MATHEMATICAL ANALAYSIS 1

Lecture 10

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Applications

Vibrations

the tension force in the spring is *ks* , where *k* is the spring constant. The force due to gravity pulling down on the spring is *mg* , and equilibrium requires that

$$ks = mg.$$

Let *y*, with positive direction downward, denote the displacement position of the object away from the equilibrium position at any time *t* after the motion has started. Then the forces acting on the object are

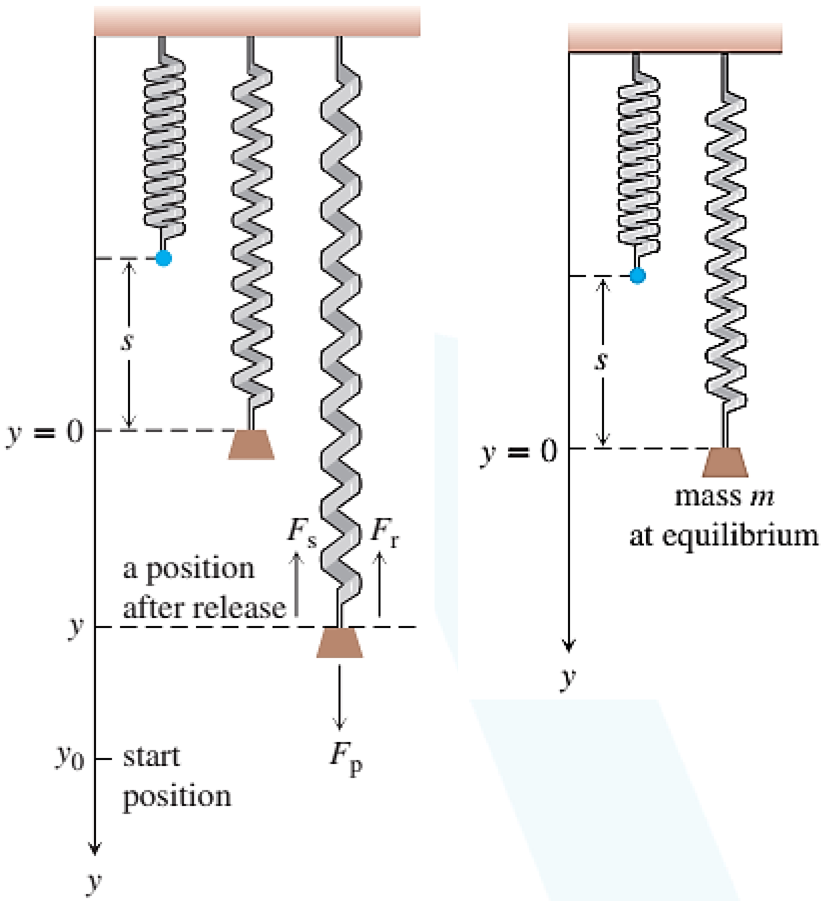
$F_p = mg,$ the propulsion force due to gravity,

$F_s = k(s + y),$ the restoring force of the spring's tension,

$F_r = \delta \dfrac{dy}{dt},$ a frictional force assumed proportional to velocity.

By Newton's second law $F = ma,$

$$m\frac{d^2y}{dt^2} = mg - ks - ky - \delta\frac{dy}{dt}.$$



$mg - ks = 0 \longrightarrow m\dfrac{d^2y}{dt^2} + \delta\dfrac{dy}{dt} + ky = 0,$

Simple Harmonic Motion

Suppose first that there is no retarding frictional force. $\delta = 0$ there is no damping.

