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Calculus 2

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Calculus 2

Lecture 6

Functions of Several Variables

Chapter 4

Partial Derivatives

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introduction

So far we have dealt with the calculus of **functions of a single variable**. But, in the real world, physical quantities often **depend** **on two or more variables**,

so in this chapter we turn our attention to **functions of several variables** and extend the basic ideas **of** **differential calculus** to such functions.

Functions of Several Variables

The notation for a function **of two or more variables** is similar to that for a function of a single variable. Here are two examples.

$$z = f(\underbrace{x, y}_{\text{2 variables}}) = x^2 + xy$$ Function of two variables

and

$$w = f(\underbrace{x, y, z}_{\text{3 variables}}) = x + 2y - 3z$$ Function of three variables

Functions of Several Variables

Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.

For the function **$z = f(x, y)$**

x and y are called the **independent variables** and z is called the **dependent variable**.

[*Compare this with the notation* ***$y = f(x)$*** *for functions of a single variable*.]

Similar definitions can be given for functions **of three, four, or n variables**, where the **domains** consist of ordered triples (x_1, x_2, x_3) quadruples (x_1, x_2, x_3, x_4) and n- tuples $(x_1, x_2 ......, x_n)$. In all cases, the **range** is a set of real numbers.

In this chapter, you will study only functions **of two or three variables**

EXAMPLE 1 Evaluating a Function

The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (3, 0, 4) is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

Domains of Functions of Several Variables

EXAMPLE 2 (a) Functions of Two Variables

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

(b) Functions of Three Variables

Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

Operations on Functions of Several Variables

Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$$ Sum or difference

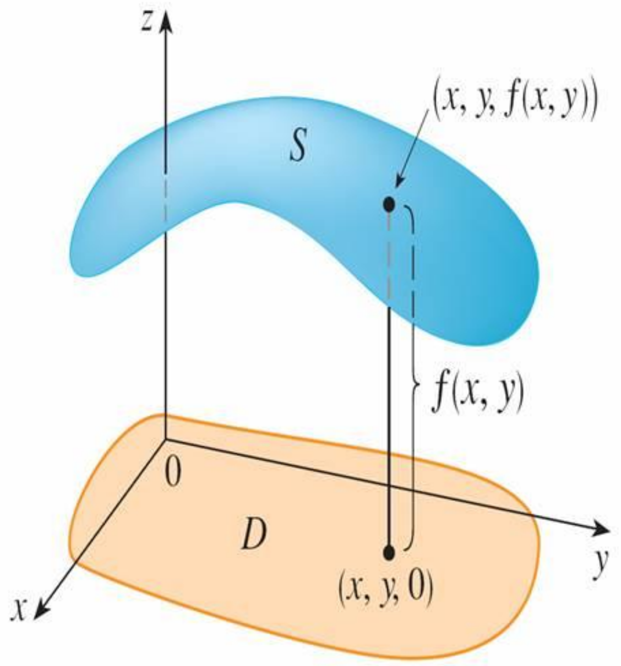
$$(fg)(x, y) = f(x, y)g(x, y)$$ Product

$$\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0$$ Quotient

The Graph of a Function of Two Variable

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of **two variables** is a **surface** S with equation $z = f(x, y)$.



