جامعة المنارة
MANARA UNIVERSITY

# MATHEMATICAL ANALAYSIS 1

# Lecture 9

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# Applications of Taylor Series

**The Binomial Series**

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

$$(1+x)^{-1} = 1 + \sum_{k=1}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots.$$

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

**EXAMPLE 4** Estimate $\int_0^1 \sin x^2\, dx$ with an error of less than 0.001.

$$\int \sin x^2\, dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \cdots$$

$$\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \cdots$$

$$\frac{1}{11 \cdot 5!} \approx 0.00076$$

$$\int_0^1 \sin x^2\, dx \approx \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

with an error of less than $10^{-6}$

$$\int_0^1 \sin x^2\, dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720} \approx 0.310268303,$$

# Evaluating Indeterminate Forms

**EXAMPLE 6** Evaluate

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

$$\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$$

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right) = -\frac{1}{2}.$$

**EXAMPLE 7** Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}=x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}.$$

$$\Rightarrow\quad \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}\right)=0.$$

# Exercises

- Find the first four terms of the binomial series for the functions

$$(1+x)^{1/3}$$

$$1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\dots$$

$$\left(1+\frac{x}{2}\right)^{-2}$$

$$1-x+\frac{3}{4}x^2-\frac{1}{2}x^3+\dots$$

$$\frac{x}{\sqrt[3]{1+x}}$$

$$x-\frac{1}{3}x^2+\frac{2}{9}x^3-\frac{14}{81}x^4+\dots$$

- Find the binomial series for the functions

$$(1+x)^4 \qquad 1+4x+6x^2+4x^3+x^4$$

$$\left(1-\frac{x}{2}\right)^4 \qquad 1-2x+\frac{3}{2}x^2-\frac{1}{2}x^3+\frac{1}{16}x^4$$

- use series to estimate the integrals' values with an error of magnitude less than $10^{-5}$.

$$\int_0^{0.4}\frac{e^{-x}-1}{x}\,dx$$

$-0.3633060.$

$$\int_0^{0.35}\sqrt[3]{1+x^2}\,dx$$

$0.3546472.$

# Exercises

- Use series to approximate the values of the integrals with an error of magnitude less than $10^{-8}$.

$$\int_0^{0.1} \frac{\sin x}{x} dx \qquad \int_0^{1} \frac{1 - \cos x}{x^2} dx$$

$0.0999444611, |E| \le \frac{(0.1)^7}{7 \cdot 7!} \approx 2.8 \times 10^{-12}$ $\qquad$ $0.4863853764, \quad |E| \le \frac{1}{11 \cdot 12!} \approx 1.9 \times 10^{-10}$

- Estimate the error if $\cos \sqrt{t}$ is approximated by $1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!}$ in the integral $\int_0^1 \cos \sqrt{t}\, dt$. $\qquad$ 0.000004960

- find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than $10^{-3}$.

$$F(x) = \int_0^x \sin t^2\, dt, \quad [0, 1] \qquad F(x) = \int_0^x \frac{\ln(1+t)}{t} dt, \quad [0, 0.5]$$

$|\text{error}| < \frac{1}{15 \cdot 7!} \approx 0.000013$ $\qquad$ $|\text{error}| < \frac{(0.5)^6}{6^2} \approx .00043$

# Exercises

- Use series to evaluate the limits

$$\lim_{y\to 0}\frac{\tan^{-1}y-\sin y}{y^3\cos y}$$

$-\frac{1}{6}$

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{1-\cos x}$$

$2$

$$\lim_{x\to 0}\frac{\sin 3x^2}{1-\cos 2x}$$

$\frac{3}{2}$

$$\lim_{x\to 0}\frac{\ln(1+x^3)}{x\cdot\sin x^2}$$

$1$

- find the sum of each series.

$$1-\frac{3^2}{4^2\cdot 2!}+\frac{3^4}{4^4\cdot 4!}-\frac{3^6}{4^6\cdot 6!}+\cdots$$

$\cos\left(\frac{3}{4}\right)$

$$\frac{\pi}{3}-\frac{\pi^3}{3^3\cdot 3!}+\frac{\pi^5}{3^5\cdot 5!}-\frac{\pi^7}{3^7\cdot 7!}+\cdots$$

