Calculus 2

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Calculus 2

Lecture 10

Multiple Integration

Iterated Integrals

$$
\begin{aligned}
f(x, y) &= \int f_x(x, y)\, dx && \text{Integrate with respect to } x. \\
&= \int 2xy\, dx && \text{Hold } y \text{ constant.} \\
&= y \int 2x\, dx && \text{Factor out constant } y. \\
&= y(x^2) + C(y) && \text{Antiderivative of } 2x \text{ is } x^2. \\
&= x^2 y + C(y). && C(y) \text{ is a function of } y.
\end{aligned}
$$

Iterated Integrals

$$\int_{1}^{2y} 2xy\,dx = x^2y\Big]_{1}^{2y} = (2y)^2y - (1)^2y = 4y^3 - y.$$

x is the variable of integration and y is fixed.

Replace x by the limits of integration.

The result is a function of y.

Similarly, you can integrate with respect to y by holding x fixed. Both procedures are summarized as follows.

Iterated Integrals

EXAMPLE 1 **Integrating with Respect to *y***

Evaluate $\int_1^x (2x^2y^{-2} + 2y)\, dy$.

Solution Considering x to be constant and integrating with respect to y produces

$$\int_1^x (2x^2y^{-2} + 2y)\, dy = \left[\frac{-2x^2}{y} + y^2\right]_1^x \qquad \text{Integrate with respect to } y.$$

$$= \left(\frac{-2x^2}{x} + x^2\right) - \left(\frac{-2x^2}{1} + 1\right)$$

$$= 3x^2 - 2x - 1.$$

Iterated Integrals

EXAMPLE 2 **The Integral of an Integral**

Evaluate $\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\, dy \right] dx.$

Solution Using the result of Example 1, you have

$$\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\, dy \right] dx = \int_1^2 (3x^2 - 2x - 1)\, dx$$

$$= \Big[x^3 - x^2 - x \Big]_1^2 \qquad \text{Integrate with respect to } x.$$

$$= 2 - (-1)$$

$$= 3.$$

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Iterated Integrals

Area of a Region in the Plane

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$ Figure 14.2 (vertically simple)

2. If R is defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$ Figure 14.3 (horizontally simple)

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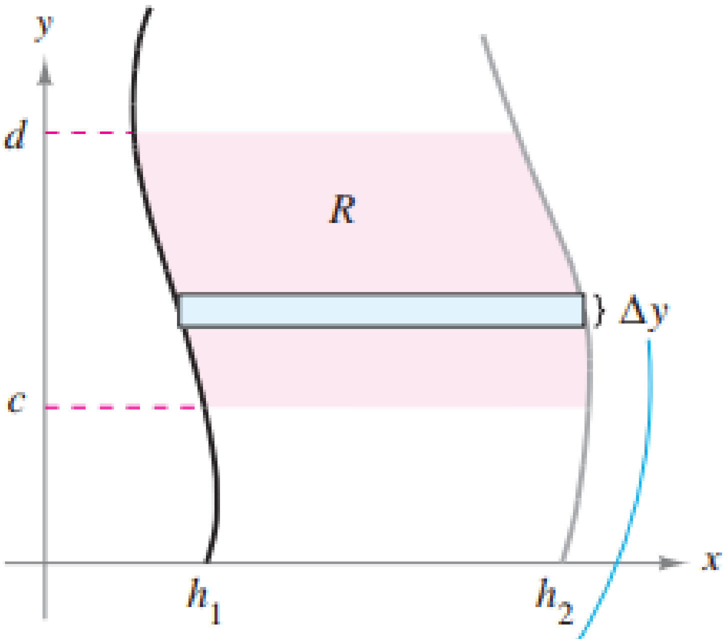
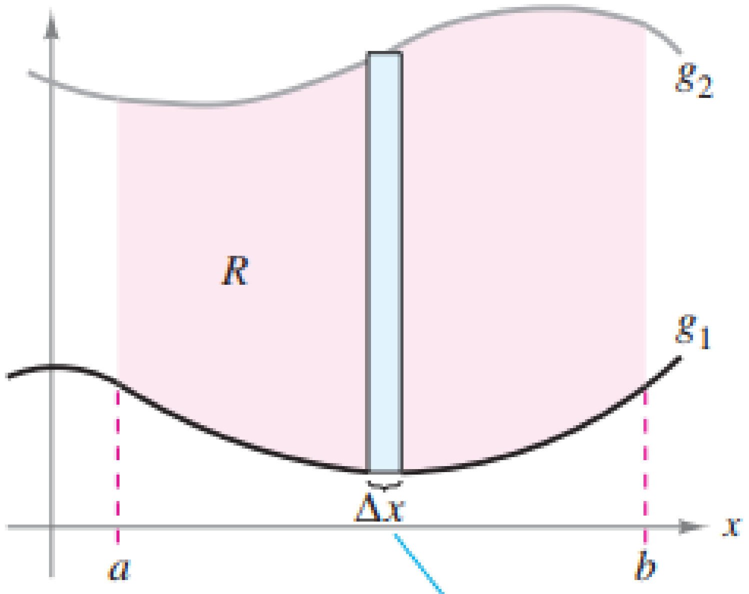
Region is bounded by
$c \le y \le d$ and
$h_1(y) \le x \le h_2(y)$



