

Lecture 4: Determinants

CEDC102: Linear Algebra

Manara University

2023-2024

The Determinant of a Matrix
Determinant and Elementary Operations
Properties of Determinants
Applications of Determinants

The Determinant of a Matrix

The determinant is a function $\det: M_n(K) \rightarrow K; A \mapsto \det(A) = |A|$

K : a field (R or C)

- **The determinant of a 2×2 matrix:**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- **Ex. 1: (The determinant of a matrix of order 2)**

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

- **Note:** The determinant of a matrix can be positive, zero, or negative

- **Minor of the entry a_{ij} :**

The determinant of the matrix determined by deleting the i -th row and j -th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

- **Cofactor of a_{ij} :**

$$C_{ij} = (-1)^{i+j} M_{ij}$$

■ Ex 2:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

- Ex 3: Find all the minors and cofactors of A

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of A

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,$$

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5,$$

$$M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2,$$

$$M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$$

$$M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,$$

$$M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3,$$

$$M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

(2) All the cofactors of A

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = + \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,$$

$$C_{12} = - \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5,$$

$$C_{13} = + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2,$$

$$C_{22} = + \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$$

$$C_{23} = - \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,$$

$$C_{32} = - \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3,$$

$$C_{33} = + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- **Theorem : (Expansion by cofactors)**

Let A is a square matrix of order n . Then the determinant of A is given by

$$(a) \det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

(Cofactor expansion along the i -th row, $i = 1, 2, \dots, n$)

or

$$(b) \det(A) = |A| = \sum_{i=1}^n a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

(Cofactor expansion along the j -th column, $j = 1, 2, \dots, n$)

■ Ex 4: The determinant of a matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

■ **Ex 5: The determinant of a matrix of order 3**

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

From Ex 3: $C_{11} = -1, C_{12} = 5, C_{13} = 4$
 $C_{21} = -2, C_{22} = -4, C_{23} = 8$
 $C_{31} = 5, C_{32} = 3, C_{33} = -6$

Sol:

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14 \end{aligned}$$

■ Ex 6: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix} = 7$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} = -8$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (0)(7) + (2)(5) + (1)(-8) = 2 \end{aligned}$$

- **Note:**

The row (or column) containing the most zeros is the best choice for expansion by cofactors.

- **Ex 7: The determinant of a matrix of order 4**

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3 \left[(0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

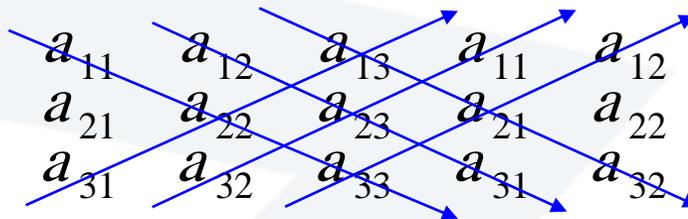
$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

$$= (3)(13)$$

$$= 39$$

- The determinant of a matrix of order 3 (Sarrus Rule)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

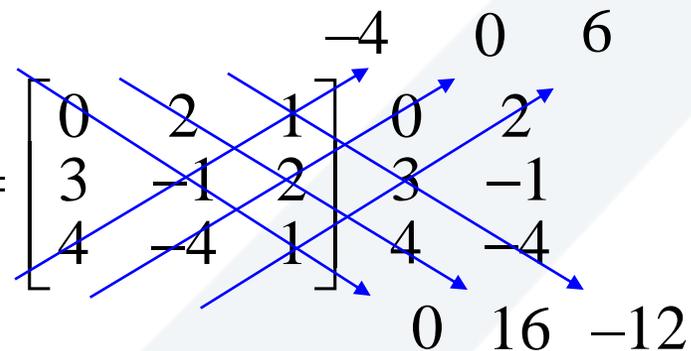


Subtract these three products.

Add these three products.

$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- Ex 8:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$$


$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 \\ - (-4 + 0 + 6) = 2$$

- **Upper triangular matrix:**

All the entries below the main diagonal are zeros.

- **Lower triangular matrix:**

All the entries above the main diagonal are zeros.

