Capacitors and Inductors

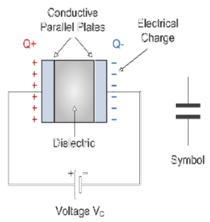
Introduction

Capacitors and inductors are passive elements, each of which has the ability to both store and deliver finite amount of energy. They differ from ideal source in the respect, since they cannot sustain a finite average power flow over an infinite time interval. Although they are classed as linear elements, the current-voltage relationships for these elements are time-dependent, leading to many interesting circuits.

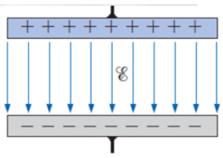
The capacitor

Just like the Resistor, the **Capacitor**, sometimes referred to as a Condenser, is a simple passive device that is used to "store electricity". The capacitor is a component which has the ability or "capacity" to store energy in the form of an electrical charge producing a potential difference (*Static Voltage*) across its plates, much like a small rechargeable battery.

The amount of potential difference present across the capacitor depends upon how much charge was deposited onto the plates by the work being done by the source voltage and also by how much capacitance the capacitor has and this is illustrated below.



Capacitance is the electrical property of a capacitor and is the measure of a capacitors ability to store an electrical charge onto its two plates with the unit of capacitance being the Farad (abbreviated to F) named after the British physicist Michael Faraday. Capacitance is defined as being that a capacitor has the capacitance of One Farad when a charge of One Coulomb is stored on the plates by a voltage of One volt. Capacitance, C is always positive and has no negative units. However, the Farad is a very large unit of measurement to use on its own so sub-multiples of the Farad are generally used such as micro-farads, nano-farads and pico-farads, for example, the capacitance is determined by



$$C = \frac{Q}{V}$$

C = farads (F)
Q = coulombs (C)
V = volts (V)

If a potential difference of V volts is applied across the two plates separated by a distance of d, the electric field strength between the plates is determined by

$$\mathscr{E} = \frac{V}{d}$$ (volts/meter, V/m)

The ratio of the flux density to the electric field intensity in the dielectric is called the **permittivity** of the dielectric:

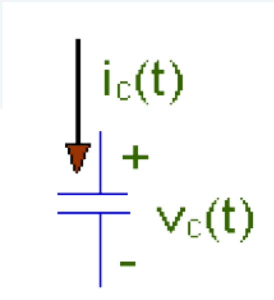
$$\epsilon = \frac{D}{\mathscr{E}}$$ (farads/meter, F/m)

For a vacuum, the value of ϵ (denoted by ϵ_0) is 8.85×10^{-12} F/m. The ratio of the permittivity of any dielectric to that of a vacuum is called the **relative permittivity, ϵ_r**. It simply compares the permittivity of the dielectric to that of air. In equation form,

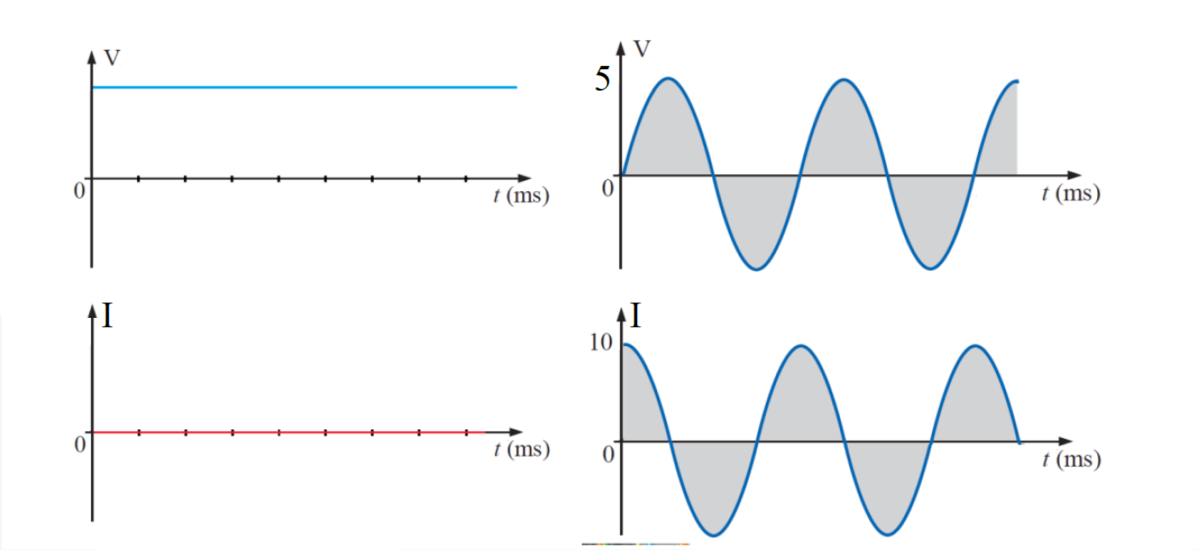
$$\epsilon_r = \frac{\epsilon}{\epsilon_o}$$