$$\text{Area} = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy$$

Region is bounded by
$a \le x \le b$ and
$g_1(x) \le y \le g_2(x)$



$$\text{Area} = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$$

Iterated Integrals

EXAMPLE 4 **Finding Area by an Iterated Integral**

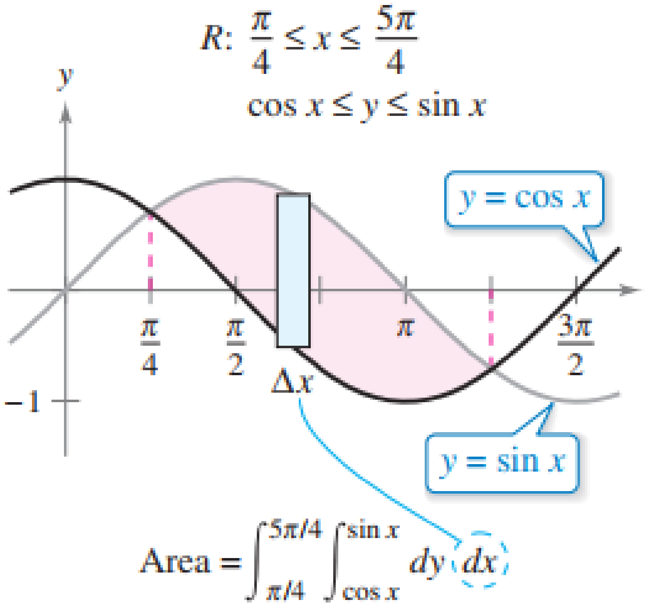
Use an iterated integral to find the area of the region bounded by the graphs of

$f(x) = \sin x$ — Sine curve forms upper boundary.

and

$g(x) = \cos x$ — Cosine curve forms lower boundary.

between $x = \pi/4$ and $x = 5\pi/4$.



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Iterated Integrals

$$\text{Area of } R = \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy\, dx$$

$$= \int_{\pi/4}^{5\pi/4} y \Big]_{\cos x}^{\sin x} dx$$

Integrate with respect to y.

$$= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

$$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

Integrate with respect to x.

