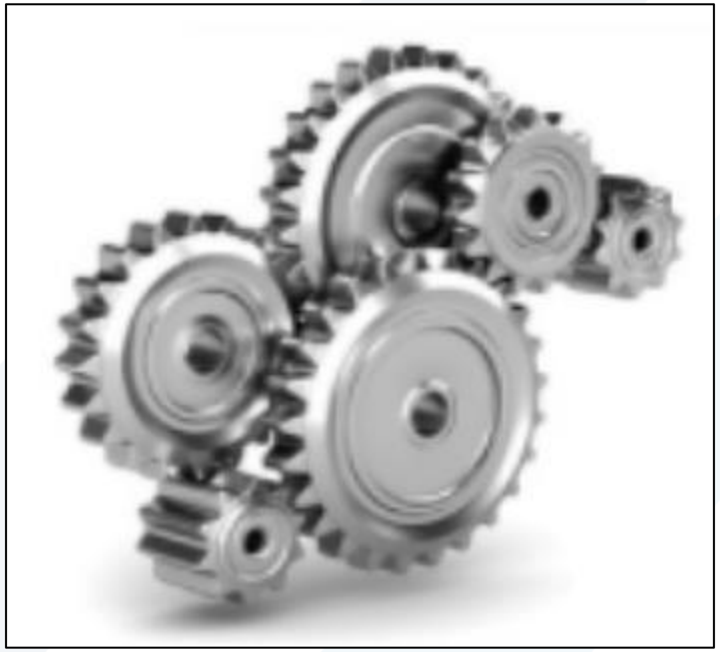
جامعة المنارة

كلية الهندسة

قسم الروبوت و الأنظمة الذكية

مقرر ديناميك الآلات و الاهتزازات الميكانيكية

Mechanical Vibrations



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Undamped Free Vibration

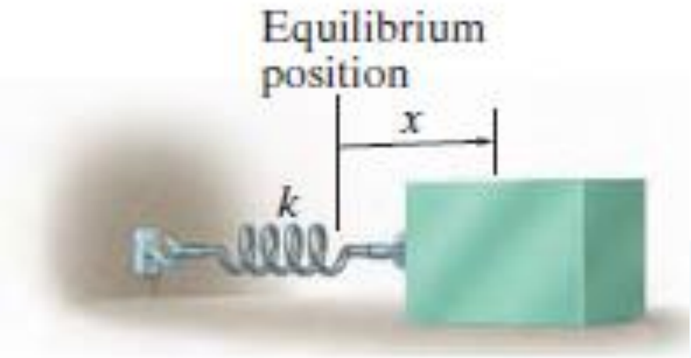
A *vibration* is the oscillating motion of a body or system of connected bodies displaced from a position of equilibrium. In general, there are two types of vibration, free and forced. *Free vibration* occurs when the motion is maintained by gravitational or elastic restoring forces, such as the swinging motion of a pendulum or the vibration of an elastic rod. *Forced vibration* is caused by an external periodic or intermittent force applied to the system. Both of these types of vibration can either be damped or undamped. *Undamped* vibrations exclude frictional effects in the analysis. Since in reality both internal and external frictional forces are present, the motion of all vibrating bodies is actually *damped*.

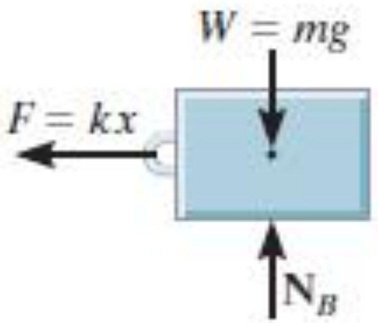
$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad -kx = m\ddot{x}$$

$$\ddot{x} + \omega_n^2 x = 0$$

The constant ω_n, generally reported in rad/s, is called the *natural frequency*, and in this case

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$



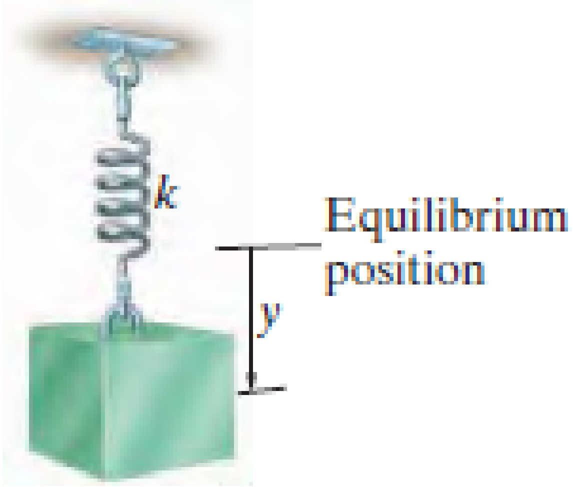


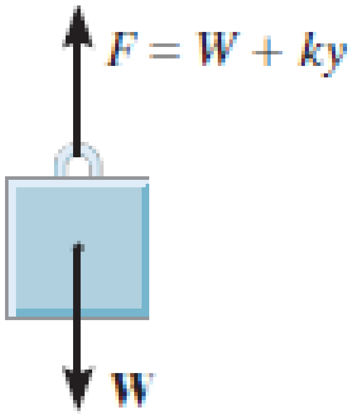
When the block is in equilibrium, the spring exerts an upward force of $F = W = mg$ on the block. Hence, when the block is displaced a distance y downward from this position, the magnitude of the spring force is $F = W + ky$,

$$+\downarrow \Sigma F_y = ma_y; \qquad -W - ky + W = m\ddot{y}$$

or

$$\ddot{y} + \omega_n^2 y = 0$$





$$\boxed{x = A \sin \omega_n t + B \cos \omega_n t}$$

Here A and B represent two constants of integration. The block's velocity and acceleration are determined by taking successive time derivatives, which yields

$$v = \dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t$$

$$a = \ddot{x} = -A\omega_n^2 \sin \omega_n t - B\omega_n^2 \cos \omega_n t$$

Substituting $x = x_1$ when $t = 0$

$B = x_1$

And since $v = v_1$ when $t = 0$,

$A = v_1/\omega_n$.