- **Diagonal matrix:**

All the entries above and below the main diagonal are zeros.

- **Note:**

A matrix that is both upper and lower triangular is called diagonal.

■ **Ex 9:**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

diagonal

■ **Theorem : (Determinant of a Triangular Matrix)**

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11} a_{22} a_{33} \cdots a_{nn}$$

■ Ex 10:

$$(a) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a) \quad |A| = (2)(-2)(1)(3) = -12$$

$$(b) \quad |B| = (-1)(3)(2)(4)(-2) = 48$$

Evaluation of a determinant using elementary operations

■ Theorem : (Elementary row operations and determinants)

Let A and B be square matrices.

$$(a) \ B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \quad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b) \ B = r_i^{(k)}(A) \quad \Rightarrow \quad \det(B) = k \det(A) \quad (\text{i.e. } |r_i^{(k)}(A)| = k|A|)$$

$$(c) \ B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \quad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$

■ Ex 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- **Notes:**

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k \det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k} \det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

- **Note:**

A row-echelon form of a square matrix is always upper triangular.

■ Ex 2:

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \det(A) = ?$$

Sol:

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{12}}{=} - \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{12}^{(-2)}}{=} - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\stackrel{r_2^{(-\frac{1}{7})}}{=} (-1) \left(\frac{1}{\frac{-1}{7}} \right) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{23}^{(-1)}}{=} (7) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} \stackrel{r_3^{(-1)}}{=} (7)(-1) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (7)(-1)(1)(1)(1) = -7$$

Determinants and elementary column operations

■ Theorem : (Elementary columns operations and determinants)

Let A and B be square matrices.

$$(a) \ B = c_{ij}(A) \Rightarrow \det(B) = -\det(A) \quad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b) \ B = c_i^{(k)}(A) \Rightarrow \det(B) = k \det(A) \quad (\text{i.e. } |c_i^{(k)}(A)| = k|A|)$$

$$(c) \ B = c_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A) \quad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

■ Ex 3:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \Rightarrow \det(A_1) = \det(c_1^{(\frac{1}{2})}(A)) = \frac{1}{2} \det(A) = \left(\frac{1}{2}\right)(-8) = -4$$

$$A_2 = c_{12}(A) \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- **Theorem : (Conditions that yield a zero determinant)**

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$

- (a) An entire row (or an entire column) consists of zeros.
- (b) Two rows (or two columns) are equal.
- (c) One row (or column) is a multiple of another row (or column).

■ Ex 4:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

■ Ex 5: (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \stackrel{C_{13}^{(2)}}{=} \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = (-3)(-1)^{3+1} \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = (-3)(-1) = 3$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \stackrel{R_{12}^{(\frac{4}{5})}}{=} \begin{vmatrix} -3 & 5 & 2 \\ -2/5 & 0 & 3/5 \\ -3 & 0 & 6 \end{vmatrix} = (5)(-1)^{1+2} \begin{vmatrix} -2/5 & 3/5 \\ -3 & 6 \end{vmatrix} = (-5)(-\frac{3}{5}) = 3$$

■ Ex 6: (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \begin{matrix} r_{24}^{(1)} \\ = \\ r_{25}^{(-1)} \end{matrix} = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1)(-1)^{2+2} \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$c_{41}^{(-3)} = \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4} \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} r_{21}^{(1)} = \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3} \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

Properties of Determinants

- **Theorem : (Determinant of a matrix product)**

$$\det (AB) = \det (A) \det (B)$$

- **Notes:**

$$(1) \det (EA) = \det (E) \det (A)$$

$$(2) \det (A + B) \neq \det (A) + \det (B)$$

$$(3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

■ Ex 1: (Determinant of a matrix product)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7, \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

Check:

$$|AB| = |A| |B|$$

$$-77 = -7 \times 11$$

- **Theorem : (Determinant of a scalar multiple of a matrix)**

If A is an $n \times n$ matrix and c is a scalar, then

$$\det (cA) = c^n \det (A)$$

- **Ex 2:**

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5 \quad \text{Find } |A|$$

Sol:

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- **Theorem : (Determinant of an invertible a matrix)**

A square matrix A is invertible (nonsingular) if and only if $\det (A) \neq 0$

- **Ex 3: (Classifying square matrices as singular or nonsingular)**

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \Rightarrow A$ has no inverse (it is singular).