The Graph of a Function

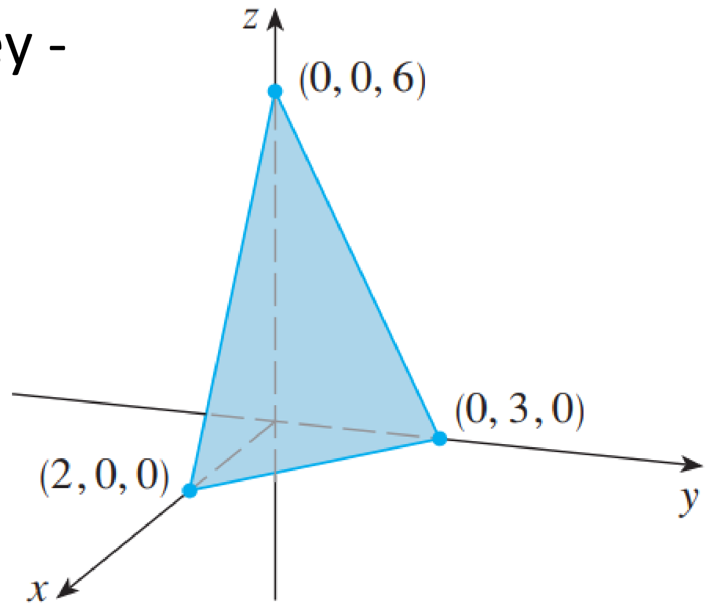
EXAMPLE: Sketch the graph of the function $f(x, y) = 6-3x-2y$

The **graph** is a plane of f has the equation

$$z = = 6-3x-2y \quad \text{or} \quad 3x + 2y + z - 6 = 0$$

which represents **a plane**.

To graph the plane we first find the intercepts. Putting y=z=0 in the equation, we get x=2 as the x-intercept. Similarly, they -intercept is 3 and the z -intercept is 6.



the function f is called as a **linear function**

The Graph of a Function

The function in Example is a special case of the function

$$f(x, y) = ax + by + c$$

is called as a **linear function**.

The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a **plane**.

In much the same way that **linear functions** of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

EXAMPLE: Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

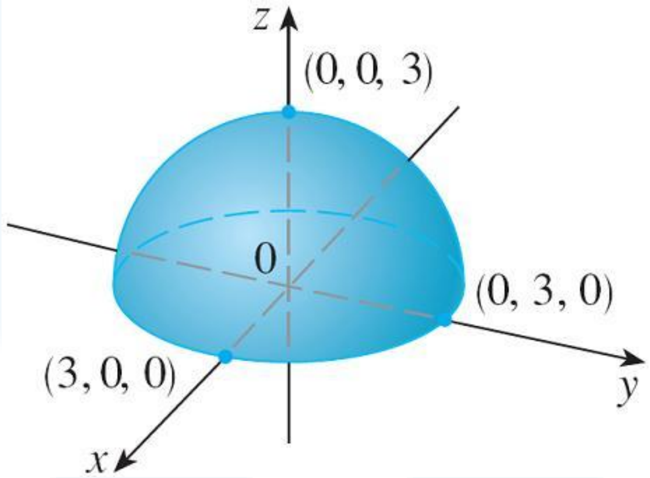
The graph has equation $z = \sqrt{9 - x^2 - y^2}$.

We square both sides of this equation to obtain

$z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$,

which we recognize as an equation of the sphere with center the origin and radius 3.

But, since $z \geq 0$, the graph of g is just **the top half of this sphere** (see Figure).



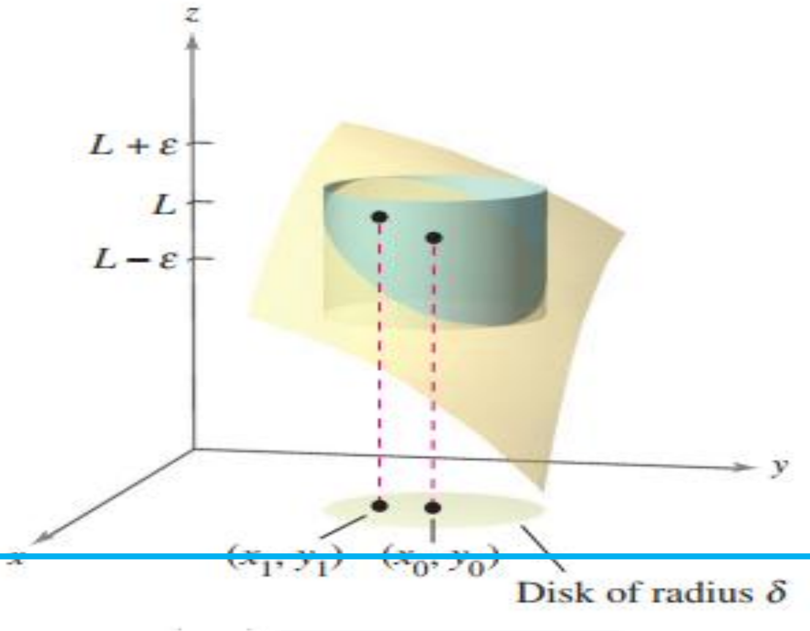
Limits and Continuity

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.