$\frac{\sqrt{3}}{2}$

$$1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\cdots$$

$-\frac{\ln(1-x)}{x}$

## Linear First-Order Differential Equations

$$y'(x)+P(x)y(x)=Q(x)$$

$$y(x)=\exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx+C\right)$$

**Bernoulli's Equation**

$$y'(x)+P(x)y(x)=Q(x)y^{n}$$

$z=y^{1-n}$ → **Linear Equation** $z'+(1-n)p(x)z=(1-n)Q(x)$

**Example** Solve the following differential equation

$$xy' - 3y = x^5 \cdot \sqrt[4]{y}$$

Solution

$$y' - \frac{3}{x}y = x^4 y^{\frac{1}{4}}$$

**Bernoulli's Equation with** $n = \frac{1}{4}$

$z = y^{\frac{3}{4}}$ → **Linear Equation** $z' - \frac{9}{4x}z = \frac{3}{4}x^4$

→ $$z = e^{-\int -\frac{9}{4x}dx}\left[\int \frac{3}{4}x^4 e^{\int -\frac{9}{4x}dx} dx + c\right] = \frac{3}{11}x^5 + cx^{\frac{9}{4}}$$

→ $$y = \left[\frac{3}{11}x^5 + cx^{\frac{9}{4}}\right]^{\frac{4}{3}}$$

# Exercises

Solve the given differential equation

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$y^3 = 1 + cx^{-3}.$$

$$3(1 + t^2)\frac{dy}{dt} = 2ty(y^3 - 1)$$

$$y^{-3} = 1 + c\left(1 + t^2\right).$$

Solve the given initial-value problem

$$x^2\frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$$

$$y^{-3} = -\tfrac{9}{5}x^{-1} + \tfrac{49}{5}x^{-6}.$$

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \tag{1}$$

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2}$$

**THEOREM 1—The Superposition Principle** If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation (2), then for any constants $c_1$ and $c_2$, the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation (2).

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \tag{1}$$

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2}$$

**THEOREM 2** If $P$, $Q$, and $R$ are continuous over the open interval $I$ and $P(x)$ is never zero on $I$, then the linear homogeneous equation (2) has two linearly independent solutions $y_1$ and $y_2$ on $I$. Moreover, if $y_1$ and $y_2$ are *any* two linearly independent solutions of Equation (2), then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where $c_1$ and $c_2$ are arbitrary constants.

# Constant-Coefficient Homogeneous Equations

$$ay'' + by' + cy = 0,$$

$$y = e^{rx}$$

$$ar^2 + br + c = 0.$$

**auxiliary equation**

**characteristic equation**

Case 1

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

**THEOREM 3** If $r_1$ and $r_2$ are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to $ay'' + by' + cy = 0$.

**EXAMPLE 1** Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

$$r^2 - r - 6 = 0,$$

$$(r - 3)(r + 2) = 0.$$

$$y = c_1e^{3x} + c_2e^{-2x}.$$

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## Case 2

**THEOREM 4** If $r$ is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

$$y_2 = p(x)e^{(-b/2a)x}$$

$$y_2 = xe^{(-b/2a)}$$

**EXAMPLE 2** Find the general solution to

$$y'' + 4y' + 4y = 0.$$

$$r^2 + 4r + 4 = 0, \Rightarrow (r+2)^2 = 0. \Rightarrow y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

## Case 3

**THEOREM 5** If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

**EXAMPLE 3** Find the general solution to the differential equation

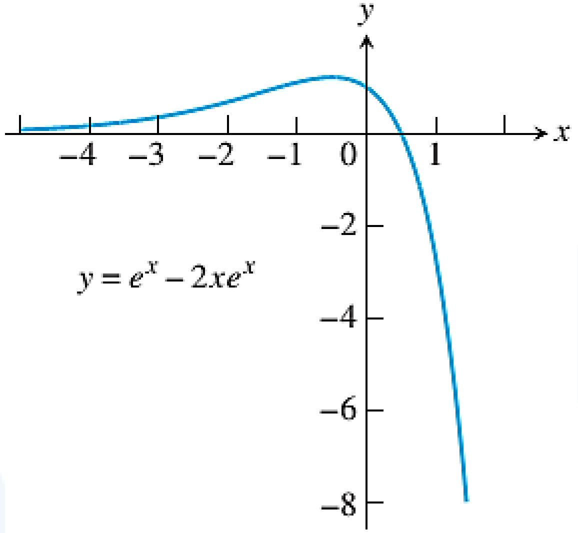
$$y'' - 4y' + 5y = 0.$$

$r^2 - 4r + 5 = 0.$ $\Rightarrow$ $r_1 = 2 + i$ and $r_2 = 2 - i.$ $\Rightarrow$ $y = e^{2x}(c_1 \cos x + c_2 \sin x).$

