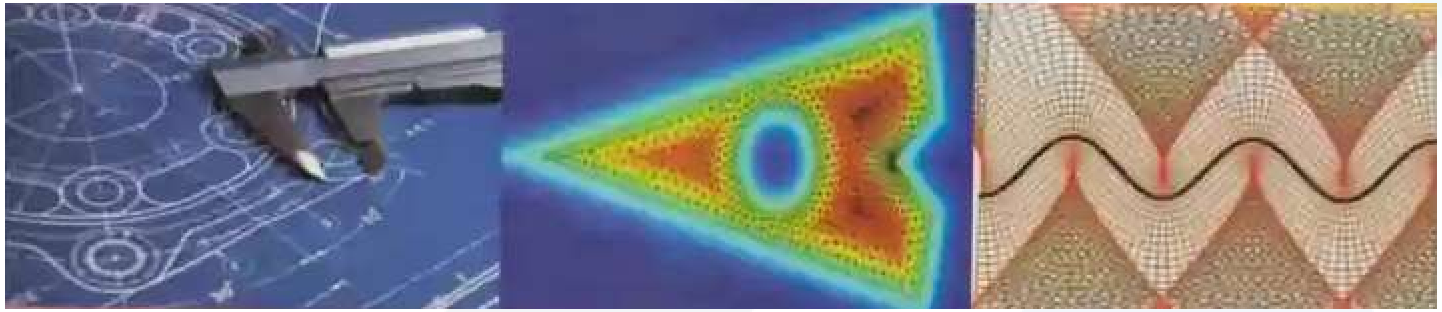
CEDC301: Engineering Mathematics

Lecture Notes 6: Laplace Transform: Part A



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Chapter 4

Laplace Transform

1. Definition of the Laplace Transform
2. The Inverse Transform and Transforms of Derivatives
3. Translation Theorems
4. Additional Operational Properties
5. The Dirac Delta Function
6. Systems of Linear Differential Equations

1. Definition of the Laplace Transform

- Laplace transform offer simple and efficient strategies for solving many science and engineering problems, including: control systems; signal processing; mechanical networks; electrical networks and communications systems.
- The purpose of the Laplace Transform is to transform ordinary differential equations (ODEs) into algebraic equations, which makes it easier to solve.
- The Laplace Transform is a generalized Fourier Transform, since it allows one to obtain transforms of functions that have no Fourier Transforms.
- One of the advantages of using the Laplace Transform to solve differential equations is that all initial conditions are automatically included during the process of transformation, so one does not have to find the homogeneous solutions and the particular solution separately.

- More importantly, the use of the unit step function (Heaviside function) and Dirac's delta make the Laplace transform particularly powerful for problems with inputs that have discontinuities or represent short impulses or complicated periodic functions.

Solving an IVP by Laplace transforms

- Definition: The Laplace transform of a function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int f(t)e^{-st}dt$$

provided that the integral converges.

where $s = \sigma + i\omega$, the independent variable of the transform.

- There are two important variants:

  Unilateral (or one-sided): $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt;$

  Bilateral (or two sided): $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt;$

- When we refer to Laplace transform (LT) without the qualifier word "bilateral" or "unilateral", we will always imply the unilateral LT.
- Note: The unilateral Laplace transform is applied for functions that are defined for $t \geq 0$.
- Example 1: Laplace transform

  Evaluate $\mathcal{L}\{1\}$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} e^{-st}dt = \lim_{a\to\infty}\int_{0^-}^{a} e^{-st}dt = \frac{1}{s}, \qquad Re\{s\} > 0$$

- Example 2: Laplace transform

  Evaluate $\mathcal{L}\{e^{at}\}$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} e^{at}e^{-st}dt = \frac{1}{s-a}, \qquad Re\{s\} > a$$

- Example 3: Laplace transform

  Evaluate $\mathcal{L}\{\sin 2t\}$

$$\mathcal{L}\{\sin 2t\} = \int_{0^-}^{\infty} \sin 2te^{-st}dt = \left.\frac{-e^{-st}\sin 2t}{s}\right|_{0^-}^{\infty} + \frac{2}{s}\int_{0^-}^{\infty} \cos 2te^{-st}dt$$

$$= \frac{2}{s}\int_{0^-}^{\infty} \cos 2te^{-st}dt, \qquad Re\{s\} > 0$$

$$= \frac{2}{s}\left[\left.\frac{-e^{-st}\cos 2t}{s}\right|_{0^-}^{\infty} - \frac{2}{s}\int_{0^-}^{\infty} \sin 2te^{-st}dt\right] = \frac{2}{s^2} - \frac{4}{s^2}\mathcal{L}\{\sin 2t\}$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}, \quad Re\{s\} > 0$$

Linearity of the Laplace Transform

- Theorem 1 (Linearity of the Laplace Transform): The LT is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b, the Laplace transform of $af + bg$ exists, and

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

- Example 4: Laplace transform of $\sin$ and $\cos$

$$\sin \omega t = \tfrac{1}{2i}\left[e^{i\omega t} - e^{-i\omega t}\right] \Rightarrow \mathcal{L}\{\sin \omega t\} = \tfrac{1}{2i}\left[\mathcal{L}\{e^{i\omega t}\} - \mathcal{L}\{e^{-i\omega t}\}\right]$$