$\omega = \sqrt{k/m} \longrightarrow y'' + \omega^2 y = 0,$ with $y(0) = y_0$ and $y'(0) = 0.$

$\longrightarrow y = c_1 \cos \omega t + c_2 \sin \omega t. \longrightarrow y = y_0 \cos \omega t$

Damped Motion

Assume now that there is friction in the spring system, $\delta \neq 0.$

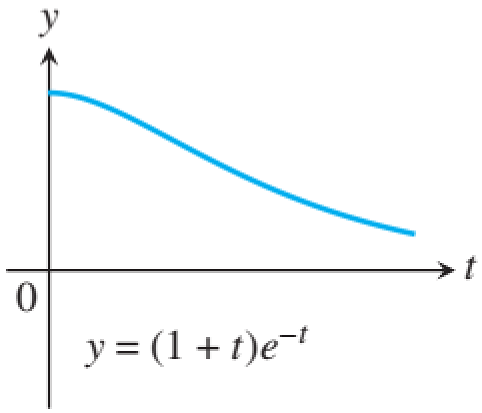
$\omega = \sqrt{k/m}$ and $2b = \delta/m \longrightarrow y'' + 2by' + \omega^2 y = 0.$

$r^2 + 2br + \omega^2 = 0, \longrightarrow r = -b \pm \sqrt{b^2 - \omega^2}.$

Case 1: $b = \omega$. The double root of the auxiliary equation is real and equals $r = \omega$. Th general solution to Equation (6) is

$$y = (c_1 + c_2 t)e^{-\omega t}.$$

This situation of motion is called **critical damping** and is not oscillatory.

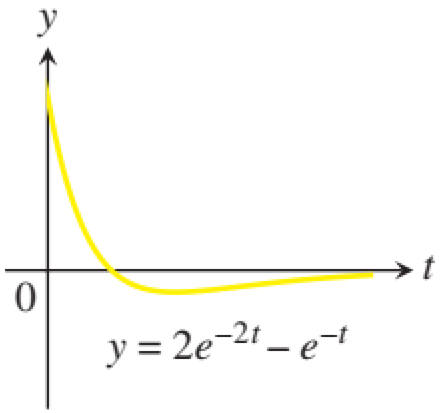


(a) Critical damping

Case 2: $b > \omega$. The roots of the auxiliary equation are real and unequal, given by $r_1 = -b + \sqrt{b^2 - \omega^2}$ and $r_2 = -b - \sqrt{b^2 - \omega^2}$. The general solution to Equation (6) is given by

$$y = c_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}.$$

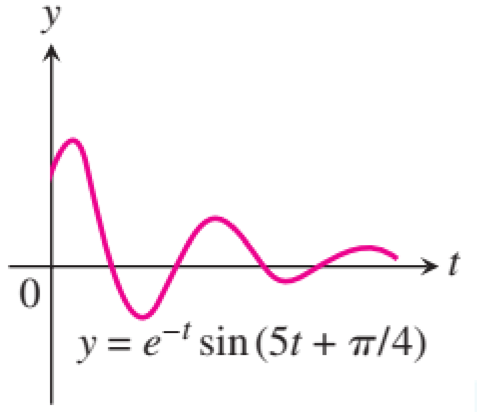
Here again the motion is not oscillatory and both r_1 and r_2 are negative. Thus y approaches zero as time goes on. This motion is referred to as **overdamping**



(b) Overdamping

Case 3: $b < \omega$. The roots to the auxiliary equation are complex and given by $r = -b \pm i\sqrt{\omega^2 - b^2}$. The general solution to Equation (6) is given by

$$y = e^{-bt}\left(c_1 \cos\sqrt{\omega^2 - b^2}\,t + c_2 \sin\sqrt{\omega^2 - b^2}\,t\right).$$



(c) Underdamping

This situation, called **underdamping**,

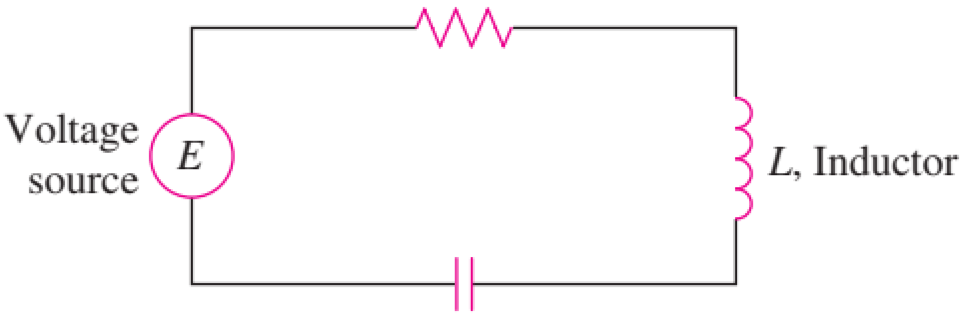
Electric Circuits

q: charge at a cross section of a conductor measured in **coulombs** (abbreviated c);