$$x = \frac{v_1}{\omega_n} \sin \omega_n t + x_1 \cos \omega_n t$$

Consider a motion represented by

$$x(t) = A\cos\omega t + B\sin\omega t$$

Such a motion is referred to as simple harmonic motion. Use of the trigonometric identity

$$\sin(\omega t + \phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi$$

gives

$$x(t) = X\sin(\omega t + \phi)$$

where

$$X = \sqrt{A^2 + B^2}$$

and

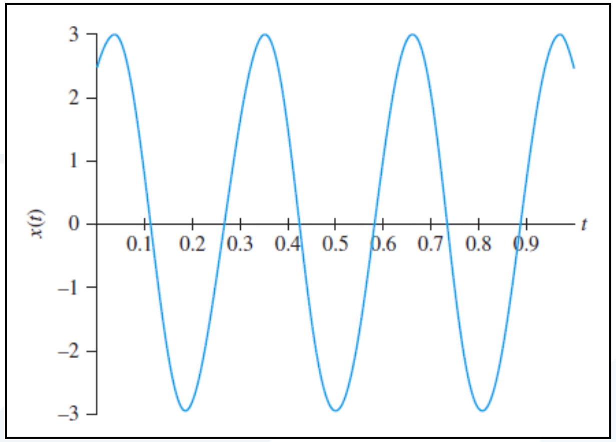
$$\phi = \tan^{-1}\left(\frac{A}{B}\right)$$

The amplitude, X, is the maximum displacement from equilibrium. The response is cyclic. The period is the time required to execute one cycle, is determined by

$$T = \frac{2\pi}{\omega}$$

and is usually measured in seconds (s). The reciprocal of the period is the number of cycles executed in one second and is called the frequency

$$f = \frac{\omega}{2\pi}$$



Thus, ω is the circular frequency measured in rad/s. The frequency also may be expressed in term of revolutions per minute (rpm) by noting that one revolution is the same as one cycle and there are 60 s in one minute,

$$\omega \text{ rpm} = (\omega \text{ rad/s})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right)$$

ϕ is the phase angle

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EXAMPLE

The response of a system is given by

$$x(t) = 0.003 \cos(30t) + 0.004 \sin(30t) \text{ m}$$

Determine (a) the amplitude of motion, (b) the period of motion, (c) the frequency in Hz, (d) the frequency in rpm, (e) the phase angle, and (f) the response in the form of Equation $x(t) = X\sin(\omega t + \phi)$

Solution

(a) The amplitude is given by

$$X = \sqrt{0.003^2 + 0.004^2} \text{ m} = 0.005 \text{ m}$$

(b) The period of motion is

$$T = \frac{2\pi}{30} \text{ s} = 0.209 \text{ s}$$

(c) The frequency in hertz is

$$f = \frac{1}{T} = \frac{1}{0.209 \text{ s}} = 4.77 \text{ Hz}$$

(d) The frequency in revolutions per minute is

$$\omega = \left(30\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 286.48\ \text{rpm}$$