## Initial Value and Boundary Value Problems

**EXAMPLE 4** Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$




$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

$$y = c_1 e^x + c_2 x e^x.$$

$$y' = c_1 e^x + c_2 (x + 1) e^x.$$

From the initial conditions

$$1 = c_1 + c_2 \cdot 0 \qquad \text{and} \qquad -1 = c_1 + c_2 \cdot 1. \Rightarrow c_1 = 1 \text{ and } c_2 = -2$$

$$y = e^x - 2xe^x.$$

## Initial Value and Boundary Value Problems

**EXAMPLE 5** Solve the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 0, \quad y\left(\frac{\pi}{12}\right) = 1.$$

$r^2 + 4 = 0.$ ⇒ $y = c_1 \cos 2x + c_2 \sin 2x.$

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y\left(\frac{\pi}{12}\right) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = 1.$$

⇒ $c_1 = 0$ and $c_2 = 2.$ ⇒ $y = 2 \sin 2x.$

## EXERCISES

$y'' + 6y' + 5y = 0, \quad y(0) = 0,\ y'(0) = 3$ $\qquad y = -\frac{3}{4}e^{-5x} + \frac{3}{4}e^{-x}$

$4y'' - 4y' + y = 0, \quad y(0) = 4,\ y'(0) = 4$ $\qquad y = 4e^{\frac{1}{2}x} + 2x\,e^{\frac{1}{2}x}$

$y'' - 2y' + 2y = 0, \quad y(0) = 0,\ y'(0) = 2$ $\qquad y = 2e^{x}\sin x$

$y'' + 2y' + y = 0, \quad y(0) = 1,\ y'(0) = 1$ $\qquad y = e^{-x} + 2x\,e^{-x}$

$y'' + 4y = 0, \quad y(0) = 0,\ y(\pi) = 1.$ $\qquad$ no solution

$y'' + 4y = 0, \quad y(0) = 0,\ y(\pi) = 0.$ $\qquad y = c_2 \sin 2x$

# Nonhomogeneous Linear Equations

$$ay'' + by' + cy = G(x),$$

**THEOREM 7** The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** $y_c$ is the general solution to the associated homogeneous equation (2) and $y_p$ is any **particular solution** to the nonhomogeneous equation (1).

# The Method of Undetermined Coefficients

**Summary of the Method of Undetermined Coefficients**

1. If $G(x) = e^{kx}P(x)$, where $P$ is a polynomial of degree $n$, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an $n$th-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x) \cos mx$ or $G(x) = e^{kx}P(x) \sin mx$, where $P$ is an $n$th-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x) \cos mx + e^{kx}R(x) \sin mx$$

where $Q$ and $R$ are $n$th-degree polynomials.

**Modification**: If any term of $y_p$ is a solution of the complementary equation, multiply $y_p$ by $x$ (or by $x^2$ if necessary).

## The Method of Undetermined Coefficients

| $g(x)$ | Form of $y_p$ |
|---|---|
| **1.** 1 (any constant) | $A$ |
| **2.** $5x + 7$ | $Ax + B$ |
| **3.** $3x^2 - 2$ | $Ax^2 + Bx + C$ |
| **4.** $x^3 - x + 1$ | $Ax^3 + Bx^2 + Cx + E$ |
| **5.** $\sin 4x$ | $A\cos 4x + B\sin 4x$ |
| **6.** $\cos 4x$ | $A\cos 4x + B\sin 4x$ |
| **7.** $e^{5x}$ | $Ae^{5x}$ |
| **8.** $(9x - 2)e^{5x}$ | $(Ax + B)e^{5x}$ |
| **9.** $x^2e^{5x}$ | $(Ax^2 + Bx + C)e^{5x}$ |
| **10.** $e^{3x}\sin 4x$ | $Ae^{3x}\cos 4x + Be^{3x}\sin 4x$ |
| **11.** $5x^2\sin 4x$ | $(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$ |
| **12.** $xe^{3x}\cos 4x$ | $(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$ |