I: current or rate of change of charge dq/dt (flow of electrons) at a cross section of a conductor measured **in amperes** (abbreviated A);

E: electric (potential) source measured **in volts** (abbreviated V);

V: difference in potential between two points along the conductor measured in **volts** (V).



$$RI + L\frac{dI}{dt} + \frac{q}{C} = E(t). \quad \Rightarrow \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Example

A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 1 lb/ft. The resistance in the spring–mass system is numerically equal to the instantaneous velocity. At $t = 0$ the weight is set in motion from a position 2 ft below its equilibrium position by giving it a downward velocity of 2 ft/sec. Write an initial value problem that models the given situation.

$mg = 16 \Rightarrow m = \frac{16}{32};$

$k = 1; \delta = 1$

$$\Rightarrow \frac{16}{32}\frac{d^2y}{dt^2} + 1\frac{dy}{dt} + 1y = 0$$

$$\Rightarrow \frac{1}{2}\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0,\ y(0) = 2,\ y'(0) = 2$$

Example

Suppose $L = 10$ henrys, $R = 10$ ohms, $C = 1/500$ farads, $E = 100$ volts, $q(0) = 10$ coulombs, and $q'(0) = i(0) = 0$. Formulate and solve an initial value problem that models the given LRC circuit. Interpret your results.

$$L = 10, R = 10, C = \frac{1}{500}, E(t) = 100$$

$$\Rightarrow 10\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 500q = 100,\ q(0) = 10, q'(0) = 0 \qquad \Rightarrow 10r^2 + 10r + 500 = 0$$

$$r = \frac{-1\pm\sqrt{1^2-4(1)(50)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{199}}{2}i \quad \Rightarrow q_c = e^{-\frac{1}{2}t}\left(c_1\cos\frac{\sqrt{199}}{2}t + c_2\sin\frac{\sqrt{199}}{2}t\right)$$

$$q_p = A \Rightarrow q_p' = 0 \Rightarrow q_p'' = 0 \Rightarrow 10(0) + 10(0) + 500A = 100 \Rightarrow 500A = 100 \Rightarrow A = \frac{1}{5}$$

$$\Rightarrow q(t) = e^{-\frac{1}{2}t}\left(c_1\cos\frac{\sqrt{199}}{2}t + c_2\sin\frac{\sqrt{199}}{2}t\right) + \frac{1}{5}$$

$$\Rightarrow q' = e^{-\frac{1}{2}t}\left[\left(-\frac{1}{2}c_1 + \frac{\sqrt{199}}{2}c_2\right)\cos\frac{\sqrt{199}}{2}t + \left(-\frac{\sqrt{199}}{2}c_1 - \frac{1}{2}c_2\right)\sin\frac{\sqrt{199}}{2}t\right]$$

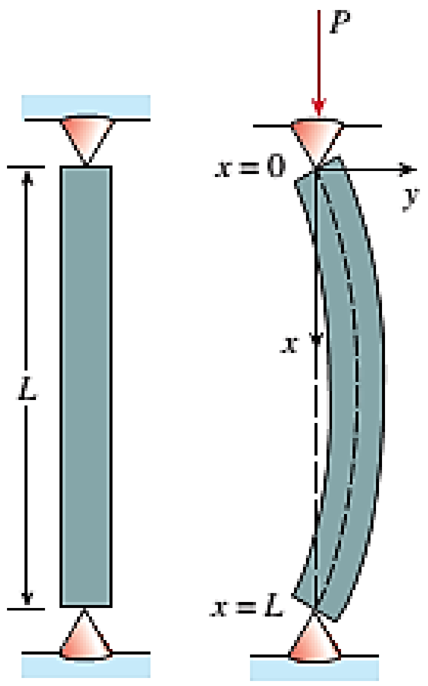
$$q(0)=10\Rightarrow c_1+\tfrac{1}{5}=10\Rightarrow c_1=\tfrac{49}{5},\ q'(0)=0\Rightarrow -\tfrac{1}{2}c_1+\frac{\sqrt{199}}{2}c_2=0\Rightarrow c_1=\tfrac{49}{5},\ c_2=\frac{49\sqrt{199}}{995}$$

