MATHEMATICAL ANALAYSIS 1

Lecture 7

Prepared by
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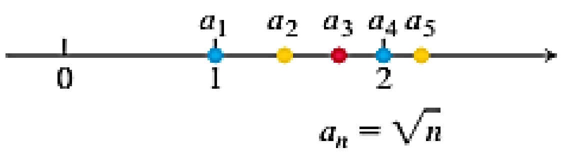
Infinite Sequences and Series

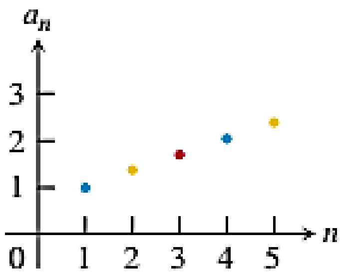
Definition A **sequence** can be thought of as a list of numbers written in a definite order:

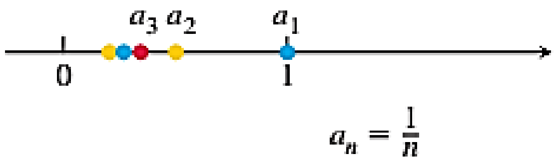
$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

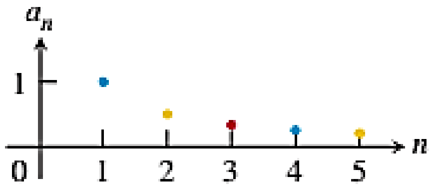
or

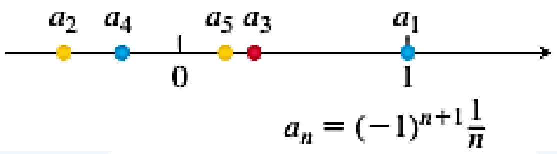
$$a_0, a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

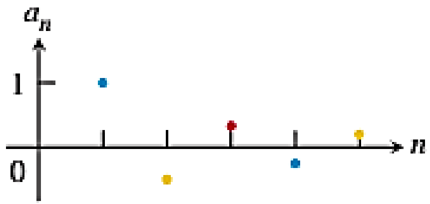












Find a formula for the general term a_n of the sequence for $n \geq 1$:

$$\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125}, \ldots$$

Solution

$$\left.\begin{aligned}
a_1 &= \frac{3}{5} &&\Rightarrow a_1 = (-1)^{1-1}\frac{1+2}{5^1} \\
a_2 &= \frac{-4}{25} &&\Rightarrow a_2 = (-1)^{2-1}\frac{2+2}{5^2} \\
a_3 &= \frac{5}{125} &&\Rightarrow a_3 = (-1)^{3-1}\frac{3+2}{5^3} \\
a_4 &= \frac{-6}{625} &&\Rightarrow a_4 = (-1)^{4-1}\frac{4+2}{5^4} \\
a_5 &= \frac{7}{3125} &&\Rightarrow a_5 = (-1)^{5-1}\frac{5+2}{5^5}
\end{aligned}\right\} \Rightarrow a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

Find a formula for the general term a_n of the sequence:

$$1, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}, \ldots$$

Solution

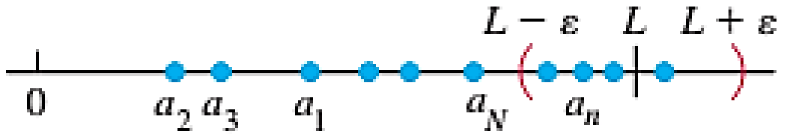
$$\left.\begin{array}{ll}
a_0 = 1 & \Rightarrow a_0 = (-1)^0 \dfrac{1}{(1)^2} \\
a_1 = \dfrac{-1}{4} & \Rightarrow a_1 = (-1)^1 \dfrac{1}{(2)^2} \\
a_2 = \dfrac{1}{9} & \Rightarrow a_2 = (-1)^2 \dfrac{1}{(3)^2} \\
a_3 = \dfrac{-1}{16} & \Rightarrow a_3 = (-1)^3 \dfrac{1}{(4)^2} \\
a_4 = \dfrac{1}{25} & \Rightarrow a_4 = (-1)^4 \dfrac{1}{(5)^2}
\end{array}\right\} \Rightarrow a_n = (-1)^n \frac{1}{(n+1)^2} \; ; \; n = 0, 1, 2, \ldots$$