The current i_c associated with a capacitance C is related to the voltage across the capacitor by

$$i_c = C\frac{dv_c}{dt}$$



Example:
Determine the current i following through the 2F capacitor for the two waveforms of following figures.



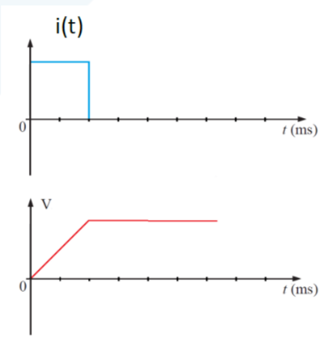
The capacitor voltage may be expressed in terms of the current by integrating i_c. We first obtain

$$dv_c = \frac{1}{C} i(t)dt$$

And then integrate between the times $t_0$and t and between the corresponding voltages $v(t_0)$ and $v(t_0)$ as.

$$v = \frac{1}{C}\int_{t_0}^{t} i(\acute{t})d\acute{t} + v(t_0)$$

Example:
Find the capacitor voltage that is associated with the current show graphically in Figure below. The value of the capacitor is 5µF.



$$v_c(t) = \frac{1}{C}\int_{0}^{t} i d\acute{t} + v(t_0)$$

$v_c(t) = \mathbf{2 \times 10^5 \times \mathit{t}}$ $0 < t < 2$

$v_c(t) = \mathbf{4 \times 10^5}$ $t > 2$

Energy storage

To determine the energy stored in a capacitor, we begin with the power delivered to it.

$$p = vi = Cv\frac{dv}{dt}$$

The change in the energy stored in its electric field is simply

$$\int_{t_0}^{t} p\,d\acute{t} = C\int_{t_0}^{t} v\frac{dv}{dt}d\acute{t} = C\int_{v(t_0)}^{v(t)} \acute{v}\,d\acute{v} = \frac{1}{2}C[[v(t)]^2 - [v(t_0)]^2]$$

And thus

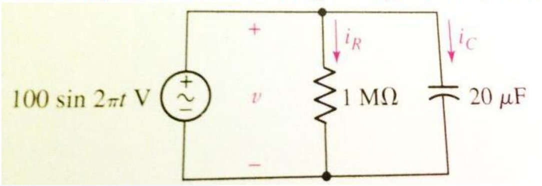
$$w_c(t) - w_c(t_0) = \frac{1}{2}C[[v(t)]^2 - [v(t_0)]^2]$$

Finally

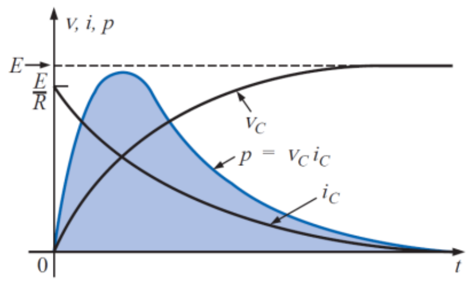
$$w_c(t) = \frac{1}{2}Cv^2$$

Example:

Find the maximum energy stored in the capacitor of following figure and the energy dissipated in the resistor over the interval $0 < t < 0.5$



Solution:



Solution:

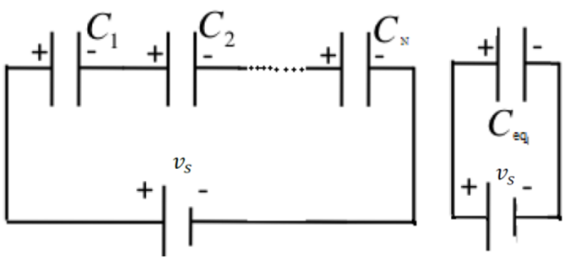
$w_c(t) = \frac{1}{2}Cv^2 = 0.1\sin^2 2\pi t \quad J$