$$\Rightarrow q(t)=e^{-\frac{1}{2}t}\left(\frac{49}{5}\cos\frac{\sqrt{199}}{2}t+\frac{49\sqrt{199}}{995}\sin\frac{\sqrt{199}}{2}t\right)+\frac{1}{5}$$

Buckling of a Thin Vertical Column

$$EI\frac{d^2y}{dx^2} = -Py \qquad \text{or} \qquad EI\frac{d^2y}{dx^2} + Py = 0,$$

Find the deflection of a thin vertical homogeneous column of length L subjected to a constant axial load P if the column is simply supported or hinged at both ends.



$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0. \qquad y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \qquad \lambda = P/EI$$

$$y_n(x) = c_2 \sin(n\pi x/L) \;\; ; \;\; \lambda_n = P/EI = n^2\pi^2/L^2$$

$$P_n = n^2\pi^2 EI/L^2$$ **critical loads**

smallest critical load $P_1 = \pi^2 EI/L^2$ **Euler load**,

Exercises

A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 1 lb/ft. The resistance in the spring–mass system is numerically equal to the instantaneous velocity. At $t = 0$ the weight is set in motion from a position 2 ft below its equilibrium position by giving it a downward velocity of 2 ft/sec. At the end of π sec, determine whether the mass is above or below the equilibrium position and by what distance.

$$y(t) = e^{-t}(2\cos t + 4\sin t).$$

0.0864 ft above equilibrium.

Exercises

A series circuit consisting of an inductor, a resistor, and a capacitor is open. There is an initial charge of 2 coulombs on the capacitor, and 3 amperes of current is present in the circuit at the instant the circuit is closed. A voltage given by $E(t) = 20\cos t$ is applied. In this circuit the voltage drops are numerically equal to the following: across the resistor to 4 times the instantaneous change in the charge, across the capacitor to 10 times the charge, and across the inductor to 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time. Determine the charge on the capacitor and the current at time $t = 10$.

$$q(t) = 2e^{-t}\sin 2t + \sin t + 2\cos t;$$

Systems of Linear Differential Equations

Linear Systems

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\
\vdots \qquad & \qquad\qquad\qquad\quad \vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t).
\end{aligned}
$$

When $f_i(t) = 0$, $i = 1, 2, \ldots, n$, the linear system is said to be **homogeneous**; otherwise it is **nonhomogeneous**.

Matrix Form of a Linear System

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}. \tag{4}$$

$$\mathbf{X}' = \mathbf{A}\mathbf{X}. \tag{5}$$

Theorem 10.1.2 Superposition Principle

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be a set of solution vectors of the homogeneous system (5) on an interval I. Then the linear combination

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k,$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution of the system on the interval.

Definition 10.1.2 Linear Dependence/Independence

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ be a set of solution vectors of the homogeneous system (5) on an interval I. We say that the set is **linearly dependent** on the interval if there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_k\mathbf{X}_k = \mathbf{0}$$

for every t in the interval. If the set of vectors is not linearly dependent on the interval, it is said to be **linearly independent**.

Theorem 10.1.3 Criterion for Linearly Independent Solutions

Let $$\mathbf{X}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \quad \ldots, \quad \mathbf{X}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

be n solution vectors of the homogeneous system (5) on an interval I. Then the set of solution vectors is linearly independent on I if and only if the **Wronskian**

$$W(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \neq 0 \tag{9}$$

for every t in the interval.