Properties of Limits of Functions of Two Variables

Limits of functions of several variables **have the same properties** regarding sums, differences, products, and quotients as do limits of functions of single variables

THEOREM 1—Properties of Limits of Functions of Two Variables
The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$

Properties of Limits of Functions of Two Variables

5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$

6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer

7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,

n a positive integer, and if n is even, we assume that $L > 0$.

Some of these properties are used in the next example.

Limits of Functions

EXAMPLE:

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

NOTE:

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

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EXAMPLE: A Limit That Does Not Exist

Show that the limit does not exist.

$$\lim_{(x, y) \to (0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2} \right)^2$$

Solution The domain of the function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2} \right)^2$$

consists of all points in the xy-plane except for the point $(0, 0)$. To show that the limit as (x, y) approaches $(0, 0)$ does not exist, consider approaching $(0, 0)$ along two different "paths," as shown in Figure 13.22. Along the x-axis, every point is of the form

$$(x, 0)$$

and the limit along this approach is

$$\lim_{(x, 0) \to (0, 0)} \left(\frac{x^2 - 0^2}{x^2 + 0^2} \right)^2 = \lim_{(x, 0) \to (0, 0)} 1^2 = 1.$$ Limit along x-axis

However, when (x, y) approaches $(0, 0)$ along the line $y = x$, you obtain

$$\lim_{(x, x) \to (0, 0)} \left(\frac{x^2 - x^2}{x^2 + x^2} \right)^2 = \lim_{(x, x) \to (0, 0)} \left(\frac{0}{2x^2} \right)^2 = 0.$$ Limit along line $y = x$

Continuity of a Function of Two Variables

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

THEOREM **Continuous Functions of Two Variables**

If k is a real number and f and g are continuous at (x_0, y_0), then the following functions are also continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, $g(x_0, y_0) \neq 0$