$P_R = \frac{V^2}{R} = \frac{10^2\sin^2 2\pi t}{10^6} \quad \text{W}$

$w_R = \int_0^{0.5} P_R\,dt = \int_0^{0.5}\frac{10^2\sin^2 2\pi t}{10^6}\,dt \quad \text{J}$

Capacitors in Series

For the following circuit the source voltage can be written as

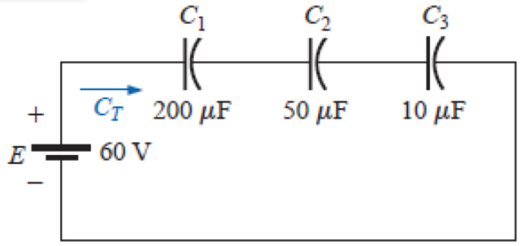


$$\frac{1}{C_{eq}} = \sum_{n=1}^{N}\frac{1}{C_n} = \frac{1}{C_1} + \frac{1}{C_2} \dots \dots + \frac{1}{C_N}$$

Example:

For the circuit shown below

1- Find the total capacitance.
2- Determine the charge on each plate.
3- Find the voltage across each capacitor.



Solution

1- $\frac{1}{C_T} = \sum_{n=1}^{3}\frac{1}{C_n} = \frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3} = \frac{1}{200\times10^{-6}}+\frac{1}{50\times10^{-6}}+\frac{1}{10\times10^{-6}} = (0.005+0.02+0.1)\times10^6 = 0.125\times10^6$

$$C_T = \frac{1}{0.125\times10^6} = 8\,\mu F$$

2- $Q_T = Q_1 = Q_2 = Q_3 = C_T E = 8\times10^{-6}\times60 = 480\mu C$

3- $v_1 = \frac{Q_1}{C_1} = \frac{480\times10^{-6}}{200\times10^{-6}} = 2.4\ V$

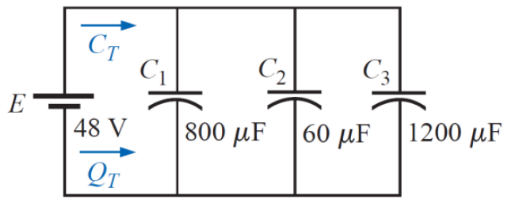
$v_2 = \frac{Q_2}{C_2} = \frac{480\times10^{-6}}{50\times10^{-6}} = 9.6\ V$

$v_3 = \frac{Q_3}{C_3} = \frac{480\times10^{-6}}{10\times10^{-6}} = 48\ V$

Example:

For the network of Figure shown below

a. Find the total capacitance.
b. Determine the charge on each plate.
c. Find the total charge.



Solution

a- $C_T = C_1 + C_2 + C_3 = 800\times10^{-6} + 60\times10^{-6} + 1200\times10^{-6} = 2060\,\mu F$

b- $Q_1 = C_1E = 800\times10^{-6}\times48 = 38.4\,mC$

$Q_2 = C_2E = 60\times10^{-6}\times48 = 2.88\,mC$

$Q_3 = C_3E = 1200\times10^{-6}\times48 = 57.6\,mC$

c- $Q_T = Q_1 + Q_2 + Q_3 = 38.4\times10^{-3} + 2.88\times10^{-3} + 57.6\times10^{-3} = 98.88\,mC$

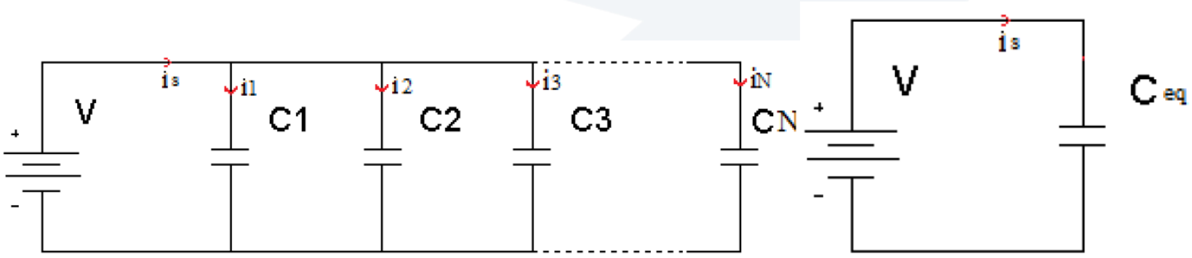
$$C_{eq} = \sum_{n=1}^{N} C_n = C_1 + C_2 + \cdots + C_N$$