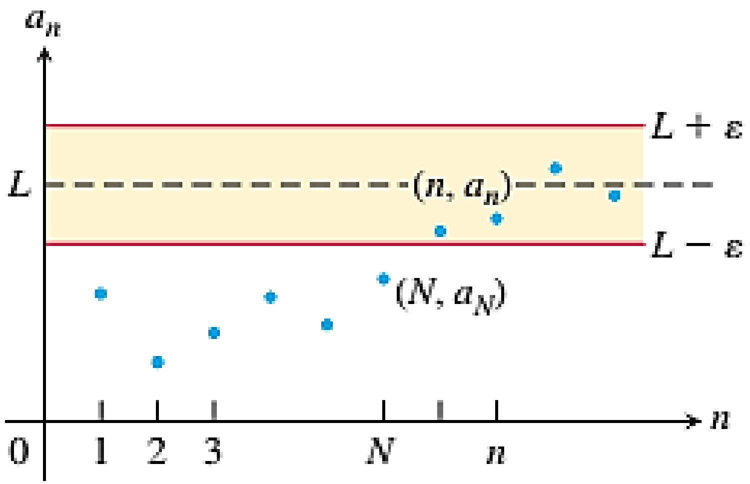
DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

If no such number L exists, we say that $\{a_n\}$ **diverges.**

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 10.2).



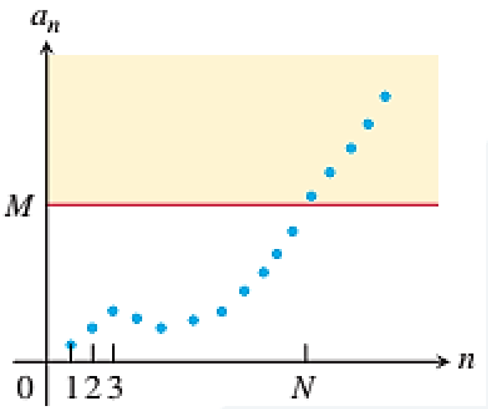


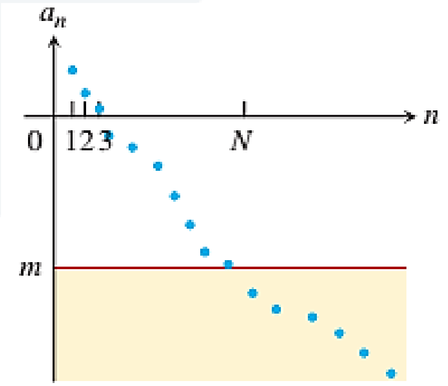
DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$





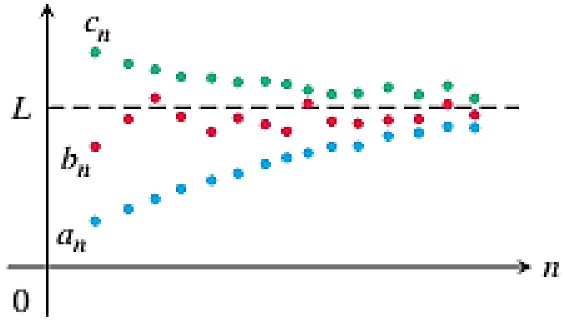
Calculating Limits of Sequences

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

Calculating Limits of Sequences

THEOREM 2—The Sandwich Theorem for Sequences
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.