$$= 2\sqrt{2}.$$

Iterated Integrals

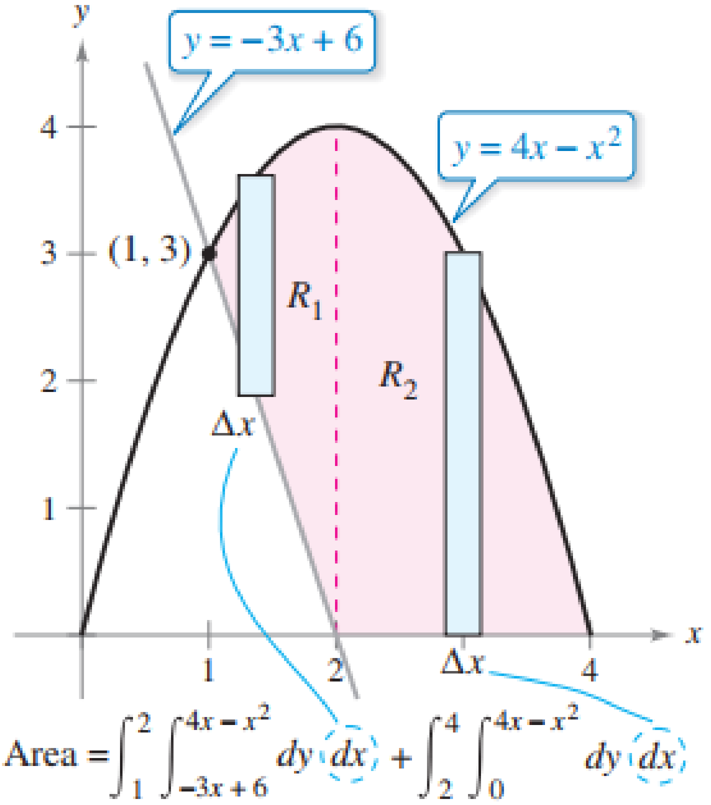
EXAMPLE 6 **An Area Represented by Two Iterated Integrals**

Find the area of the region R that lies below the parabola

$$y = 4x - x^2$$ Parabola forms upper boundary.

above the x-axis, and above the line

$$y = -3x + 6.$$ Line and x-axis form lower boundary.



$$\text{Area} = \int_1^2 \int_{-3x+6}^{4x-x^2} dy\,dx + \int_2^4 \int_0^{4x-x^2} dy\,dx$$

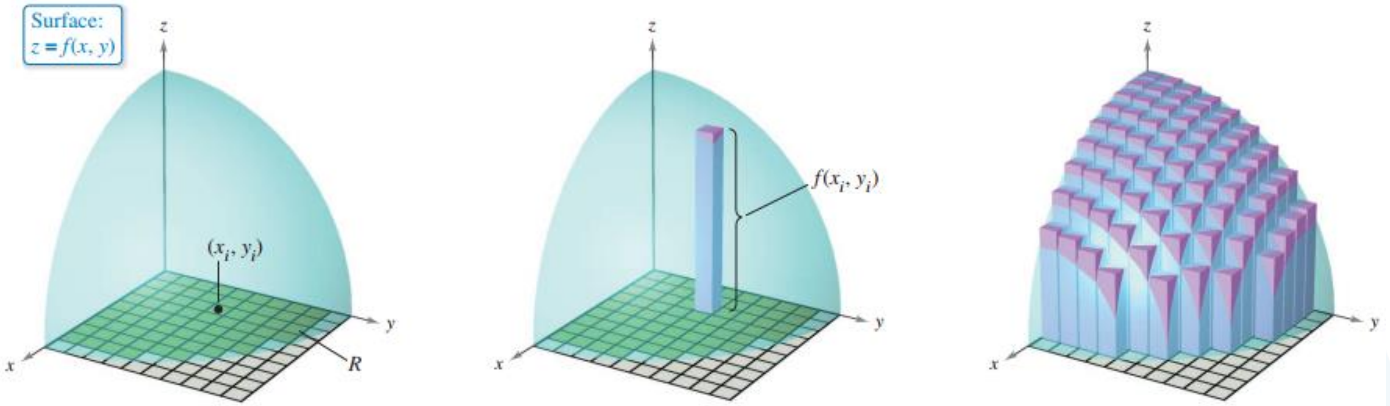
$$= \int_1^2 (4x - x^2 + 3x - 6)\,dx + \int_2^4 (4x - x^2)\,dx$$

$$= \left[\frac{7x^2}{2} - \frac{x^3}{3} - 6x\right]_1^2 + \left[2x^2 - \frac{x^3}{3}\right]_2^4$$

$$= \left(14 - \frac{8}{3} - 12 - \frac{7}{2} + \frac{1}{3} + 6\right) + \left(32 - \frac{64}{3} - 8 + \frac{8}{3}\right)$$

$$= \frac{15}{2}.$$

Double Integrals and Volume



Definition of Double Integral

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is

$$\int\int_R f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

Double Integrals and Volume

THEOREM 14.1 Properties of Double Integrals

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$

2. $\displaystyle\iint_R [f(x, y) \pm g(x, y)]\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$