Capacitors in parallel

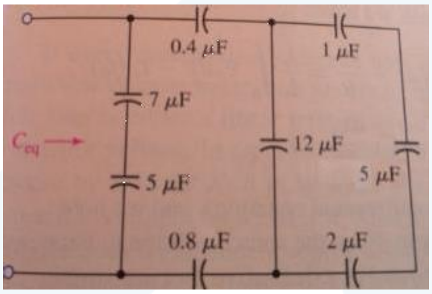
The circuits of following figure enable us to establish the value of capacitor which is equivalent to parallel capacitors are

$$i_s = \sum_{n=1}^{N} i_n = \sum_{n=1}^{N} C_n\frac{dv}{dt} = C_{eq}\frac{dv}{dt}$$

$$C_{eq}=\sum_{n=1}^{N} C_n = C_1 + C_2 + \cdots + C_N$$



Example:
Find C_{eq} for the following network



$$C_1 = \frac{1\times5\times2}{1\times5+1\times2+2\times5} = \frac{10}{17}\mu F$$

$$C_2 = \frac{10}{17} + 12 = 12.588\mu F$$

$$C_3 = \frac{12.588\times0.4\times0.8}{12.588\times0.4+0.4\times0.8+12.588\times0.8} = 0.261\mu F$$

$$C_4 = \frac{7\times5}{7+5} = \frac{35}{12}\mu F$$

$$C_{eq} = C_3 + C_4 = 3.18\mu F$$

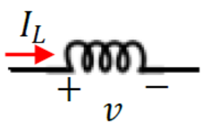
Important Characteristics of an Ideal Capacitor

1- There is no current through a capacitor if the voltage across it is not change with time. A capacitor is an open circuit to d.c.
2- A finite amount of energy can be stored in the capacitor even if the current trough the capacitor is zero, such as when the voltage across it is constant.
3- It is impossible to change the voltage across the capacitor by a finite amount in zero time, as this requires an infinite current through the capacitor.
4- A capacitor never dissipate energy, but only stores it. Although this is true for the mathematical model, it is not true for a physical capacitor due to finite resistor associated with the dielectric as well as the packaging.

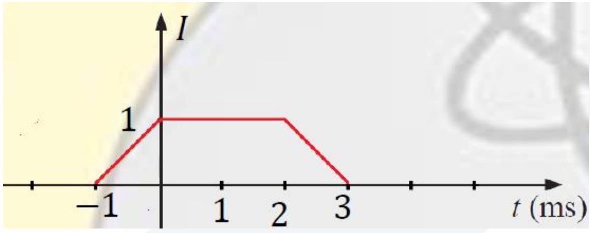
The Inductor

In the early 1800s the Danish science Oersted showed that a current carrying conductor produced a magnetic f Shortly thereafter, Ampers made some careful measurements which demonstrated that this magnetic field was line related to the current which produced it. The English experimental Michael faraday and the American inventor Jo Henry discovered that a changing magnetic field could induce a voltage in a neighboring circuit. They showed this voltage was proportional to the time rate of change of the current producing the magnetic field. Mathemati can be expressed as

$$v = L\frac{di}{dt}$$



Given the waveform of the current in a 3H inductor as shown in figure below, determine the inductor voltage and sketch it.

