Problem sets 7 : Eigenvalues- Eigenvectors

CEDC102: Linear Algebra and Matrix Theory

Manara University

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Problem 1

Suppose that A is a 3×3 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$ and corresponding eigenvectors x_1, x_2, x_3.

(a) Give eigenvectors and eigenvalues of $(A^2 - 3A + 4I)^{-1}$

(b) For what value(s), if any, of the scalar μ is $B = A^2 - 3A + \mu I$ **singular**, which corresponds to B having one or more eigenvalues equal to ______.

(c) $A^n x$ for large n is very nearly parallel to ______ unless x is ______.

Problem 2

Consider the following recurrence:

$$f_n = \frac{f_{n-2} - f_{n-1}}{2}$$

Suppose that we start it with $f_0 = 0$ and $f_1 = 1$. Then the first few terms in the sequence are:

$$f_0, f_1, f_2, \ldots = 0, 1, -\frac{1}{2}, \frac{3}{4}, -\frac{5}{8}, \frac{11}{16}, -\frac{21}{32}, \frac{43}{64}, -\frac{85}{128}, \frac{171}{256}, -\frac{341}{512}, \frac{683}{1024}, -\frac{1365}{2048}, \frac{2731}{4096}, \ldots$$

(a) Similar to the Fibonacci sequence from class, we can write this recurrence in matrix form:

$$\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = A \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix}$$

for what matrix A?

(b) Therefore, write a formula for $f_n = \_\_^T A^n \_\_$: fill in the blanks with two column vectors.

(c) Find the eigenvalues of A and corresponding eigenvectors.

(d) From (c), what should the ratio f_n/f_{n-1} approach for large n? Check that your prediction matches what the terms in the sequence above appear to be doing.

(e) Give an *exact* formula for f_n, using (b) and writing ___ in the basis of ___.

Problem 3

In class, we saw that $\boldsymbol{o} = [1, 1, \ldots, 1, 1]$ is an eigenvector of M^T with eigenvalue $\lambda = 1$ for any Markov matrix M.

(a) If x_k is an eigenvector of M ($Mx_k = \lambda_k x_k$) for any *other* eigenvalue $\lambda_k \neq 1$ of M, show that we must have $o^T x_k = 0$: it must be *orthogonal* to o. (Hint: use $o^T = o^T M$.)

(b) If we expand an arbitrary x in an eigenvector basis $x = c_1 x_1 + \cdots + c_m x_m$, letting x_m be a steady-state eigenvector ($\lambda_m = 1$) and supposing all of the other eigenvalues are $\neq$ show that $o^T x$ gives us a simple formula for $c_m =$ ________.

(c) Hence, if all other eigenvalues have magnitude < 1, then $M^n x \to$ ________ (simple formula in o, x, x_m) as $n \to \infty$.

Problem 4

Consider $A = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}$ and $B = \begin{pmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{pmatrix}$. For this problem you keep in mind the diagonalization of matrices like A and B.

(a) Which of A^n or B^n (or both, or neither) go $\rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as $n \rightarrow \infty$?

(b) For what values of the real scalar μ is $\sqrt{A - \mu I}$ a real matrix?

Problem 5

Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the ______ eigenvectors as A. Its eigenvalues are ______ by 1.

Problem 6

The eigenvalues of A equal the eigenvalues of A^T. This is because $\det(A - \lambda I)$ equals $\det(A^T - \lambda I)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are *not* the same.

Problem 7

If A has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $X\Lambda X^{-1}$ to find A. No other matrix has the same λ's and x's.

Problem 8

Suppose that M is a diagonalizable $m \times m$ Markov matrix with all positive entries. For each of the following, say whether the ODE solutions at large times t are expected to be **exponentially growing**, **exponentially decaying**, **oscillating forever**, or **approaching a nonzero constant**, for a randomly chosen initial condition $x(0)$ — or **multiple possibilities depending on M**. Justify your answers

a. $\frac{dx}{dt} = Mx$
b. $\frac{dx}{dt} = (M - I)x$
c. $\frac{dx}{dt} = (M^2 - M - I)x$

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Problem 9

In class we showed that, for a complex-conjugate pair of eigenvalues $\lambda_1 = a + ib$ and $\lambda_2 = \overline{\lambda_1} = a - ib$, eigenvectors x_1 and $x_2 = \overline{x_1}$, and (scalar) coefficients c_1 and $c_2 = \overline{c_1}$, we can write

$$c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{\lambda_1 t} x_1 + \overline{c_1 e^{\lambda_1 t} x_1} = 2\mathrm{Re}\left[c_1 e^{\lambda_1 t} x_1\right] = 2e^{at}\mathrm{Re}\left[c_1 e^{ibt} x_1\right]$$

which turned into a vector of terms proportional to $re^{at}\cos(bt+\phi)$, where the amplitude r and phase ϕ depended on the coefficient c_1 and the eigenvector x_1 components.

Derive that we can **alternatively** write

$$c_1 e^{\lambda_1 t} x_1 + \overline{c_1 e^{\lambda_1 t} x_1} = e^{at}\left(v_1\cos(bt) + v_2\sin(bt)\right)$$

for some vectors $v_1 =$ _____, $v_2 =$ _____ in terms of c_1 and x_1. (Hint: break $c_1 x_1$ into its real and imaginary parts, i.e. write $c_1 x_1 = (\text{real part}) + i(\text{imag part})$.)

Problem 10

$\frac{dx}{dt} = Ax$ has the solution

$$x(t) = v_1 e^{-3t}\cos(2t) + v_2 e^{-t} + v_3 e^{-3t}\sin(2t)$$

for some nonzero real constant vectors v_1, v_2, v_3, and some initial condition $x(0)$. Help construct A, v_1, v_2, v_3, and $x(0)$:

(a) Write down a numerical formula for a possible real matrix A such that A is as small in size as possible and where A contains no zero entries.

Your formula can be left as a product of some matrices and/or matrix inverses you don't need to multiply them out or invert any matrices, but you should give possible numeric values for all of the matrices in your formula. (You don't need to explicitly check that your A has no zero entries as long as zero entries seem unlikely. e.g. the inverse of a matrix with no special structure probably has no zero entries. It wouldn't hurt to check in Julia, however.)

(b) Using the numbers that you chose from the formula in your previous part, give possible corresponding (numeric) values for $x(0)$, v_1, v_2, and v_3.