Problem sets 6 : Determinants

CEDC102: Linear Algebra and Matrix Theory

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Problem 1. If the entries in every row of A add to zero, solve $Ax = 0$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

Problem 2

The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0|, \; C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \; C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \; C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

(a) What are these determinants C_1, C_2, C_3, C_4?
(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

Problem 3

Compute the determinants of

$$A = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & d & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & a & a^2 & a^3 \\ a & 1 & a & a^2 \\ a^2 & a & 1 & a \\ a^3 & a^2 & a & 1 \end{bmatrix}$$

Problem 4

This problem shows in two ways that $\det A = 0$:

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & x_{44} & x_{45} \\ 0 & 0 & 0 & x_{54} & x_{55} \end{bmatrix}$$

(a) How do you know that the rows are linearly dependent?

(b) Explain why all 120 terms are zero in the big formula for $\det A$.

Problem 5

If a 4×4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$, $\det(-A)$, $\det(A^2)$, and $\det(A^{-1})$.

Problem 6

True or **false** (give a *reason* if true or a 2×2 *counter-example* if false), using the properties of determinants. A and B are square matrices.

(a) If A is not invertible then AB is not invertible.

(b) The determinant of A is always the product of its pivots.

(c) $\det(A - B)$ always equals $\det(A) - \det(B)$.

(d) AB and BA must have the same determinant.

Problem 7

(a) If Q is a unitary matrix, from the properties of determinants explain why $\det Q$ must be ______ or ______.

(b) If P is a 3×3 projection matrix onto a 2d subspace, then explain why its determinant must be ______.

(c) If A is a 5×5 matrix that is anti-symmetric ($A^T = -A$), then explain why its determinant must be ______.