Theorem 10.1.5 General Solution—Homogeneous Systems

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be a fundamental set of solutions of the homogeneous system (5) on an interval I. Then the **general solution** of the system on the interval is

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Nonhomogeneous Systems

Theorem 10.1.6 General Solution—Nonhomogeneous Systems

Let $\mathbf{X}_p$ be a given solution of the nonhomogeneous system (4) on an interval I, and let

$$\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$$

denote the general solution on the same interval of the associated homogeneous system (5). Then the **general solution** of the nonhomogeneous system on the interval is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p.$$

The general solution $\mathbf{X}_c$ of the associated homogeneous system (5) is called the **complementary function** of the nonhomogeneous system (4).

we are prompted to ask whether we can always find a solution of the form

$$\mathbf{X} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K}e^{\lambda t}$$

for the general homogeneous linear first-order system

$$\mathbf{X}' = \mathbf{A}\mathbf{X},$$

where the coefficient matrix $\mathbf{A}$ is an $n \times n$ matrix of constants.

Distinct Real Eigenvalues

Theorem 10.2.1 General Solution—Homogeneous Systems

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct real eigenvalues of the coefficient matrix $\mathbf{A}$ of the homogeneous system (2), and let $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ be the corresponding eigenvectors. Then the **general solution** of (2) on the interval $(-\infty, \infty)$ is given by

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t}.$$

EXAMPLE 1

Solve

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 2x + y.$$

SOLUTION We first find the eigenvalues and eigenvectors of the matrix of coefficients. From the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$$

we see that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

Now for $\lambda_1 = -1$

$$3k_1 + 3k_2 = 0$$
$$2k_1 + 2k_2 = 0.$$

$$\Rightarrow \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 4$, we have

$$-2k_1 + 3k_2 = 0$$
$$2k_1 - 3k_2 = 0$$

$$\Rightarrow \quad \mathbf{K}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$\Rightarrow \quad \mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t},$$

$$\Rightarrow \quad \mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

EXAMPLE 2

Solve

$$\frac{dx}{dt} = -4x + y + z$$

$$\frac{dy}{dt} = x + 5y - z$$

$$\frac{dz}{dt} = y - 3z.$$

SOLUTION Using the cofactors of the third row, we find

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = -(\lambda + 3)(\lambda + 4)(\lambda - 5) = 0,$$

$\Rightarrow$ the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = 5$.

For $\lambda_1 = -3$, Gauss–Jordan elimination gives

$$(\mathbf{A} + 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \longrightarrow \mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t}.$$

Similarly, for $\lambda_2 = -4$,

$$(\mathbf{A} + 4\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \mathbf{K}_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} e^{-4t}.$$

Finally, when $\lambda_3 = 5$, the augmented matrices

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -9 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \mathbf{K}_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}, \quad \mathbf{X}_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}.$$

The general solution

$$\mathbf{X} = c_1\begin{pmatrix}1\\0\\1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}10\\-1\\1\end{pmatrix}e^{-4t} + c_3\begin{pmatrix}1\\8\\1\end{pmatrix}e^{5t}.$$

Repeated Eigenvalues

In general, if m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^{m+1}$ is not a factor, then λ_1 is said to be an **eigenvalue of multiplicity m**. The next three examples illustrate the following cases:

(*i*) For some $n \times n$ matrices $\mathbf{A}$ it may be possible to find m linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ corresponding to an eigenvalue λ_1 of multiplicity $m \le n$. In this case, the general solution of the system contains the linear combination

$$c_1\mathbf{K}_1e^{\lambda_1 t} + c_2\mathbf{K}_2e^{\lambda_1 t} + \cdots + c_m\mathbf{K}_me^{\lambda_1 t}.$$

(*ii*) If there is only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_{11}e^{\lambda_1 t}\\
\mathbf{X}_2 &= \mathbf{K}_{21}te^{\lambda_1 t} + \mathbf{K}_{22}e^{\lambda_1 t}\\
&\vdots\\
\mathbf{X}_m &= \mathbf{K}_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + \mathbf{K}_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \cdots + \mathbf{K}_{mm}e^{\lambda_1 t},
\end{aligned}$$

where $\mathbf{K}_{ij}$ are column vectors, can always be found.