$|B| = -12 \neq 0 \Rightarrow B$ has an inverse (it is nonsingular).

- **Theorem : (Determinant of an inverse matrix)**

If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

- **Theorem : (Determinant of a transpose)**

If A is a square matrix, then $\det(A^T) = \det(A)$.

- **Ex 4:**

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad (a) \quad |A^{-1}| = ? \quad (b) \quad |A^T| = ?$$

Sol:

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 \quad \Rightarrow \quad |A^{-1}| = \frac{1}{|A|} = \frac{1}{4}, \quad |A^T| = |A| = 4$$

- **Equivalent conditions for a nonsingular matrix:**

If A is an $n \times n$ matrix, then the following statements are equivalent.

- (1) A is invertible.
- (2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix \mathbf{b} .
- (3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- (4) A is row-equivalent to I_n
- (5) A can be written as the product of elementary matrices.
- (6) $\det(A) \neq 0$

- **Ex 5: Which of the following system has a unique solution?**

$$(a) \quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b) \quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

(a) $A\mathbf{x} = \mathbf{b} \Rightarrow |A| = 0$. This system does not have a unique solution.

(b) $B\mathbf{x} = \mathbf{b} \Rightarrow |B| = -12 \neq 0$. This system has a unique solution.

Applications of Determinants

- **The Adjoint of a Matrix** $\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$

- **The Inverse of a Matrix Using its Adjoint**

If A is an $n \times n$ invertible matrix, then $A^{-1} = \frac{1}{\det A} \text{adj}(A)$

- **Ex 1: Find the inverse of a matrix using its adjoint**

Use the adjoint of A to find A^{-1} , where $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

Sol:

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{vmatrix} = 3$$

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{3} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{bmatrix}$$

■ Cramer's Rule

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$A\mathbf{x} = \mathbf{b}, \quad A = [a_{ij}]_{n \times n} = [A^{(1)}, A^{(2)}, \dots, A^{(n)}], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0 \quad (\text{this system has a unique solution})$$

$$A_j = \left[A^{(1)}, A^{(2)}, \dots, A^{(j-1)}, \mathbf{b}, A^{(j+1)}, \dots, A^{(n)} \right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & \mathbf{b}_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & \mathbf{b}_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & \mathbf{b}_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

(i.e. $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$)

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \quad j = 1, 2, \dots, n$$

- Ex 2: Use Cramer's rule to solve the system of linear equations

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x \quad \quad + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10$$

$$\det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15,$$

$$\det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5}, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2}, \quad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

- **Determinants as Area or Volume**

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is

$$\text{Area} = |\det(A)|$$

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is

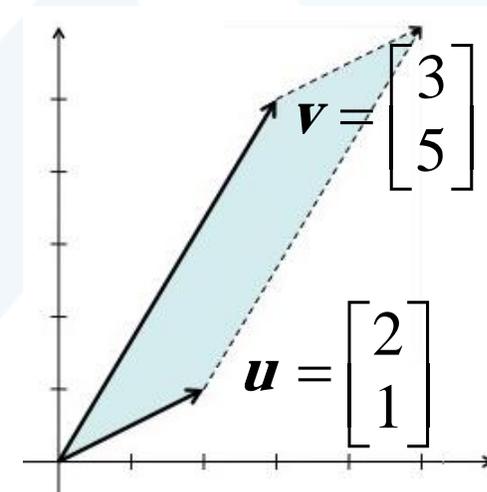
$$\text{Volume} = |\det(A)|$$

- **Ex 3:**

Find area of parallelogram formed by $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Sol:

$$\text{Area} = |\det(A)| = \left| \det \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \right| = |10 - 3| = 7$$



■ **Ex 4:**

Find volume of parallelepiped formed by $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Sol:

$$\text{Volume} = |\det(A)| = \left| \det \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix} \right| = |2 + 3 + 4 - (1 + 1 + 24)| = 17$$