3. $\displaystyle\iint_R f(x, y)\, dA \geq 0, \quad \text{if } f(x, y) \geq 0$

4. $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA, \quad \text{if } f(x, y) \geq g(x, y)$

5. $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.

Double Integrals and Volume

Volume of a Solid Region

If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f is

$$V = \int_R\int f(x, y)\, dA.$$

Double Integrals and Volume

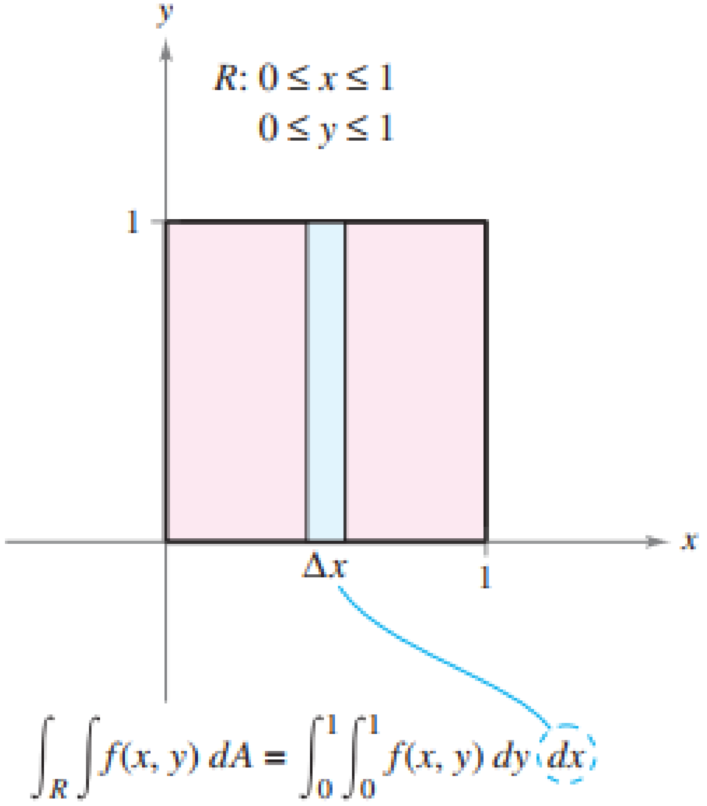
EXAMPLE

Evaluate

$$\int_R\int\left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dA$$

where R is the region given by

$$0 \le x \le 1, \quad 0 \le y \le 1.$$



Double Integrals and Volume

Solution

$$\iint_R \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dA = \int_0^1 \int_0^1 \left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right) dy\, dx$$

$$= \int_0^1 \left[\left(1 - \frac{1}{2}x^2\right)y - \frac{y^3}{6}\right]_0^1 dx$$

$$= \int_0^1 \left(\frac{5}{6} - \frac{1}{2}x^2\right) dx$$

$$= \left[\frac{5}{6}x - \frac{x^3}{6}\right]_0^1$$

$$= \frac{2}{3}$$

Double Integrals and Volume

EXAMPLE Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \leq x \leq 1, 0 \leq y \leq 2$.