EXAMPLE 3

$$\text{Solve} \quad \mathbf{X}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\mathbf{X}.$$

SOLUTION Expanding the determinant in the characteristic equation

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

yields $-(\lambda+1)^2(\lambda-5)=0$. We see that $\lambda_1=\lambda_2=-1$ and $\lambda_3=5$.

For $\lambda_1=-1$, Gauss–Jordan elimination immediately gives

$$(\mathbf{A}+\mathbf{I}|\mathbf{0}) = \left(\begin{array}{rrr|r} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \xRightarrow{\text{row operations}} \left(\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$$\Rightarrow \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad \Rightarrow \quad \mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} \text{ and } \mathbf{X}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}.$$

Lastly, for $\lambda_3 = 5$, the reduction

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{rrr|r} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We conclude that the general solution of the system is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}.$$

Second Solution

Now suppose that λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t}, \tag{12}$$

where $$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}.$$

To see this we substitute (12) into the system $\mathbf{X}' = \mathbf{AX}$ and simplify:

$$(\mathbf{AK} - \lambda_1\mathbf{K})te^{\lambda_1 t} + (\mathbf{AP} - \lambda_1\mathbf{P} - \mathbf{K})e^{\lambda_1 t} = \mathbf{0}.$$

Since this last equation is to hold for all values of t, we must have

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = \mathbf{0} \tag{13}$$

and $$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}. \tag{14}$$

EXAMPLE 4

$$\mathbf{X}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\mathbf{X}$$

$(\lambda + 3)^2 = 0 \longrightarrow \lambda_1 = \lambda_2 = -3$ is a root of *multiplicity two*

$$\mathbf{K}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathbf{X}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$$

Identifying $\mathbf{K} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $\longrightarrow$ $(\mathbf{A} + 3\mathbf{I})\mathbf{P} = \mathbf{K}$ or $\begin{aligned} 6p_1 - 18p_2 &= 3 \\ 2p_1 - 6p_2 &= 1. \end{aligned}$

$\longrightarrow \mathbf{P} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \longrightarrow \mathbf{X}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}.$

$$\longrightarrow \mathbf{X} = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}\right]$$

Eigenvalue of Multiplicity Three

When the coefficient matrix $\mathbf{A}$ has only one eigenvector associated with an eigenvalue λ_1 of multiplicity three, we can find a solution of the form (12) and a third solution of the form

$$\mathbf{X}_3 = \mathbf{K}\frac{t^2}{2}e^{\lambda_1 t} + \mathbf{P}te^{\lambda_1 t} + \mathbf{Q}e^{\lambda_1 t}, \tag{15}$$

where

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}, \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}.$$

By substituting (15) into the system $\mathbf{X}' = \mathbf{AX}$, we find that the column vectors $\mathbf{K}$, $\mathbf{P}$, and $\mathbf{Q}$ must satisfy

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = \mathbf{0} \tag{16}$$

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K} \tag{17}$$

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}. \tag{18}$$

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EXAMPLE 5

Solve $\quad \mathbf{X}' = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix}\mathbf{X}.$

SOLUTION The characteristic equation $(\lambda - 2)^3 = 0$ shows that $\lambda_1 = 2$ is an eigenvalue of multiplicity three. By solving $(\mathbf{A} - 2\mathbf{I})\mathbf{K} = \mathbf{0}$ we find the single eigenvector

$$\mathbf{K} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We next solve the systems $(\mathbf{A} - 2\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - 2\mathbf{I})\mathbf{Q} = \mathbf{P}$ in succession and find that

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}.$$

$$\Rightarrow \quad \mathbf{X} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{2t} + c_2\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{2t}\right] + c_3\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\frac{t^2}{2}e^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 0 \\ -\frac{6}{5} \\ \frac{1}{5} \end{pmatrix}e^{2t}\right].$$