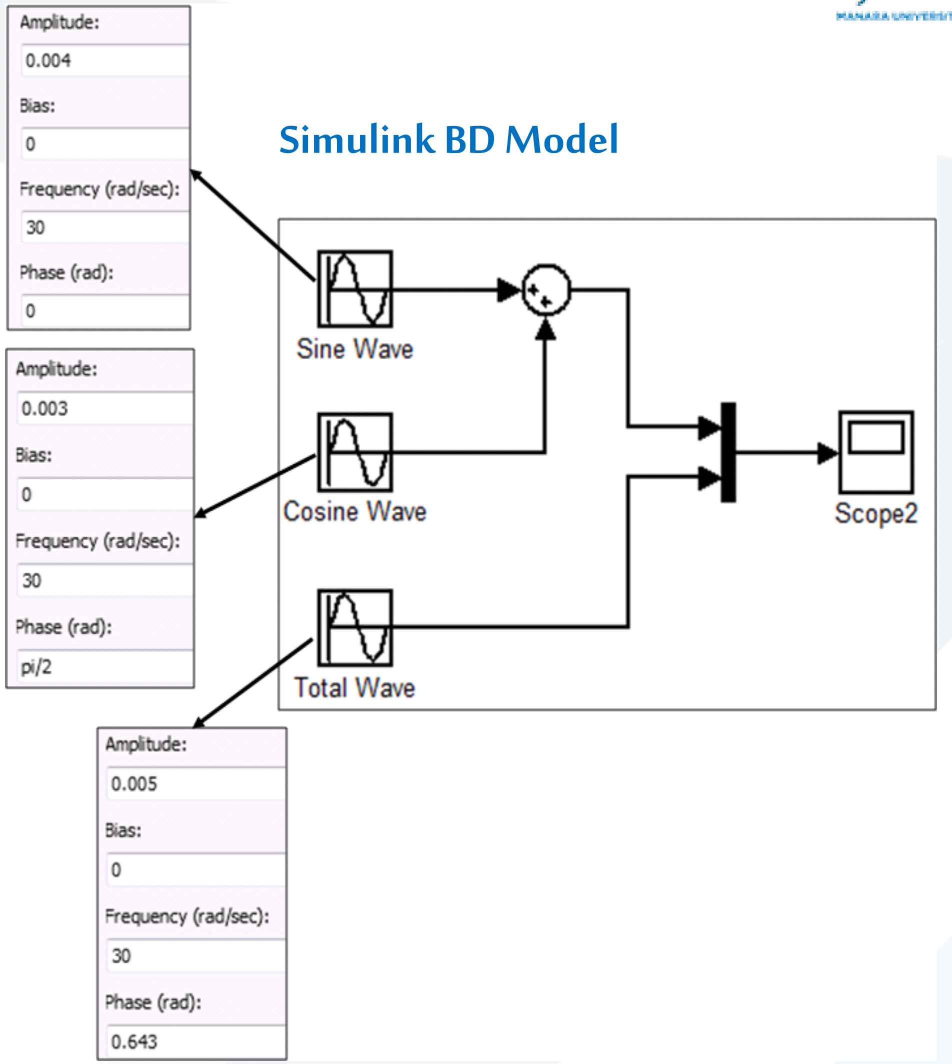
(e) The phase angle is

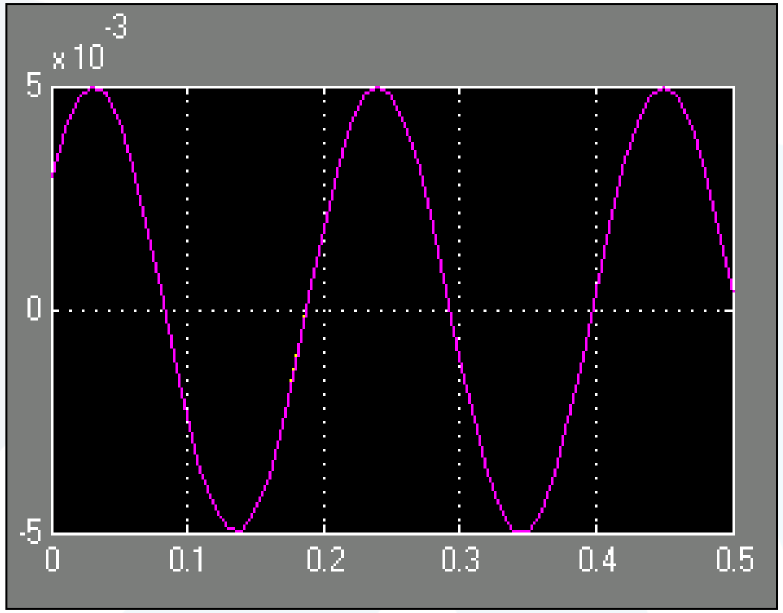
$$\phi = \tan^{-1}\left(\frac{0.003}{0.004}\right) = 0.643\ \text{rad}$$

