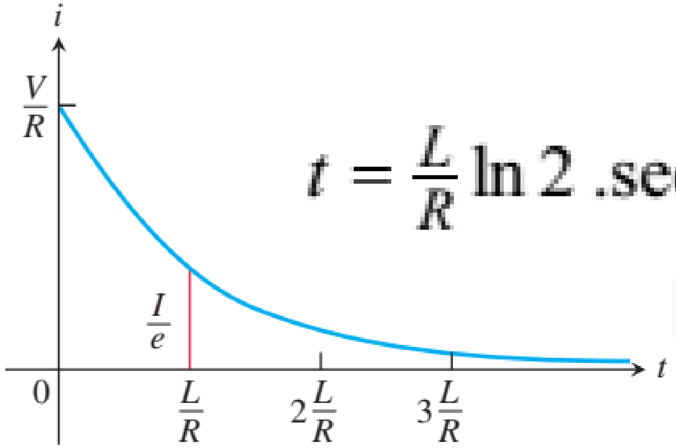
$$t = \frac{L}{R}\ln 2 \ \text{ sec}$$

Exercises

- **Current in an open RL circuit** If the switch is thrown open after the current in an RL circuit has built up to its steady-state value $I = V/R$, the decaying current (see accompanying figure) obeys the equation

$$L\frac{di}{dt} + Ri = 0$$

$i = Ie^{-Rt/L}$ amp



a. Solve the equation to express i as a function of t.

b. How long after the switch is thrown will it take the current to fall to half its original value?

$t = \frac{L}{R}\ln 2$ .sec

c. Show that the value of the current when $t = L/R$ is I/e. (The significance of this time is explained in the next exercise.)

- A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The k in Equation (1) is about 3.9 kg/sec.

$$v = v_0 e^{-(k/m)t}.$$

a. About how far will the cyclist coast before reaching a complete stop? 168.5 meters.

b. How long will it take the cyclist's speed to drop to 1 m/sec? 41.13 seconds.

Exercises

A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.

a. At what time will the tank be full?

b. At the time the tank is full, how many pounds of concentrate will it contain?

$$V = 100 + 2t$$

$$\frac{dy}{dt} + \frac{3}{2(t+50)}y = \frac{5}{2}$$

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