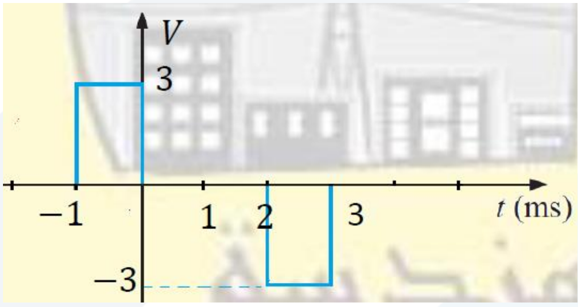


$$V = L\frac{di}{dt}$$

$$V = 3 \times 1 = 3 \quad -1 < t < 0$$

$$V = 0 \quad 0 < t < 2$$

$$V = 3 \times (-1) = -3 \quad 2 < t < 3$$



To calculate the inductor current, rewrite the voltage expression as

$$di = \frac{1}{L} v dt$$

$$\int_{i(t_0)}^{i(t)} d\acute{i} = \frac{1}{L}\int_{t_0}^{t} v d\acute{t}$$

$$i(t) - i(t_0) = \frac{1}{L}\int_{t_0}^{t} v d\acute{t}$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v d\acute{t} + i(t_0)$$

Example:

The voltage across a 2H inductor is known to be $6cos5t\ V$. Determine the resulting inductor current if $i\left(t = -\frac{\pi}{2}\right) = 1A$.

Example:

A 100 mH inductor has voltage $v_L = 2e^{-3t}V$ across its terminals. Determine the resulting inductor current if $i_L(-0.5) = 1A$.

The observed power is given by the current-voltage product

$$p = vi = Li\frac{di}{dt}$$

The energy w_L accepted by the inductor is stored in the magnetic field around the coil. The change in tis energy is expressed by the integral of the power over the desired time interval:

$$\int_{t_0}^{t} p d\acute{t} = L\int_{t_0}^{t} i\frac{di}{d\acute{t}}d\acute{t} = L\int_{i(t_0)}^{i(t)} \acute{i}d\acute{i} = \frac{1}{2}L[[i(t)]^2 - [i(t_0)]^2]$$

Thus

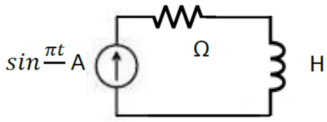
$$w_L(t) - w_L(t_0) = \frac{1}{2}L[[i(t)]^2 - [i(t_0)]^2]$$

When $t_0 = -\infty$ and $i(t_0) = 0$, so that the energy can be expressed as

$$w_L(t) = \frac{1}{2}LI^2$$

Example:

Find the maximum energy stored in the inductor of following figure and calculate how much energy is dissipated in the resistor in the time during which the energy is being stored in, and then recovered from, the inductor.



Solution

The energy stored in the inductor is

$w_L(t) = \frac{1}{2}Li^2 = 216sin^2\frac{\pi t}{6}$ J

At $t = 0$, $w_L = 0$

At $t = 3\ s$, $w_L = 216$ J

The power dissipated in the resistor is

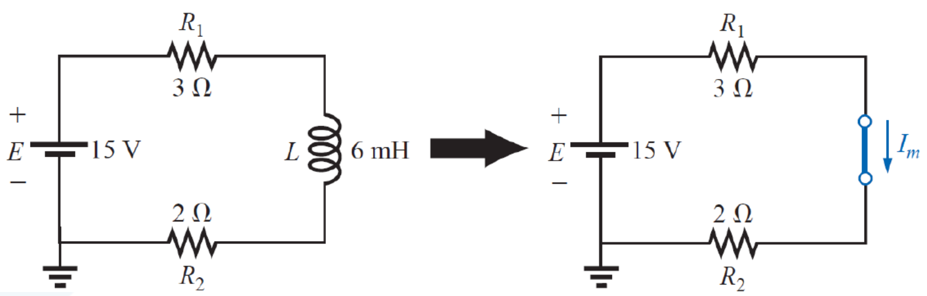
$p_R = i^2R = 14.4sin^2\frac{\pi t}{6}$ w

The energy converted into heat in the resistor within 6 s interval is

$w_R = \int_0^6 14.4sin^2\frac{\pi t}{6}dt = \int_0^6 \frac{14.4}{2}\left(1 - cos\frac{\pi}{3}t\right)dt = 43.2\ J$

Example:

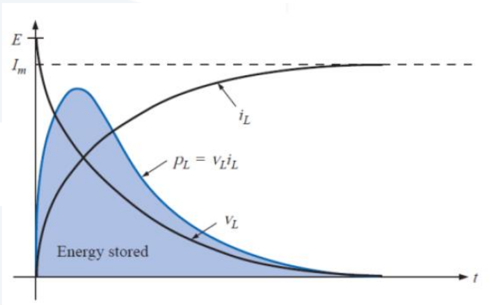
Find the energy stored by the inductor in the circuit shown below when the current through it has reached its final value.