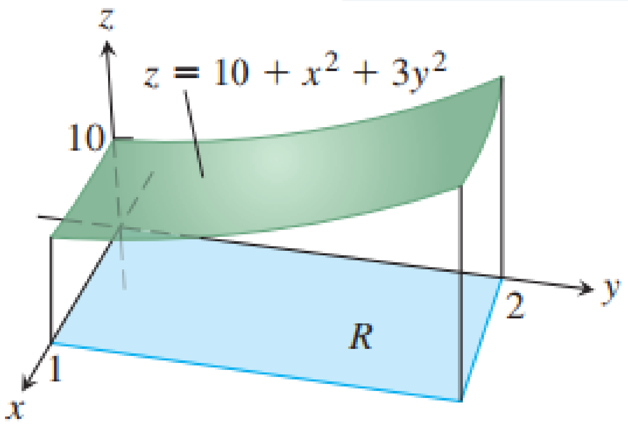


FIGURE The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R

Double Integrals and Volume

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \left[20x + \frac{2}{3}x^3 + 8x \right]_0^1 = \frac{86}{3}.$$

Triple Integrals

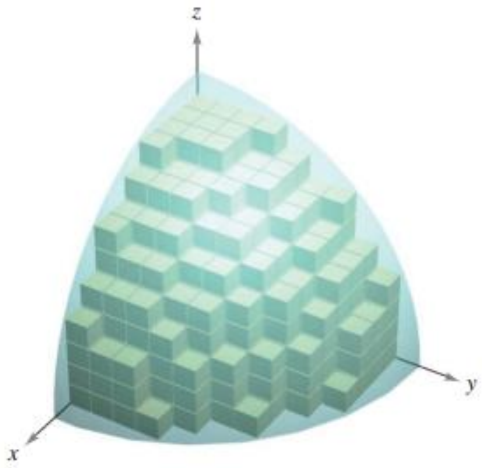
Definition of Triple Integral

If f is continuous over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV.$$



Volume of $Q \approx \sum_{i=1}^{n} \Delta V_i$

1. $$\iiint_Q cf(x, y, z)\, dV = c\iiint_Q f(x, y, z)\, dV$$

2. $$\iiint_Q [f(x, y, z) \pm g(x, y, z)]\, dV = \iiint_Q f(x, y, z)\, dV \pm \iiint_Q g(x, y, z)\, dV$$

3. $$\iiint_Q f(x, y, z)\, dV = \iiint_{Q_1} f(x, y, z)\, dV + \iiint_{Q_2} f(x, y, z)\, dV$$

In the properties above, Q is the union of two nonoverlapping solid subregions Q_1 and Q_2. If the solid region Q is simple, then the triple integral $\iiint f(x, y, z)\, dV$ can be evaluated with an iterated integral using one of the six possible orders of integration:

$dx\, dy\, dz \quad dy\, dx\, dz \quad dz\, dx\, dy$

$dx\, dz\, dy \quad dy\, dz\, dx \quad dz\, dy\, dx.$

Surface Area

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\, dz\, dy\, dx.$$

Solution For the first integration, hold x and y constant and integrate with respect to z.

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\, dz\, dy\, dx = \int_0^2 \int_0^x e^x(yz + z^2)\Big]_0^{x+y} dy\, dx$$

$$= \int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\, dy\, dx$$

For the second integration, hold x constant and integrate with respect to y.

Evaluation by Iterated Integrals

For the second integration, hold x constant and integrate with respect to y.

$$\int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\, dy\, dx = \int_0^2 \left[e^x\left(x^2y + \frac{3xy^2}{2} + \frac{2y^3}{3} \right) \right]_0^x dx$$

$$= \frac{19}{6} \int_0^2 x^3 e^x\, dx$$

Finally, integrate with respect to x.