(a) $\dfrac{\cos n}{n} \to 0$ $\qquad -\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n};$

(b) $\dfrac{1}{2^n} \to 0$ $\qquad 0 \le \dfrac{1}{2^n} \le \dfrac{1}{n};$

(c) $(-1)^n \dfrac{1}{n} \to 0$ $\qquad -\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}.$

(d) If $|a_n| \to 0$, then $a_n \to 0$ $\qquad -|a_n| \le a_n \le |a_n|.$

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Calculating Limits of Sequences

THEOREM 3—The Continuous Function Theorem for Sequences
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

EXAMPLE 5 Show that $\sqrt{(n+1)/n} \to 1$.

$f(x) = \sqrt{x}$ and $L = 1$

Using L'Hôpital's Rule

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n\to\infty} a_n = L \qquad \text{whenever} \qquad \lim_{x\to\infty} f(x) = L.$$

EXAMPLE 8 Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\ln\left(\frac{n+1}{n-1}\right).$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) \qquad \infty\cdot 0 \text{ form}$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0} \text{ form}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

The sequence $\{a_n\}$ converges to e^2.

THEOREM 5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le \ldots$. The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

THEOREM 6—The Monotonic Sequence Theorem
If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

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Exercises

find a formula for the nth term of the sequence.

$$\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \ldots$$

$$a_n = \frac{3n+2}{n!}, \quad n = 1, 2, \ldots$$

$$-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \ldots$$

$$a_n = \frac{2n-5}{n(n+1)}, \quad n = 1, 2, \ldots$$

Which of the sequences $\{a_n\}$ in Exercises 31–100 converge, and which diverge? Find the limit of each convergent sequence.

$$a_n = \frac{3^n}{n^3}$$

∞

$$a_n = (n+4)^{1/(n+4)}$$

1

$$a_n = \left(\frac{1}{n}\right)^{1/(\ln n)}$$

e^{-1}

$$\left(\frac{3n+1}{3n-1}\right)^n$$

$e^{2/3}$

$$\left(3^n + 5^n\right)^{1/n}$$

5

Exercises

In Exercises 101–108, assume that each sequence converges and find its limit.

$$a_1 = 2, \quad a_{n+1} = \frac{72}{1 + a_n}$$

8

$$2,\ 2 + \frac{1}{2},\ 2 + \frac{1}{2 + \frac{1}{2}},\ 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}},\ \ldots$$

$$1 + \sqrt{2}$$

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

EXAMPLE 5 Find the sum of the "telescoping" series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

$$s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

$$s_k = 1 - \frac{1}{k+1}. \longrightarrow \lim_{k \to \infty} s_k = 1 \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

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Geometric Series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

$$s_n - rs_n = a - ar^n$$

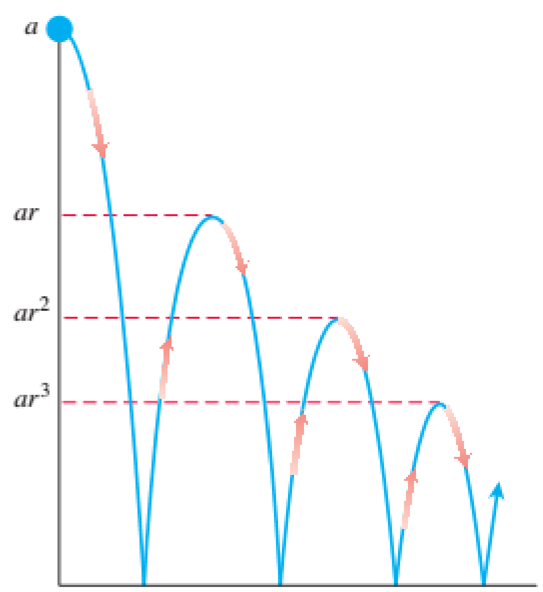
$$s_n = \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1).$$