(f) Written in the form of Equation, the response is

$$x(t) = 0.005\sin(30t + 0.643)\ \text{m}$$

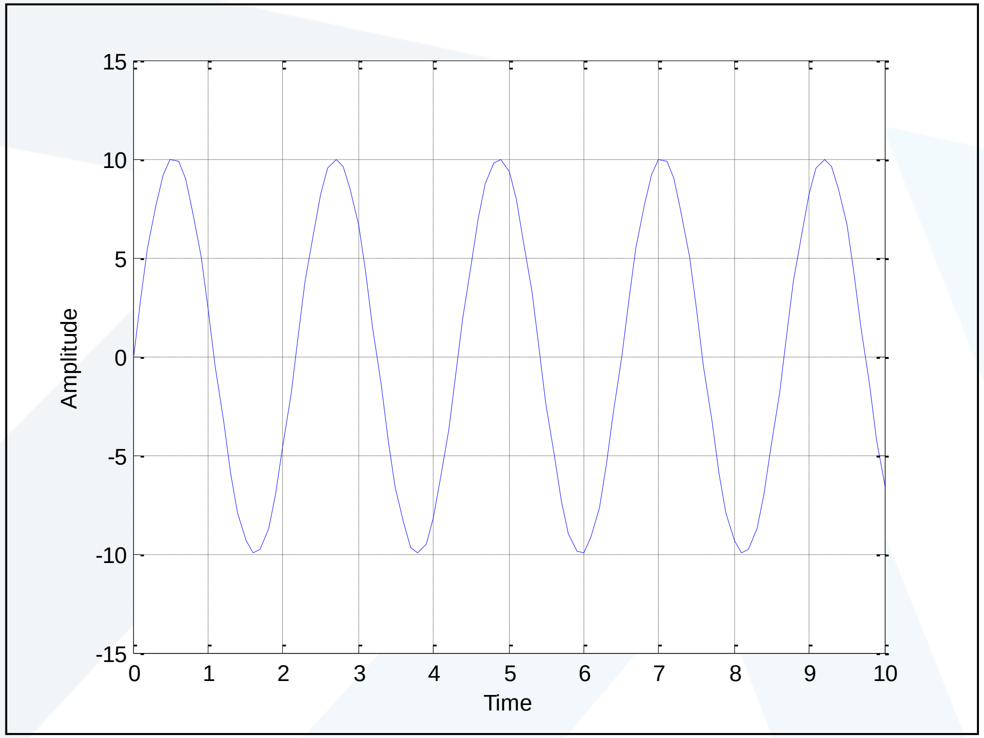
Simulink BD Model



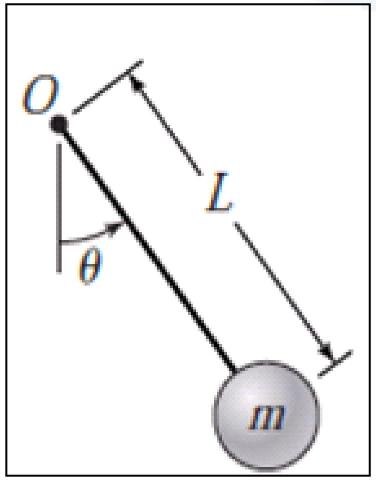


Analysis with Matlab

```
wmax=3;
w=1;
X=10;
phy=0;
b=0;
t=0:0.1:10;
while w < wmax
     x=X*sin(w*t+phy)+b;
     plot(t,x)
     xlabel('Time')
     ylabel('Amplitude')
     grid
     axis([0,10,-15,15]);
     w = w + 0.1;
     drawnow
end
```



EXAMPLE



$$\sin\theta \approx \theta$$

$$L\ddot{\theta} + g\theta = 0$$

$$LD^2\theta + g\theta = 0$$

$$(L\lambda^2 + g)e^{\lambda t} = 0$$

$$\lambda^2 = -\frac{g}{L} \quad \lambda_{1,2} = \pm i\sqrt{\frac{g}{L}}$$

$$\theta = e^0 (A\cos\sqrt{\frac{g}{L}}t + B\sin\sqrt{\frac{g}{L}}t)$$

$$\theta(t=0) = \theta_0 \qquad \theta_0 = A$$

$$\theta' = -A\sqrt{\frac{g}{L}}\sin\sqrt{\frac{g}{L}}t + B\sqrt{\frac{g}{L}}\cos\sqrt{\frac{g}{L}}t$$

$$\theta'(t=0) = \theta'_0$$

$$\theta'_0 = B\sqrt{\frac{g}{L}} \quad B = \frac{\theta'_0}{\sqrt{\frac{g}{L}}}$$

$$\theta = \theta_0 \cos\sqrt{\frac{g}{L}}t + \frac{\theta'_0}{\sqrt{\frac{g}{L}}}\sin\sqrt{\frac{g}{L}}t)$$

$$\theta = A\sin(wt + \varphi) + b$$

$$w = \sqrt{\frac{g}{L}}$$

$$A = \sqrt{\theta_0{}^2 + (\frac{\theta'_0}{\sqrt{\frac{g}{L}}})^2}$$

$$\varphi = \arctan(\frac{\theta_0}{\frac{\theta'_0}{\sqrt{\frac{g}{L}}}})$$

$$\theta = \sqrt{\theta_0{}^2 + (\frac{\theta'_0}{\sqrt{\frac{g}{L}}})^2}\,.\sin(\sqrt{\frac{g}{L}}t + \arctan(\frac{\theta_0}{\frac{\theta'_0}{\sqrt{\frac{g}{L}}}}))$$

Procedure for Analysis

As in the case of the block and spring, the natural frequency ω_n of a body or system of connected bodies having a single degree of freedom can be determined using the following procedure:

Free-Body Diagram.

- Draw the free-body diagram of the body when the body is displaced a *small amount* from its equilibrium position.
- Locate the body with respect to its equilibrium position by using an appropriate *inertial coordinate* q. The acceleration of the body's mass center $\mathbf{a}_G$ or the body's angular acceleration $\boldsymbol{\alpha}$ should have an assumed sense of direction which is in the *positive direction* of the position coordinate.
- If the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then it may be beneficial to also draw the kinetic diagram since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$, and thereby makes it convenient for visualizing the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

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Equation of Motion.

- Apply the equation of motion to relate the elastic or gravitational *restoring* forces and couple moments acting on the body to the body's accelerated motion.