Continuity of a Function of Two Variables

- Theorem establishes the continuity of **polynomial** and **rational functions** at every point in their domains.
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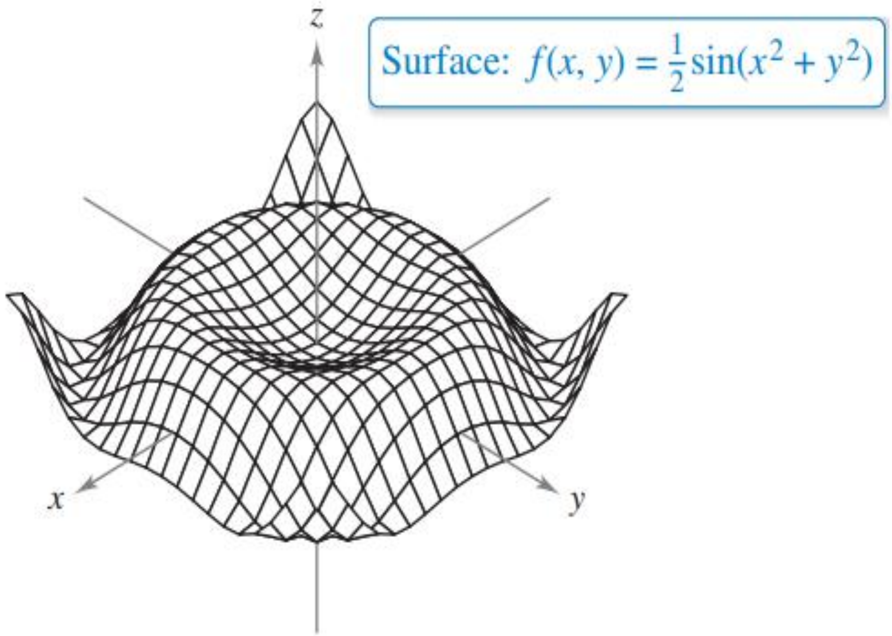
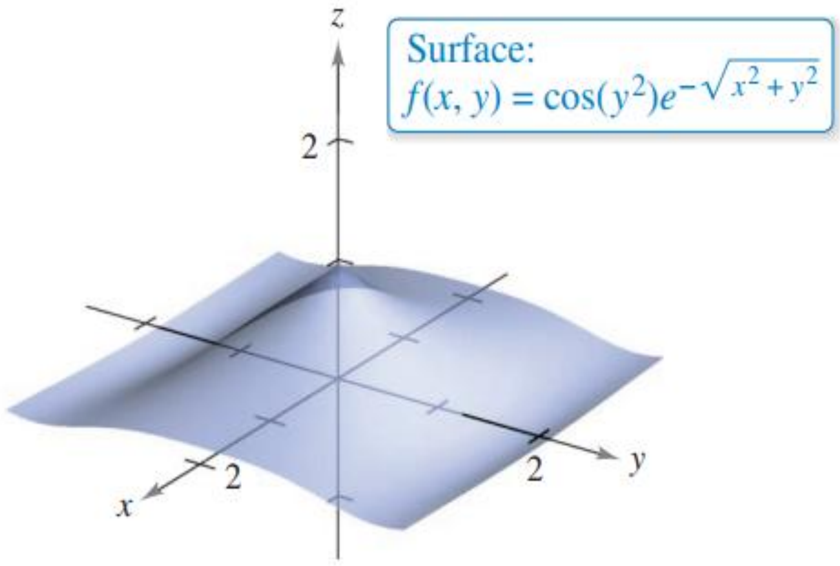
- Furthermore, the continuity of other types of functions can be extended naturally from one to two variables.

- For instance, the functions whose graphs are shown in Figures are continuous at every point in the plane.

Continuity of a Function of Two Variables



The function f is continuous at every point in the plane.



The function f is continuous at every point in the plane.

Continuity of a Function of Two Variables

EXAMPLE: Discuss the continuity of each function.

a. $f(x, y) = \dfrac{x - 2y}{x^2 + y^2}$ **b.** $g(x, y) = \dfrac{2}{y - x^2}$

Solution

a. Because a rational function is continuous at every point in its domain, you can conclude that f is continuous at each point in the xy-plane except at (0, 0), as shown in Figure 13.25.