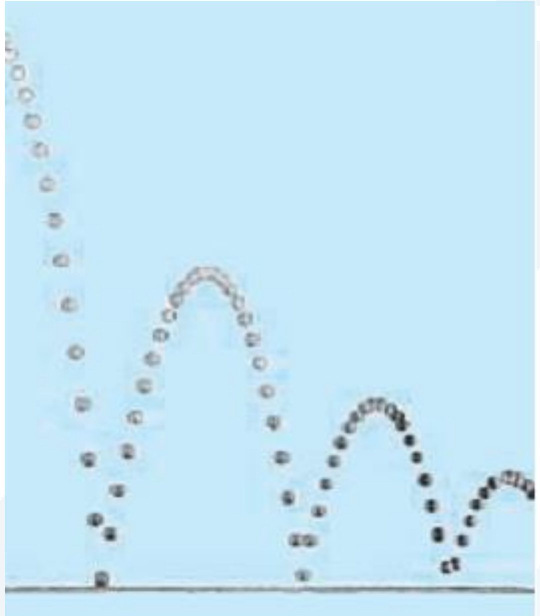
EXAMPLE 2 The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to $\dfrac{a}{1-r} = \dfrac{5}{1+(1/4)} = 4.$

EXAMPLE 3 You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down (Figure 10.10).





$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{\text{This sum is } 2ar/(1-r).}$$

$$= a + \frac{2ar}{1-r} = a\frac{1+r}{1-r}.$$

If $a = 6$ m and $r = 2/3$

$$s = 6 \cdot \frac{1+(2/3)}{1-(2/3)} = 6\left(\frac{5/3}{1/3}\right) = 30 \text{ m.}$$

EXAMPLE 4 Express the repeating decimal 5.232323 . . . as the ratio of two integers.

$$5.232323\ldots = 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)}_{1/(1-0.01)}$$

$$= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99}$$

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The *n*th-Term Test for a Divergent Series

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The *n*th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$.

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

Combining Series

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number k).

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

Caution Remember that $\sum(a_n + b_n)$ can converge *even if* both $\sum a_n$ and $\sum b_n$ diverge. For example, $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverge, whereas $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0. ●

EXAMPLE 9 Find the sums of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{4}{2^n}$

(a) $$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \frac{4}{5}$$

(b) $$\sum_{n=0}^{\infty} \frac{4}{2^n} = 8$$

Exercises

In Exercises 1–6, find a formula for the nth partial sum of each series and use it to find the series' sum if the series converges.

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

$$\frac{2\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\left(\frac{1}{3}\right)} \qquad 3$$

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$$

$$\frac{1}{2} - \frac{1}{n+2} \qquad \frac{1}{2}$$

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Exercises

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

$$0.\overline{234} = 0.234\,234\,234\ldots$$

$$\frac{234}{999}$$

Find the sum of each series

$$\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$$

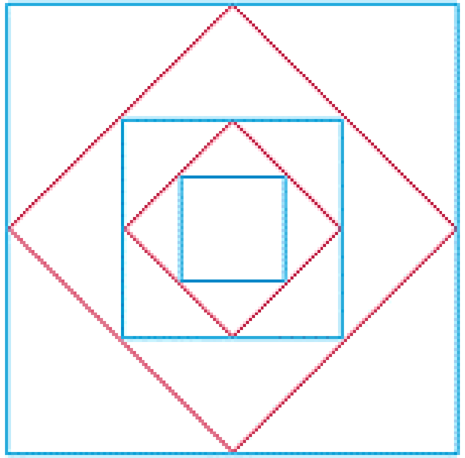
$$-\frac{1}{\ln 2}$$

$$\sum_{n=1}^{\infty}\left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$$

$$-\frac{\pi}{4}$$

Exercises

The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m^2. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



$8\ m^2$

The Integral Test

THEOREM 9—The Integral Test
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

the improper integral converges if p >1 and diverges if $p \leq 1$

EXAMPLE 5 Determine the convergence or divergence of the series.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

converges

diverges

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ $\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \frac{1}{2e}.$