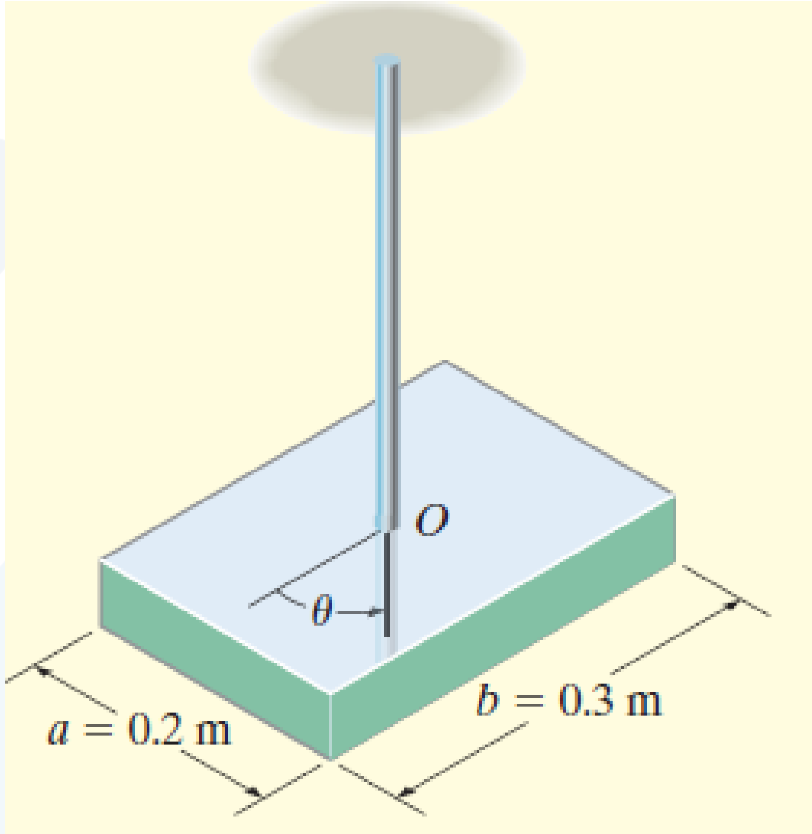
Kinematics.

- Using kinematics, express the body's accelerated motion in terms of the second time derivative of the position coordinate, $\ddot{q}$.
- Substitute the result into the equation of motion and determine ω_n by rearranging the terms so that the resulting equation is in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE

The 10-kg rectangular plate shown is suspended at its center from a rod having a torsional stiffness $k = 1.5$ N.m/rad.

Determine the natural period of vibration of the plate when it is given a small angular displacement θ in the plane of the plate.



SOLUTION

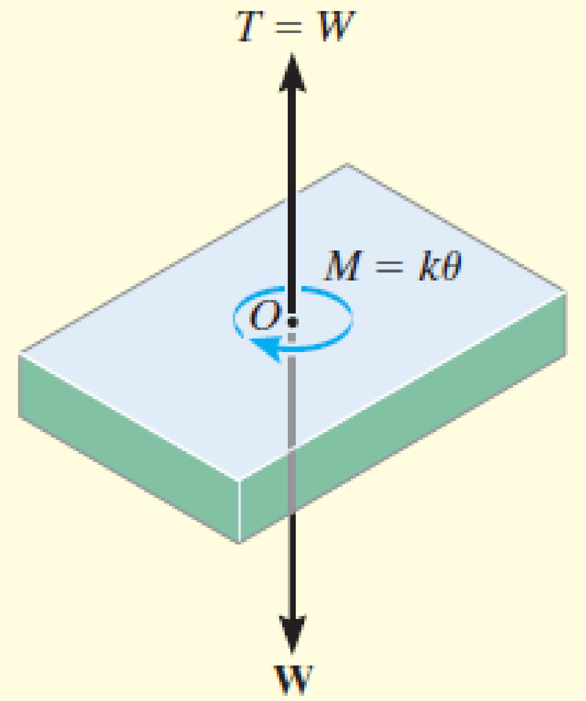
Free-Body Diagram. Since the plate is displaced in its own plane, the torsional *restoring* moment created by the rod is $M = k\theta$.
This moment acts in the direction opposite to the angular displacement θ.
The angular acceleration $\ddot{\theta}$ acts in the direction of *positive* θ.

Equation of Motion.

$$\Sigma M_O = I_O\alpha; \qquad -k\theta = I_O\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{k}{I_O}\theta = 0$$



Since this equation is in the "standard form," the natural frequency is $\omega_n = \sqrt{k/I_O}$.

$$I_O = \tfrac{1}{12}m(a^2 + b^2).$$

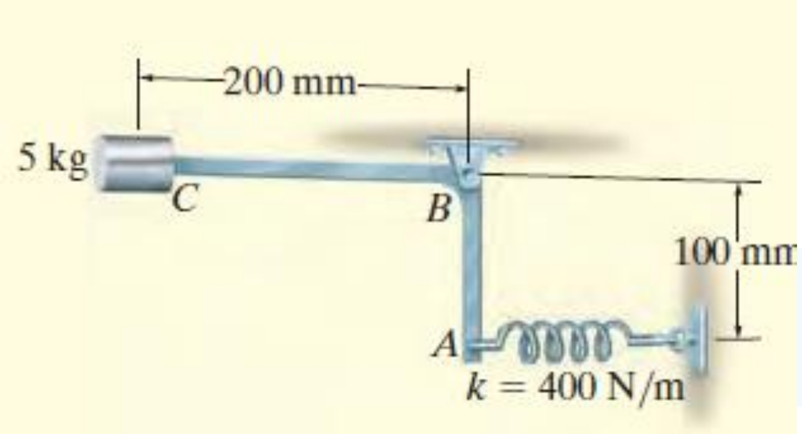
$$I_O = \frac{1}{12}(10\text{ kg})\left[(0.2\text{ m})^2 + (0.3\text{ m})^2\right] = 0.1083\text{ kg}\cdot\text{m}^2$$

The natural period of vibration is therefore,

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_O}{k}} = 2\pi\sqrt{\frac{0.1083}{1.5}} = 1.69\text{ s} \qquad \textit{Ans.}$$

EXAMPLE

The bent rod shown has a negligible mass and supports a 5-kg collar at its end. If the rod is in the equilibrium position shown, determine the natural period of vibration for the system.