Complex Eigenvalues

Theorem 10.2.2 Solutions Corresponding to a Complex Eigenvalue

Let $\mathbf{A}$ be the coefficient matrix having real entries of the homogeneous system (2), and let $\mathbf{K}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, α and β real. Then

$$\mathbf{K}_1 e^{\lambda_1 t} \quad \text{and} \quad \overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t}$$

are solutions of (2).

$$\mathbf{K}_1 e^{\lambda_1 t} = \mathbf{K}_1 e^{\alpha t} e^{i\beta t} = \mathbf{K}_1 e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

$$\overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t} = \overline{\mathbf{K}}_1 e^{\alpha t} e^{-i\beta t} = \overline{\mathbf{K}}_1 e^{\alpha t}(\cos \beta t - i \sin \beta t).$$

By the superposition principle, Theorem 10.1.2, the following vectors are also solutions:

$$\mathbf{X}_1 = \frac{1}{2}(\mathbf{K}_1 e^{\lambda_1 t} + \overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t}) = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1)\, e^{\alpha t}\cos \beta t - \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1) e^{\alpha t}\sin \beta t$$

$$\mathbf{X}_2 = \frac{i}{2}(-\mathbf{K}_1 e^{\lambda_1 t} + \overline{\mathbf{K}}_1 e^{\overline{\lambda}_1 t}) = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1) e^{\alpha t}\cos \beta t + \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1) e^{\alpha t}\sin \beta t.$$

For *any* complex number $z = a + ib$, both $\frac{1}{2}(z + \bar{z}) = a$ and $\frac{i}{2}(-z + \bar{z}) = b$ are *real* numbers.

Therefore, the entries in the column vectors $\frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1)$ and $\frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1)$ are real numbers. By defining

$$\mathbf{B}_1 = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1) \quad \text{and} \quad \mathbf{B}_2 = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1), \tag{22}$$

Theorem 10.2.3 Real Solutions Corresponding to a Complex Eigenvalue

Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous system (2), and let $\mathbf{B}_1$ and $\mathbf{B}_2$ denote the column vectors defined in (22). Then

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{B}_1 \cos \beta t - \mathbf{B}_2 \sin \beta t]e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos \beta t + \mathbf{B}_1 \sin \beta t]e^{\alpha t} \end{aligned} \tag{23}$$

are linearly independent solutions of (2) on $(-\infty, \infty)$.

The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ in (22) are often denoted by

$$\mathbf{B}_1 = \mathrm{Re}(\mathbf{K}_1) \quad \text{and} \quad \mathbf{B}_2 = \mathrm{Im}(\mathbf{K}_1) \tag{24}$$

EXAMPLE 6

Solve the initial-value problem

$$\mathbf{X}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

SOLUTION First we obtain the eigenvalues from

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4 = 0.$$

The eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = \overline{\lambda_1} = -2i$. For λ_1 the system

$$\begin{aligned} (2-2i)k_1 + \qquad\qquad 8k_2 &= 0 \\ -k_1 + (-2-2i)k_2 &= 0 \end{aligned} \qquad \Rightarrow \qquad \mathbf{K}_1 = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

$$\mathbf{B}_1 = \mathrm{Re}(\mathbf{K}_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_2 = \mathrm{Im}(\mathbf{K}_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

$$\mathbf{X} = c_1\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 2t\right]$$

$$= c_1\begin{pmatrix} 2\cos 2t - 2\sin 2t \\ -\cos 2t \end{pmatrix} + c_2\begin{pmatrix} 2\cos 2t + 2\sin 2t \\ -\sin 2t \end{pmatrix}.$$

