التحليل الرياضي ١

محاضرة 8 ميكاترونيكس

عملي

Prepared by
Dr. Sami INJROU

التكامل

تكامل التوابع المثلثية

تمارين

1 احسب التكاملات الآتية:

- $\int \cos^3 x \sin x \, dx$
- $\int \sin^3 x \cos^3 x \, dx$
- $\int 16 \sin^2 x \cos^2 x \, dx$
- $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$
- $\int \cos 3x \cos 4x \, dx$

الحل

$\int \cos^3 x \sin x \, dx$

$$\int \cos^3 x \sin x \, dx = -\int \cos^3 x(-\sin x)\, dx = -\tfrac{1}{4}\cos^4 x + C$$

$\int \sin^3 x \cos^3 x \, dx$

$$\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x \cos^2 x \cos x dx = \int \sin^3 x \left(1-\sin^2 x\right)\cos x \, dx$$

$$= \int \sin^3 x \cos x \, dx - \int \sin^5 x \cos x \, dx = \tfrac{1}{4}\sin^4 x - \tfrac{1}{6}\sin^6 x + C$$

$\int 16 \sin^2 x \cos^2 x \, dx$

$$= 16\int \left(\tfrac{1-\cos 2x}{2}\right)\left(\tfrac{1+\cos 2x}{2}\right) dx = 4\int \left(1-\cos^2 2x\right) dx = 4\int dx - 4\int \left(\tfrac{1+\cos 4x}{2}\right) dx$$

$$= 4x - 2\int dx - 2\int \cos 4x \, dx = 4x - 2x - \tfrac{1}{2}\sin 4x + C = 2x - \tfrac{1}{2}\sin 4x + C = 2x - \sin 2x \cos 2x + C$$

$$= 2x - 2\sin x \cos x\,(2\cos^2 x - 1) + C = 2x - 4\sin x \cos^3 x + 2\sin x \cos x + C$$

تمارين

$$\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x \, dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$$

$$\int \cos 3x \cos 4x \, dx = \frac{1}{2}\int (\cos(-x) + \cos 7x)\, dx = \frac{1}{2}\int (\cos x + \cos 7x)\, dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + C$$

التكامل

تكامل التوابع الكسرية

تمارين

1 احسب التكاملات الآتية:

- $\int \frac{x+4}{x^2+5x-6}\,dx$
- $\int_4^8 \frac{y\,dy}{y^2-2y-3}$
- $\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy$

الحل

$$\int \frac{x+4}{x^2+5x-6}\,dx$$

$$\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1)+B(x+6);\ x=1 \Rightarrow B = \frac{5}{7};\ x=-6 \Rightarrow A = \frac{-2}{-7} = \frac{2}{7};$$

$$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3}$$

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1)+B(y-3);\ y=-1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4};\ y=3 \Rightarrow A = \frac{3}{4};$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$$

$$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$$

تمارين

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy$$

$$\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By; \quad y=0 \Rightarrow A=4;\ y=-1 \Rightarrow B = \frac{3}{-1} = -3;$$

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^{1} \frac{dy}{y} - 3\int_{1/2}^{1} \frac{dy}{y+1} = \left[4\ln|y| - 3\ln|y+1|\right]_{1/2}^{1} = (4\ln 1 - 3\ln 2) - \left(4\ln\tfrac{1}{2} - 3\ln\tfrac{3}{2}\right)$$

$$= \ln\tfrac{1}{8} - \ln\tfrac{1}{16} + \ln\tfrac{27}{8} = \ln\left(\tfrac{27}{8}\cdot\tfrac{1}{8}\cdot 16\right) = \ln\tfrac{27}{4}$$

2 احسب التكاملات الآتية:

- $\displaystyle\int_{0}^{1} \frac{x^3\,dx}{x^2+2x+1}$
- $\displaystyle\int_{-1}^{0} \frac{x^3\,dx}{x^2-2x+1}$
- $\displaystyle\int \frac{dx}{(x^2-1)^2}$

الحل

$$\int_{0}^{1} \frac{x^3\,dx}{x^2+2x+1}$$

بالقسمة المطولة نحصل على

$$\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$$

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B = Ax + (A+B)$$

$$\Rightarrow A=3,\ A+B=2 \Rightarrow A=3,\ B=-1;$$

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$$\int_0^1 \frac{x^3dx}{x^2+2x+1} = \int_0^1 (x-2)dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1 = \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$$

$$\int_{-1}^{0} \frac{x^3\,dx}{x^2 - 2x + 1}$$

بالقسمة المطولة نحصل على $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$

$$\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1)+B = Ax+(-A+B) \Rightarrow A=3, -A+B=-2 \Rightarrow A=3, B=1;$$

$$\int_{-1}^{0} \frac{x^3dx}{x^2-2x+1} = \int_{-1}^{0} (x+2)dx + 3\int_{-1}^{0} \frac{dx}{x-1} + \int_{-1}^{0} \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^{0}$$

$$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$$

تمارين

$$\int \frac{dx}{(x^2-1)^2}$$

$$\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$$

$$x=-1 \Rightarrow C = \tfrac{1}{4}; \quad x = 1 \Rightarrow D = \tfrac{1}{4};$$

$$\text{coefficient of } x^3 = A + B \Rightarrow A + B = 0;$$

$$\text{constant} = A - B + C + D \Rightarrow A - B + C + D = 1$$

$$\Rightarrow A - B = \tfrac{1}{2} \Rightarrow A = \tfrac{1}{4} \Rightarrow B = -\tfrac{1}{4};$$