$$ay'' + by' + cy = G(x).$$

| If $G(x)$ has a term that is a constant multiple of . . . | And if | Then include this expression in the trial function for $y_p$. |
|---|---|---|
| $e^{rx}$ | $r$ is not a root of the auxiliary equation | $Ae^{rx}$ |
| | $r$ is a single root of the auxiliary equation | $Axe^{rx}$ |
| | $r$ is a double root of the auxiliary equation | $Ax^2e^{rx}$ |
| $\sin kx$, $\cos kx$ | $ki$ is not a root of the auxiliary equation | $B\cos kx + C\sin kx$ |
| $px^2 + qx + m$ | 0 is not a root of the auxiliary equation | $Dx^2 + Ex + F$ |
| | 0 is a single root of the auxiliary equation | $Dx^3 + Ex^2 + Fx$ |
| | 0 is a double root of the auxiliary equation | $Dx^4 + Ex^3 + Fx^2$ |

**EXAMPLE 1** Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

$$y_c = c_1 e^{-x} + c_2 e^{3x}.$$

$$y_p = Ax^2 + Bx + C. \quad \Rightarrow \quad 2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

$$-3A = -1, \qquad -4A - 3B = 0, \qquad \text{and} \qquad 2A - 2B - 3C = 1.$$

$$\Rightarrow \quad A = 1/3, B = -4/9, \text{ and } C = 5/27. \quad \Rightarrow \quad y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

# The Method of Variation of Parameters

$$y = v_1 y_1 + v_2 y_2,$$ with $$v_1' y_1 + v_2' y_2 = 0.$$

$$y' = v_1 y_1' + v_2 y_2',$$
$$y'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'.$$

Substituting in the nonhomogeneous differential equation

$$v_1(a y_1'' + b y_1' + c y_1) + v_2(a y_2'' + b y_2' + c y_2) + a(v_1' y_1' + v_2' y_2') = G(x).$$

$$\Rightarrow \quad a(v_1' y_1' + v_2' y_2') = G(x).$$

$$\Rightarrow \quad \begin{aligned} v_1' y_1 + v_2' y_2 &= 0, \\ v_1' y_1' + v_2' y_2' &= \frac{G(x)}{a} \end{aligned}$$

# The Method of Variation of Parameters

**Variation of Parameters Procedure**

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation

$$ay'' + by' + cy = 0$$

   to find the functions $y_1$ and $y_2$.
2. Solve the equations

$$v_1'y_1 + v_2'y_2 = 0,$$
$$v_1'y_1' + v_2'y_2' = \frac{G(x)}{a}$$

   simultaneously for the derivative functions $v_1'$ and $v_2'$.
3. Integrate $v_1'$ and $v_2'$ to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.
4. Write down the particular solution to nonhomogeneous equation (1) as

$$y_p = v_1y_1 + v_2y_2.$$

**EXAMPLE 6** Find the general solution to the equation

$$y'' + y = \tan x.$$

$$y'' + y = 0 \quad \Rightarrow \quad y_c = c_1 \cos x + c_2 \sin x.$$

$$v_1' \cos x + v_2' \sin x = 0,$$

$$-v_1' \sin x + v_2' \cos x = \tan x.$$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}. \qquad v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x.$$

$$v_1(x) = \int \frac{-\sin^2 x}{\cos x}\, dx = -\ln|\sec x + \tan x| + \sin x, \qquad v_2(x) = \int \sin x\, dx = -\cos x.$$

$$y_p = [-\ln|\sec x + \tan x| + \sin x] \cos x + (-\cos x) \sin x \quad \Rightarrow \quad y = c_1 \cos x + c_2 \sin x - (\cos x) \ln|\sec x + \tan x|.$$

## EXERCISES

$y'' - y' - 6y = e^{-x} - 7\cos x$

$y = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{4}e^{-x} + \frac{7}{50}\sin x + \frac{49}{50}\cos x$

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = e^{3x} - 12x$

$y = c_1 + c_2 e^{3x} + \frac{1}{3}x\, e^{3x} + 2x^2 + \frac{4}{3}x$

$\frac{d^2y}{dx^2} + y = \sec x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

$y'' - y' = 2^x$

$y = c_1 + c_2 e^x + \frac{\ln 2 - 1}{\ln 2(\ln 2 - 1)} 2^x$

$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x \cos x, \quad x > 0$

$y = c_1 + c_2 e^x + \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x$