$$\frac{19}{6} \int_0^2 x^3 e^x\, dx = \frac{19}{6} \Big[e^x(x^3 - 3x^2 + 6x - 6) \Big]_0^2$$

$$= 19\left(\frac{e^2}{3} + 1 \right)$$

Evaluation by Iterated Integrals

Evalute

$$\int_0^{\sqrt{\pi/2}} \int_0^y \int_1^3 \sin(y^2)\, dz\, dx\, dy$$

$$\begin{aligned}
\int_0^{\sqrt{\pi/2}} \int_0^y \int_1^3 \sin(y^2)\, dz\, dx\, dy &= \int_0^{\sqrt{\pi/2}} \int_0^y z \sin(y^2) \Big]_1^3 \, dx\, dy \\
&= 2\int_0^{\sqrt{\pi/2}} \int_0^y \sin(y^2)\, dx\, dy \\
&= 2\int_0^{\sqrt{\pi/2}} x \sin(y^2) \Big]_0^y \, dy \\
&= 2\int_0^{\sqrt{\pi/2}} y \sin(y^2)\, dy \\
&= -\cos(y^2) \Big]_0^{\sqrt{\pi/2}} \\
&= 1.
\end{aligned}$$

Evaluating an Iterated Integral In Exercises 11–30, evaluate the iterated integral.

11. $\int_0^1 \int_0^2 (x + y)\, dy\, dx$

12. $\int_{-1}^1 \int_{-2}^2 (x^2 - y^2)\, dy\, dx$

13. $\int_1^2 \int_0^4 (x^2 - 2y^2)\, dx\, dy$

14. $\int_{-1}^2 \int_1^3 (x + y^2)\, dx\, dy$

15. $\int_0^{\pi/2} \int_0^1 y \cos x\, dy\, dx$

16. $\int_0^{\ln 4} \int_0^{\ln 3} e^{x+y}\, dy\, dx$

17. $\int_0^{\pi} \int_0^{\sin x} (1 + \cos x)\, dy\, dx$

18. $\int_1^4 \int_1^{\sqrt{x}} 2ye^{-x}\, dy\, dx$

19. $\int_0^1 \int_0^x \sqrt{1 - x^2}\, dy\, dx$

20. $\int_{-4}^4 \int_0^{x^2} \sqrt{64 - x^3}\, dy\, dx$

21. $\int_{-1}^5 \int_0^{3y} \left(3 + x^2 + \frac{1}{4}y^2\right) dx\, dy$

22. $\int_0^2 \int_y^{2y} (10 + 2x^2 + 2y^2)\, dx\, dy$

23. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x + y)\, dx\, dy$

24. $\int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y\, dx\, dy$

Evaluate the double integral

9. $\displaystyle\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$

10. $\displaystyle\int_0^{\pi}\int_0^{\sin\theta} r^2\,dr\,d\theta$

11. $\displaystyle\int_0^{2\pi}\int_0^{6} 3r^2\sin\theta\,dr\,d\theta$

12. $\displaystyle\int_0^{\pi/4}\int_0^{4} r^2\sin\theta\cos\theta\,dr\,d\theta$

13. $\displaystyle\int_0^{\pi/2}\int_2^{3} \sqrt{9-r^2}\,r\,dr\,d\theta$

14. $\displaystyle\int_0^{\pi/2}\int_0^{3} re^{-r^2}\,dr\,d\theta$

15. $\displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} \theta r\,dr\,d\theta$

16. $\displaystyle\int_0^{\pi/2}\int_0^{1-\cos\theta} (\sin\theta)r\,dr\,d\theta$

Evaluating a Triple Iterated Integral In Exercises 1–8, evaluate the triple iterated integral.

1. $\int_0^3 \int_0^2 \int_0^1 (x + y + z)\, dx\, dz\, dy$

2. $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 x^2y^2z^2\, dx\, dy\, dz$

3. $\int_0^1 \int_0^x \int_0^{xy} x\, dz\, dy\, dx$

4. $\int_0^9 \int_0^{y/3} \int_0^{\sqrt{y^2-9x^2}} z\, dz\, dx\, dy$

5. $\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2}\, dy\, dx\, dz$

6. $\int_1^4 \int_1^{e^2} \int_0^{1/xz} \ln z\, dy\, dz\, dx$

7. $\int_0^4 \int_0^{\pi/2} \int_0^{1-x} x \cos y\, dz\, dy\, dx$

8. $\int_0^{\pi/2} \int_0^{y/2} \int_0^{1/y} \sin y\, dz\, dx\, dy$

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Thank you for your attention