b. The function

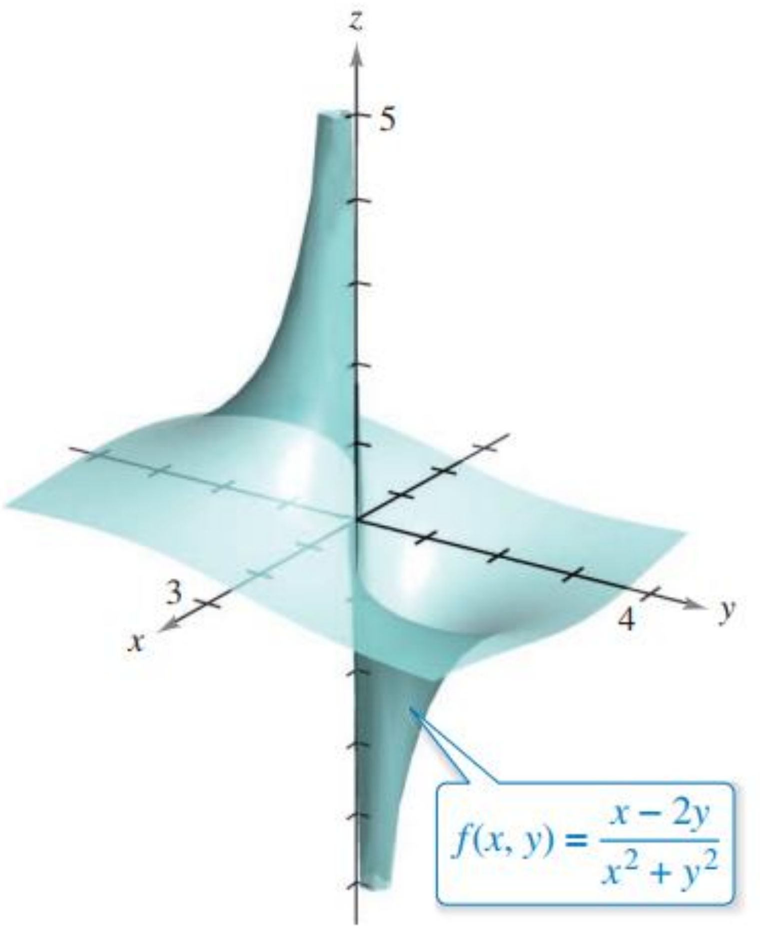
$$g(x, y) = \frac{2}{y - x^2}$$

is continuous except at the points at which the denominator is 0, which is given by the equation

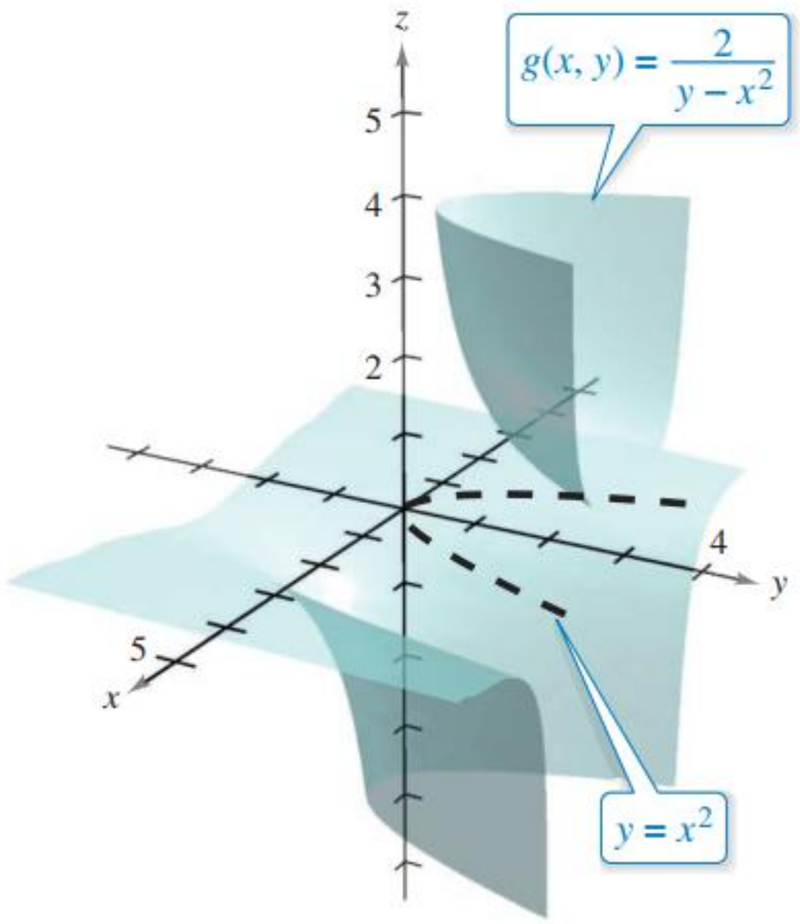
$$y - x^2 = 0.$$

So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$.

Continuity of a Function of Two Variables



The function f is not continuous at (0, 0).



The function g is not continuous on the parabola $y = x^2$.

Functions of More Than Two Variables

NOTE: The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients and powers all **extend** to functions of three or more variables. Functions like

$$\ln(x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are **continuous** throughout their domains, and limits like

$$\lim_{P \to (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

Thank you for your attention