SOLUTION

Free-Body and Kinetic Diagrams. Here the rod is displaced by a small angle θ from the equilibrium position. Since the spring is subjected to an initial compression of x_{st} for equilibrium, then the spring exerts a force of $F_s = kx - kx_{st}$ on the rod.
$\mathbf{a}_y$ must act *upward*, which is in accordance with positive θ

Equation of Motion. Moments will be summed about point B to eliminate the unknown reaction at this point. Since θ is small,

$$\circlearrowleft + \Sigma M_B = \Sigma(\mathcal{M}_k)_B;$$

$$kx(0.1\text{ m}) - kx_{st}(0.1\text{ m}) + 49.05\text{ N}(0.2\text{ m}) = -(5\text{ kg})a_y(0.2\text{ m})$$

The second term on the left side, $-kx_{st}(0.1 \text{ m})$, represents the moment created by the spring force which is necessary to hold the collar in *equilibrium*, i.e., at $x = 0$. Since this moment is equal and opposite to the moment 49.05 N(0.2 m) created by the weight of the collar, these two terms cancel in the above equation, so that

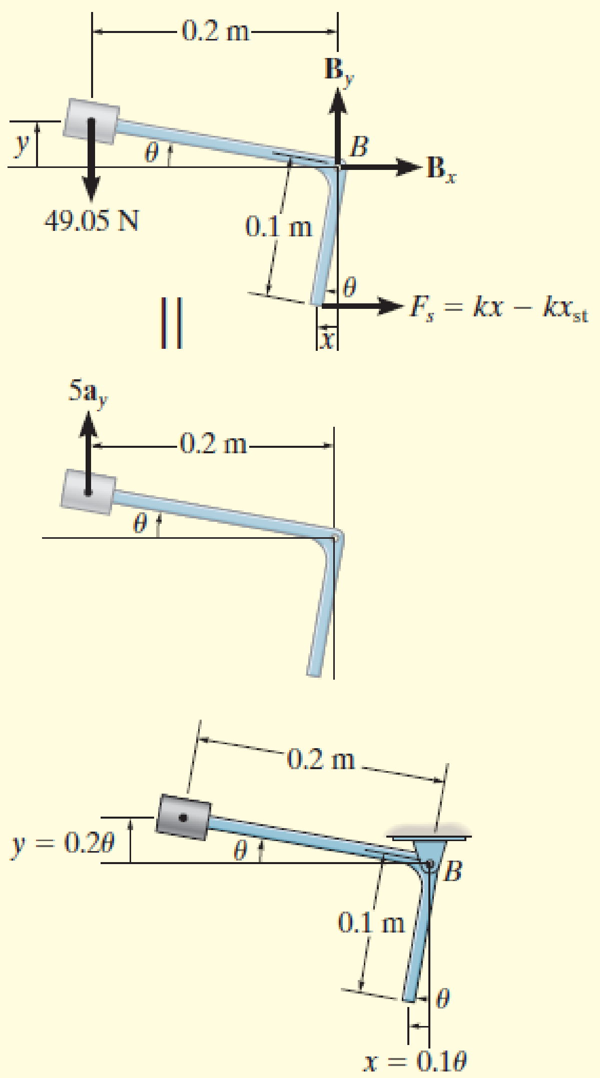
$$kx(0.1) = -5a_y(0.2) \qquad (1)$$

Kinematics. The deformation of the spring and the position of the collar can be related to the angle θ. Since θ is small, $x = (0.1 \text{ m})\theta$ and $y = (0.2 \text{ m})\theta$. Therefore, $a_y = \ddot{y} = 0.2\ddot{\theta}$. Substituting into Eq. 1 yields

$$400(0.1\theta)\,0.1 = -5(0.2\ddot{\theta})0.2$$

Rewriting this equation in the "standard form" gives

$$\ddot{\theta} + 20\theta = 0$$



Compared with $\ddot{x} + \omega_n^2 x = 0$, we have

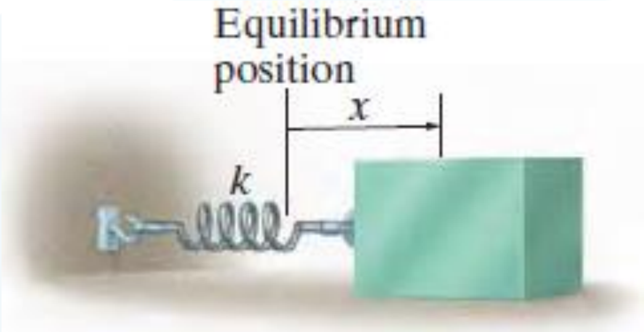
$$\omega_n^2 = 20 \quad \omega_n = 4.47 \text{ rad/s}$$

The natural period of vibration is therefore

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.47} = 1.40 \text{ s} \qquad \textit{Ans.}$$

Energy Methods

The simple harmonic motion of a body, discussed in the previous section, is due only to gravitational and elastic restoring forces acting on the body. Since these forces are *conservative*, it is also possible to use the conservation of energy equation to obtain the body's natural frequency or period of vibration. To show how to do this, consider again the block and spring model . When the block is displaced x from the equilibrium position, the kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$ and the potential energy is $V = \frac{1}{2}kx^2$. Since energy is conserved, it is necessary that



$$T + V = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant}$$

The differential equation describing the *accelerated motion* of the block can be obtained by *differentiating* this equation with respect to time; i.e.,

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\dot{x}(m\ddot{x} + kx) = 0 \qquad \ddot{x} + \omega_n^2 x = 0 \quad \omega_n = \sqrt{k/m}$$

Procedure for Analysis

The natural frequency ω_n of a body or system of connected bodies can be determined by applying the conservation of energy equation using the following procedure.