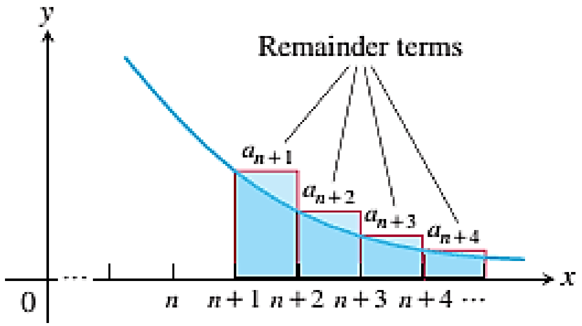
(b) $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$ $\int_1^{\infty} \frac{dx}{2^{\ln x}} = \int_0^{\infty} \left(\frac{e}{2}\right)^u du$

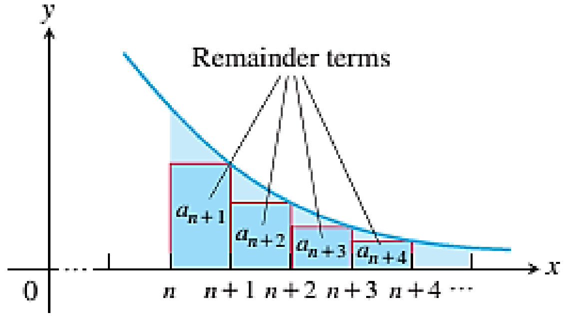
$$= \lim_{b\to\infty} \frac{1}{\ln\left(\frac{e}{2}\right)}\left(\left(\frac{e}{2}\right)^b - 1\right) = \infty.$$

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that Σa_n converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \qquad (1)$$





$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx.$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_{n}^{\infty} f(x)\,dx.$$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le s_n + \int_{n}^{\infty} f(x)\,dx \qquad (2)$$

EXAMPLE 6 Estimate the sum of the series $\Sigma(1/n^2)$ using the inequalities in (2) and $n = 10$.

$$\int_n^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}.$$

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}.$$

$$s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977,$$

$$1.64068 \le S \le 1.64977.$$

If we approximate the sum S by the midpoint of this interval, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452.$$

Exercises

Applying the Integral Test

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$$

Estimate the value of $\sum_{n=1}^{\infty} (1/n^3)$ to within 0.01 of its exact value.

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Comparison Tests

THEOREM 10—Direct Comparison Test
Let $\sum a_n$ and $\sum b_n$ be two series with $0 \le a_n \le b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

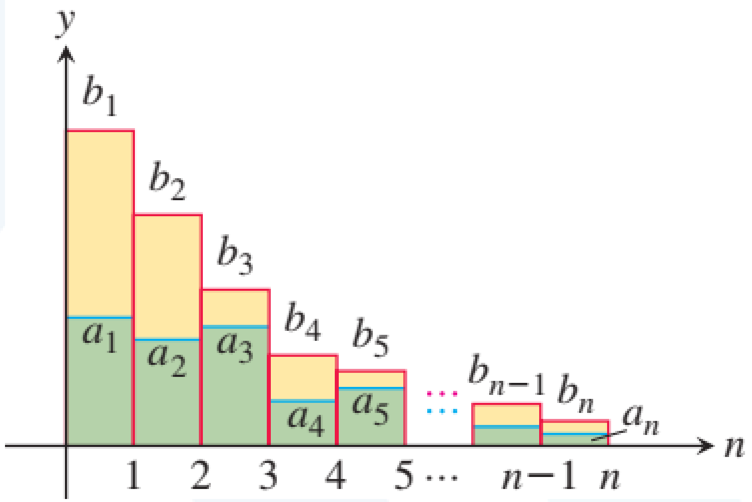
diverges

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$$



The Limit Comparison Test

THEOREM 11—Limit Comparison Test
Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$$\sum_{n=2}^{\infty} \frac{1 + n \ln n}{n^2 + 5}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

EXAMPLE 3 Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} = \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.$$

convergent

Exercises

In Exercises 1–8, use the Direct Comparison Test to determine if each series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}} \qquad \sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$$

In Exercises 9–16, use the Limit Comparison Test to determine if each series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)} \qquad \sum_{n=1}^{\infty} \frac{2^n}{3+4^n} \qquad \sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n} \qquad \sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$$

$$\sum_{n=2}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$$