Solution

$$I_m = \frac{E}{R_1 + R_2} = \frac{15}{3+2} = 3\ A$$

$$w_L(t) = \frac{1}{2} L i^2 = \frac{1}{2} \times 6 \times 10^{-3} \times 3^2 = 27\ mJ$$



Inductors in series

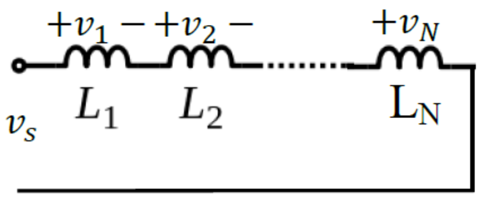
$$v_s = v_1 + v_2 + \cdots .. + v_N$$

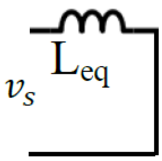
$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots .. + L_N \frac{di}{dt}$$

$$= (L_1 + L_2 + \cdots .. + L_N) \frac{di}{dt}$$

$$v_s = L_{eq} \frac{di}{dt}$$

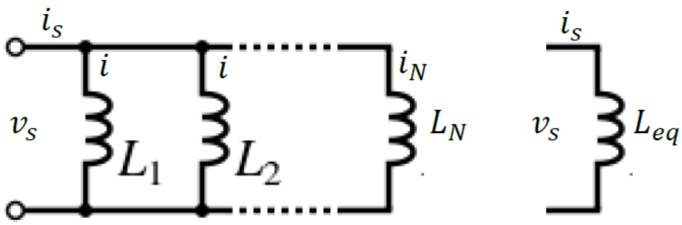
$$L_{eq} = L_1 + L_2 + \cdots .. + L_N$$





Inductors in Parallel

The combination of a number of parallel inductors is accomplished by writing the single nodal equation for the original circuit

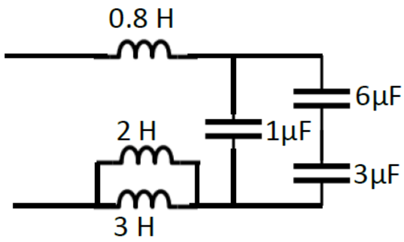


$$\frac{1}{L_{eq}} = \sum_{n=1}^{N} \frac{1}{L_n} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

Example:
Simplify the network using series-parallel combination.

$$C_{eq} = \frac{6 \times 3}{9} + 1 = 3\mu F$$

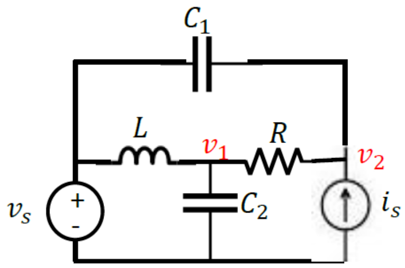
$$L_{eq} = \frac{2 \times 3}{5} + 0.8 = 2H$$



Important Characteristics of an Ideal Capacitor

1- There is no voltage across an inductor if the current through it is not changing wit time. An inductor is therefore a short circuit to dc.
2- A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero, such as when the current through it is constant.
3- It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor.
4- The inductor never dissipates energy, but only stories it. Although this is true for the mathematical model, it is not true for a physical inductor due to series resistance.

Example:
Write appropriate nodal equations for the following figure.



At node 1

$$\frac{1}{L}\int_{t_0}^{t}(v_1 - v_s)d\acute{t} + i_L(t_0) + \frac{v_1 - v_2}{R} + C_2\frac{dv_1}{dt} = 0$$

$$\frac{v_1}{R} + C_2\frac{dv_1}{dt}\frac{1}{L}\int_{t_0}^{t} v_1 d\acute{t} - \frac{v_2}{R} = \frac{1}{L}\int_{t_0}^{t} v_s d\acute{t} - i_L(t_0)$$

At node 2

$$C_1\frac{d(v_2 - v_s)}{dt} + \frac{v_2 - v_1}{R} + i_s = 0$$

$$-\frac{v_1}{R} + \frac{v_2}{R} + C_1\frac{dv_2}{dt} = C_1\frac{dv_s}{dt} + i_s$$