Energy Equation.

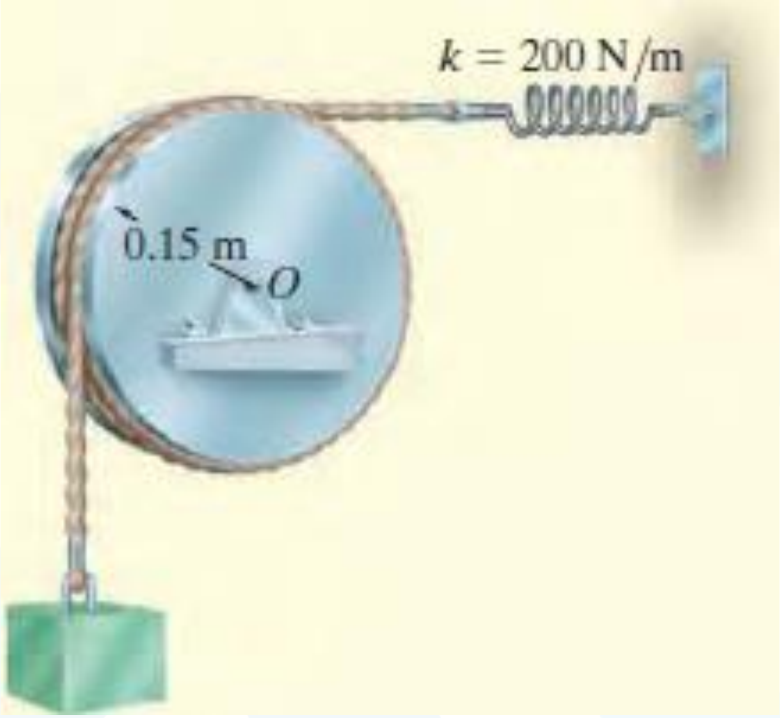
- Draw the body when it is displaced by a *small amount* from its equilibrium position and define the location of the body from its equilibrium position by an appropriate position coordinate q.
- Formulate the conservation of energy for the body, $T + V =$ constant, in terms of the position coordinate.
- In general, the kinetic energy must account for both the body's translational and rotational motion, $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$
- The potential energy is the sum of the gravitational and elastic potential energies of the body, $V = V_g + V_e$. In particular, V_g should be measured from a datum for which $q = 0$ (equilibrium position).

Time Derivative.

- Take the time derivative of the energy equation using the chain rule of calculus and factor out the common terms. The resulting differential equation represents the equation of motion for the system. The natural frequency of ω_n is obtained after rearranging the terms in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE

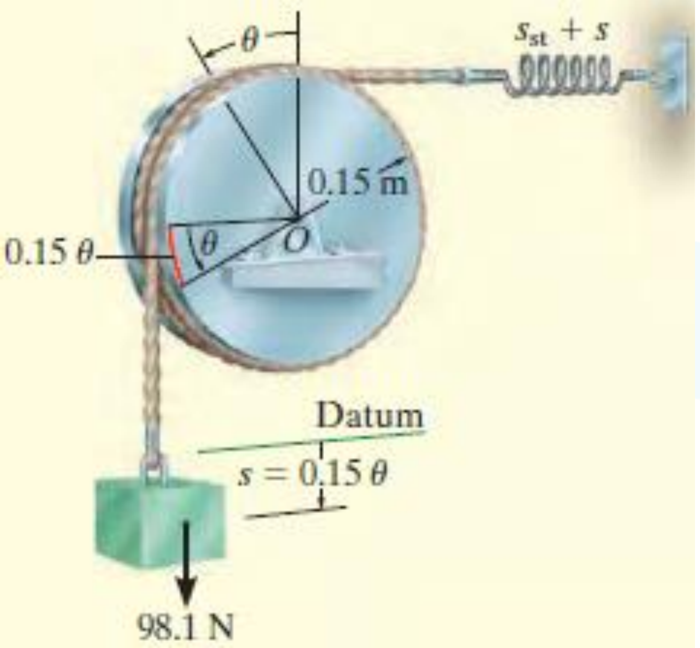
A 10-kg block is suspended from a cord wrapped around a 5-kg disk, as shown. If the spring has a stiffness $k = 200$ N/m, determine the natural period of vibration for the system.