EXERCISES

Find the general solution of the given system

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = 4x + 3y$$

$$\mathbf{X}' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}\mathbf{X}$$

$$\frac{dx}{dt} = 3x - y$$
$$\frac{dy}{dt} = 9x - 3y$$

$$\frac{dx}{dt} = 3x - y - z$$
$$\frac{dy}{dt} = x + y - z$$
$$\frac{dz}{dt} = x - y + z$$

$$\mathbf{X}' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}\mathbf{X}$$

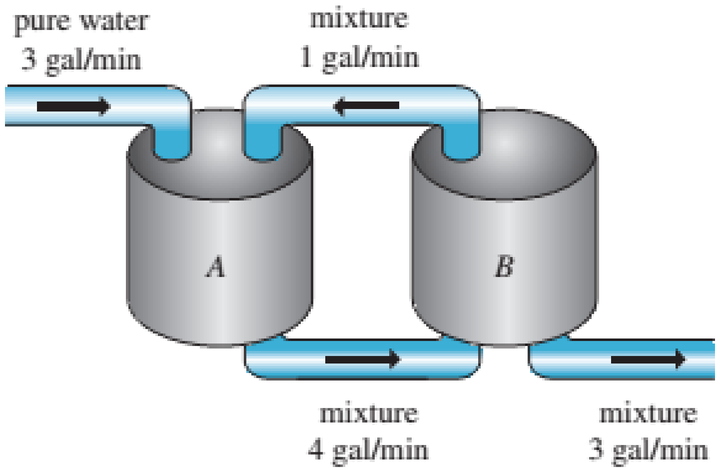
$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}\mathbf{X}$$

$$\frac{dx}{dt} = 6x - y$$
$$\frac{dy}{dt} = 5x + 2y$$

Applications

Mixtures

Consider the two tanks. Let us suppose for the sake of discussion that tank A contains 50 gallons of water in which 25 pounds of salt is dissolved. Suppose tank B contains 50 gallons of pure water. Liquid is pumped in and out of the tanks as indicated in the figure; the mixture exchanged between the two tanks and the liquid pumped out of tank B is assumed to be well stirred. We wish to construct a mathematical model that describes the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t.



$$\frac{dx_1}{dt} = \underbrace{(3\text{ gal/min})\cdot(0\text{ lb/gal}) + (1\text{ gal/min})\cdot\left(\frac{x_2}{50}\text{ lb/gal}\right)}_{\text{input rate of salt}} - \underbrace{(4\text{ gal/min})\cdot\left(\frac{x_1}{50}\text{ lb/gal}\right)}_{\text{output rate of salt}}$$

$$= -\frac{2}{25}x_1 + \frac{1}{50}x_2.$$

$$\frac{dx_2}{dt} = 4\cdot\frac{x_1}{50} - 3\cdot\frac{x_2}{50} - 1\cdot\frac{x_2}{50} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

$$\frac{dx_1}{dt} = -\frac{2}{25}x_1 + \frac{1}{50}x_2$$

$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2.$$

Electrical Networks

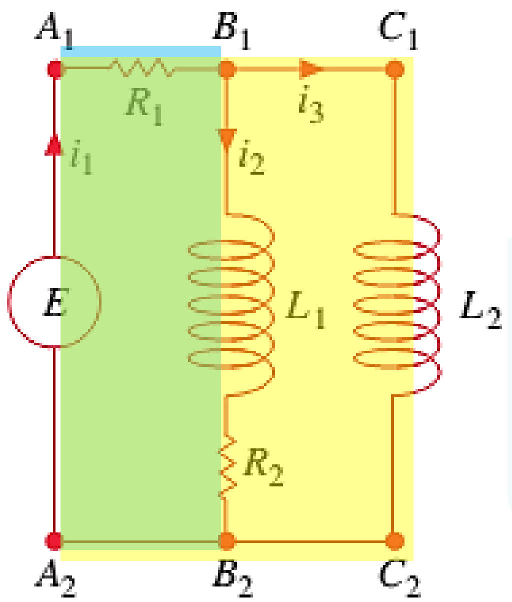
Kirchhoff's first law

$$i_1(t) = i_2(t) + i_3(t)$$

Kirchhoff's second law

$$E(t) = i_1R_1 + L_1\frac{di_2}{dt} + i_2R_2$$

$$E(t) = i_1R_1 + L_2\frac{di_3}{dt}$$



$$L_1\frac{di_2}{dt} + (R_1 + R_2)i_2 + R_1i_3 = E(t)$$

$$L_2\frac{di_3}{dt} + R_1i_2 + R_1i_3 = E(t)$$