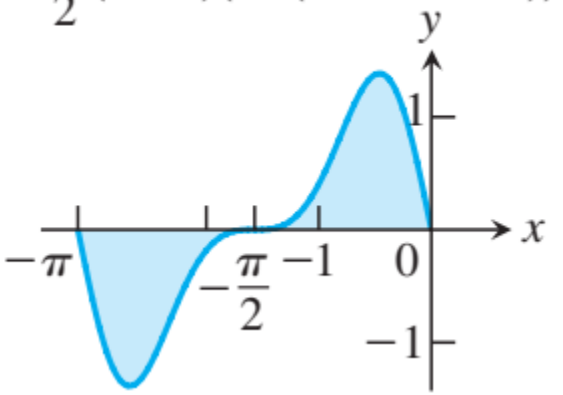
$$\int \frac{dx}{(x^2-1)^2} = \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$$

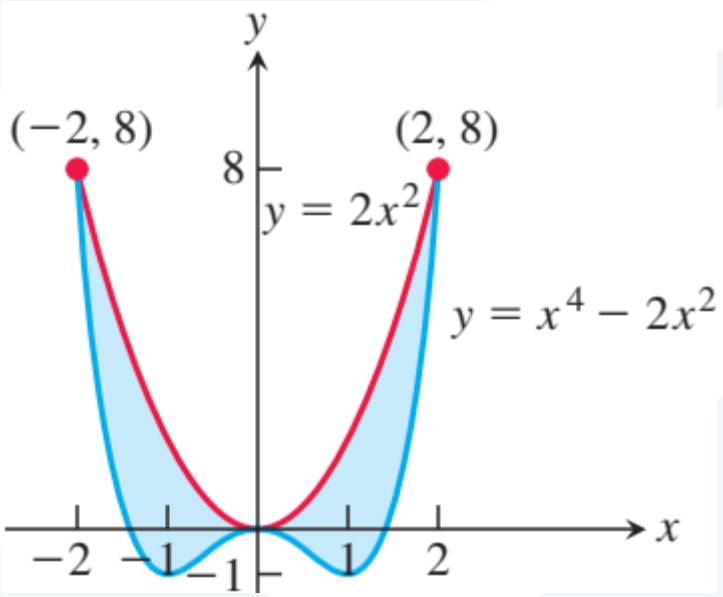
تطبيقات التكامل

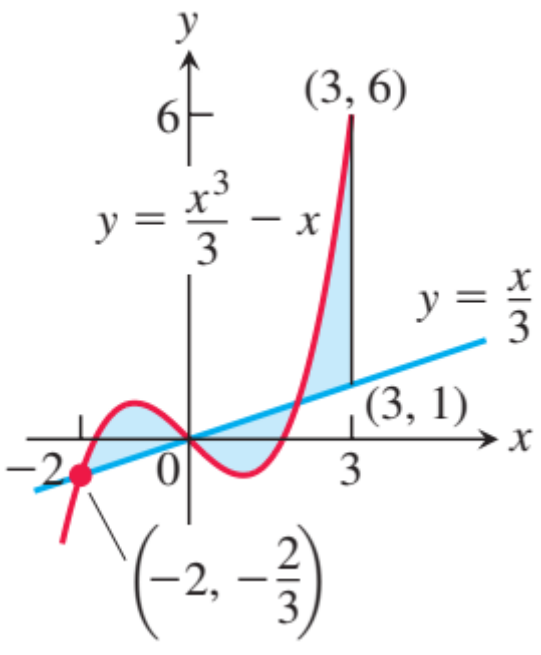
تمارين

1 احسب مساحة المنطقة المظللة لكل مما يلي:

- $y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi \sin x))$



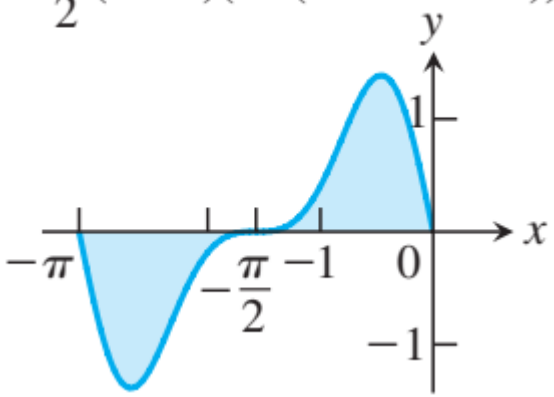
- 

- 

تمارين

الحل

$$y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi \sin x))$$



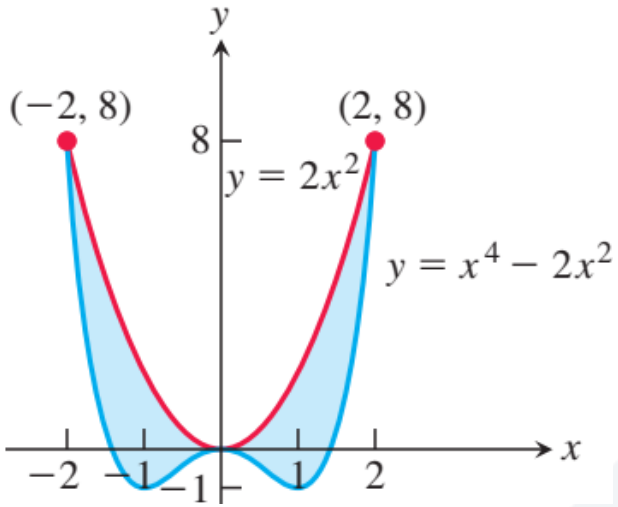
$$u = \pi + \pi \sin x \Rightarrow du = \pi \cos x\, dx \Rightarrow \frac{1}{\pi}\, du = \cos x\, dx$$

$$x = -\frac{\pi}{2} \Rightarrow u = \pi + \pi \sin\left(-\frac{\pi}{2}\right) = 0,\ x = 0 \Rightarrow u = \pi$$

بسبب التناظر حول $x = -\frac{\pi}{2}$

$$A = 2\int_{-\pi/2}^{0} \frac{\pi}{2}(\cos x)(\sin(\pi + \pi \sin x))\, dx = 2\int_{0}^{\pi} \frac{\pi}{2}(\sin u)\left(\frac{1}{\pi}\, du\right)$$

$$= \int_{0}^{\pi} \sin u\, du = [-\cos u]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2$$

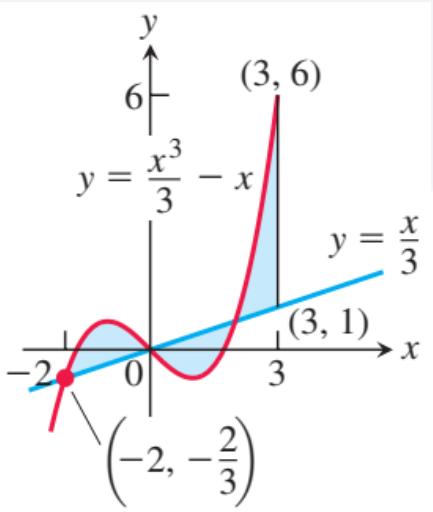


$$a = -2,\ b = 2;\ f(x) - g(x) = 2x^2 - (x^4 - 2x^2) = 4x^2 - x^4$$

$$A = \int_{-2}^{2} (4x^2 - x^4)\, dx = \left[\frac{4x^2}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right]$$

$$= \frac{64}{3} - \frac{64}{5} = \frac{320-192}{15} = \frac{128}{15}$$

تمارين



$$\text{AREA} = \text{A1} + \text{A2} + \text{A3}$$

$$\text{A1}: a = -2 \text{ and } b = 0: \quad f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{x^3}{3} - \frac{4}{3}x = \frac{1}{3}(x^3 - 4x)$$

$$\Rightarrow \text{A1} = \frac{1}{3}\int_{-2}^{0}(x^3 - 4x)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} = 0 - \frac{1}{3}(4 - 8) = \frac{4}{3};$$

$$\text{A2}: \quad a = 0$$

لإيجاد b علينا إيجاد نقطة تقاطع $y = \frac{x^3}{3} - x$ و $y = \frac{x}{3}$

$$\frac{x^3}{3} - x = \frac{x}{3} \Rightarrow \frac{x^3}{3} - \frac{4}{3}x = 0 \Rightarrow \frac{x}{3}(x-2)(x+2) = 0 \quad \Rightarrow x = -2, x = 0, x = 2 \quad \Rightarrow \quad b = 2$$

$$f(x) - g(x) = \frac{x}{3} - \left(\frac{x^3}{3} - x\right) = -\frac{1}{3}(x^3 - 4x) \Rightarrow \text{A2} = -\frac{1}{3}\int_{0}^{2}(x^3 - 4x)\,dx = \frac{1}{3}\int_{0}^{2}(4x - x^3) = \frac{1}{3}\left[2x^2 - \frac{x^4}{4}\right]_{0}^{2} = \frac{1}{3}(8 - 4) = \frac{4}{3};$$

$$\text{A3}: \ a = 2 \text{ and } b = 3: \ f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{1}{3}(x^3 - 4x)$$

$$\Rightarrow \text{A3} = \frac{1}{3}\int_{2}^{3}(x^3 - 4x)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - 2x^2\right]_{2}^{3} = \frac{1}{3}\left[\left(\frac{81}{4} - 2\cdot 9\right) - \left(\frac{16}{4} - 8\right)\right] = \frac{1}{3}\left(\frac{81}{4} - 14\right) = \frac{25}{12};$$

$$\text{AREA} = \text{A1} + \text{A2} + \text{A3} = \frac{4}{3} + \frac{4}{3} + \frac{25}{12} = \frac{32+25}{12} = \frac{19}{4}$$