SOLUTION

Energy Equation. A diagram of the block and disk when they are displaced by respective amounts s and θ from the equilibrium position is shown. Since $s = (0.15\text{ m})\theta$, then $v_b \approx \dot{s} = (0.15\text{ m})\dot{\theta}$.
Thus, the kinetic energy of the system is

$$\begin{aligned} T &= \tfrac{1}{2}m_b v_b^2 + \tfrac{1}{2}I_O\omega_d^2 \\ &= \tfrac{1}{2}(10\text{ kg})[(0.15\text{ m})\dot{\theta}]^2 + \tfrac{1}{2}\left[\tfrac{1}{2}(5\text{ kg})(0.15\text{ m})^2\right](\dot{\theta})^2 \\ &= 0.1406(\dot{\theta})^2 \end{aligned}$$



Establishing the datum at the equilibrium position of the block and realizing that the spring stretches s_{st} for equilibrium, the potential energy is

$$V = \tfrac{1}{2}k(s_{st} + s)^2 - Ws$$
$$= \tfrac{1}{2}(200\ \text{N/m})[s_{st} + (0.15\ \text{m})\theta]^2 - 98.1\ \text{N}[(0.15\ \text{m})\theta]$$

The total energy for the system is therefore,

$$T + V = 0.1406(\dot{\theta})^2 + 100(s_{st} + 0.15\theta)^2 - 14.715\theta$$

Time Derivative.

$$0.28125(\dot{\theta})\ddot{\theta} + 200(s_{st} + 0.15\theta)0.15\dot{\theta} - 14.72\dot{\theta} = 0$$

Since $s_{st} = 98.1/200 = 0.4905$ m, the above equation reduces to the "standard form"

$$\ddot{\theta} + 16\theta = 0$$

so that

$$\omega_n = \sqrt{16} = 4 \text{ rad/s}$$

Thus,

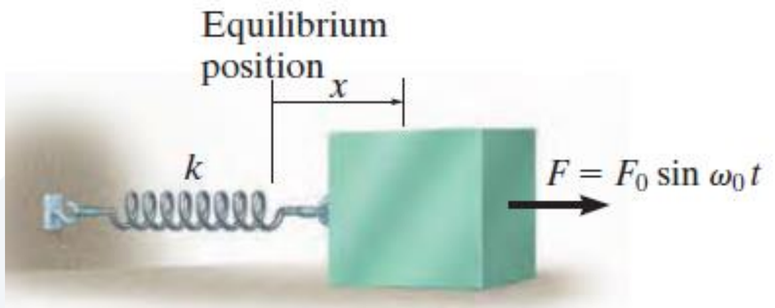
$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4} = 1.57 \text{ s} \qquad \textit{Ans.}$$

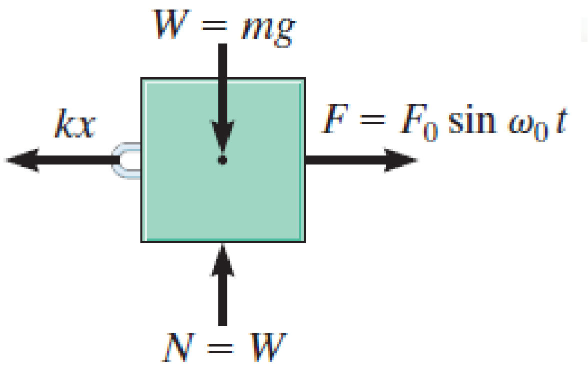
Undamped Forced Vibration

Undamped forced vibration is considered to be one of the most important types of vibrating motion in engineering. Its principles can be used to describe the motion of many types of machines and structures.

Periodic Force. The block and spring shown provide a convenient model which represents the vibrational characteristics of a system subjected to a periodic force $F = F_0 \sin \omega_0 t$. This force has an amplitude of F_0 and a *forcing frequency* ω_0. The free-body diagram for the block when it is displaced a distance x is shown.

Applying the equation of motion, we have





$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad F_0 \sin \omega_0 t - kx = m\ddot{x}$$

or

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega_0 t$$

This equation is a nonhomogeneous second-order differential equation. The general solution consists of a complementary solution, x_c, *plus* a particular solution, x_p.

The *complementary solution* is determined by setting the term on the right side of equal to zero and solving the resulting homogeneous equation.

$$x_c = C \sin(\omega_n t + \phi)$$

where ω_n is the natural frequency, $\omega_n = \sqrt{k/m}$

Since the motion is periodic, the *particular solution* can be determined by assuming a solution of the form

$$x_p = X \sin \omega_0 t$$

where X is a constant. Taking the second time derivative and substituting yields

$$-X\omega_0^2 \sin \omega_0 t + \frac{k}{m}(X \sin \omega_0 t) = \frac{F_0}{m} \sin \omega_0 t$$

Factoring out $\sin \omega_0 t$ and solving for X gives

$$X = \frac{F_0/m}{(k/m) - \omega_0^2} = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2}$$

$$\boxed{x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t}$$

The *general solution* is therefore the sum of two sine functions having different frequencies.

$$x = x_c + x_p = C \sin(\omega_n t + \phi) + \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t$$

The *complementary solution* x_c defines the *free vibration*, which depends on the natural frequency $\omega_n = \sqrt{k/m}$ and the constants C and ϕ. The *particular solution* x_p describes the *forced vibration* of the block caused by the applied force $F = F_0 \sin \omega_0 t$. Since all vibrating systems are subject to *friction*, the free vibration, x_c, will in time dampen out. For this reason the free vibration is referred to as *transient*, and the forced vibration is called *steady-state*, since it is the only vibration that remains.

Note that if the force or displacement is applied with a frequency close to the natural frequency of the system, i.e., $\omega_0/\omega_n \approx 1$, the amplitude of vibration of the block becomes extremely large.
This condition is called *resonance*, and in practice, resonating vibrations can cause tremendous stress and rapid failure of parts.

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