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{2i}\left[\frac{1}{s-\omega i} - \frac{1}{s+\omega i}\right] = \frac{\omega}{s^2+\omega^2}, \quad Re\{s\} > 0$$

$$\cos\omega t = \frac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right] \Rightarrow \mathcal{L}\{\cos\omega t\} = \frac{1}{2}\left[\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}\right]$$

$$\mathcal{L}\{\cos\omega t\} = \frac{1}{2}\left[\frac{1}{s-\omega i} + \frac{1}{s+\omega i}\right] = \frac{s}{s^2+\omega^2}, \qquad Re\{s\} > 0$$

Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

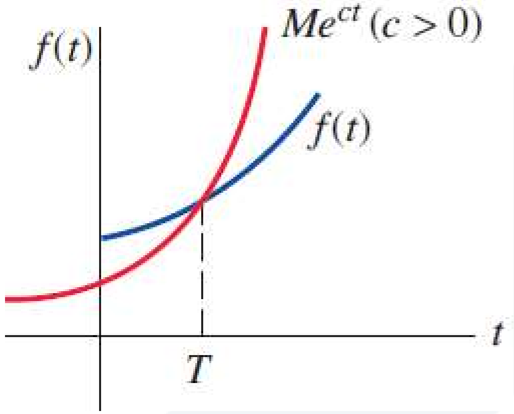
	$f(t)$	$\mathcal{L}(f)$	
1	1	$1/s$	$Re\{s\} > 0$
2	$t^n,\ n = 0, 1, 2, \ldots$	$n!/s^{n+1}$	$Re\{s\} > 0$
3	$t^a,\ a > 0$	$\Gamma(a+1)/s^{a+1}$	$Re\{s\} > 0$
4	e^{at}	$1/(s-a)$	$Re\{s\} > a$
5	$t^n e^{at}$	$n!/(s-a)^{n+1}$	$Re\{s\} > a$

	$f(t)$	$\mathcal{L}(f)$	
6	$\cos \omega t$	$s/(s^2 + \omega^2)$	$Re\{s\} > 0$
7	$\sin \omega t$	$\omega/(s^2 + \omega^2)$	$Re\{s\} > 0$
8	$\cosh at$	$s/(s^2 - a^2)$	$Re\{s\} > 0$
9	$\sinh at$	$a/(s^2 - a^2)$	$Re\{s\} > 0$
10	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$	$Re\{s\} > a$
11	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$	$Re\{s\} > a$

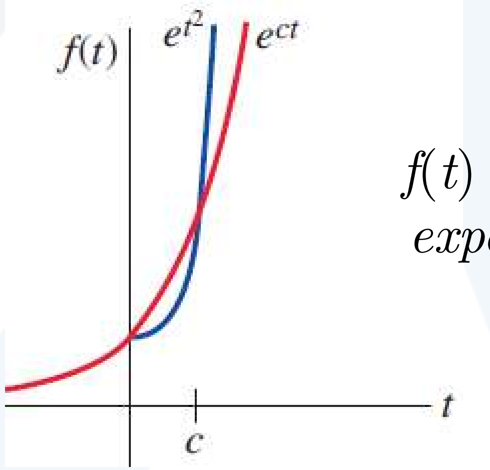
Γ is the so-called gamma function: $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt, \ a > 0$

Existence of Laplace Transforms

- Definition: A function f is said to be of exponential order if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.
- If f is an increasing function, then the condition $|f(t)| \leq Me^{ct}$ for all $t > T$, simply states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}, where c is a positive constant.



Function f is of exponential order



$f(t) = e^{t^2}$ *is not of exponential order*

- Theorem 2 (Sufficient Conditions for Existence): If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

2. The Inverse Transform and Transforms of Derivatives

Inverse Transforms

- If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the inverse Laplace transform of $F(s)$ and write $f(t) = \mathcal{L}^{-1}\{F(s)\}$. For example:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t, \quad \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

- Note: Partial fractions play an important role in finding inverse Laplace transforms.

▪ Example 5: Partial Fractions and Linearity

Evaluate $\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Transforms of Derivatives

- Theorem 3 (Transform of a Derivative): If f, f', ..., $f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$$

where $F(s) = \mathcal{L}\{f(t)\}$.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0^-) - f'(0^-)$$

Solving Linear ODEs

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t), \quad y(0^-) = y_0,\ y'(0^-) = y_1,\ \ldots,\ y^{(n-1)}(0^-) = y_{n-1}$$

where the coefficients a_i, $i = 0, 1, ..., n$ and $y_0, y_1, . . . , y_{n-1}$ are constants.

$$a_n \mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1}\mathcal{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$a_n[s^n Y(s) - s^{n-1}y(0^-) - \cdots - y^{(n-1)}(0^-)]$$
$$+a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0^-) - \cdots - y^{(n-2)}(0^-)] + \cdots + a_0 Y(s) = G(s)$$

where $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$.

- Note: The Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$.

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}, \quad P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0, \quad \deg Q(s) \le n-1$$

- Example 6: Solving a First-Order IVP

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t, \qquad y(0) = 6$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\}$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4} \Rightarrow Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)}$$

$$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$y(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

- Example 7: Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}, \quad y(0^-) = 1, \quad y'(0^-) = 5$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\left\{e^{-4t}\right\}$$

$$s^2Y(s) - sy(0^-) - y'(0^-) - 3[sY(s) - y(0^-)] + 2Y(s) = \frac{1}{s+4}$$

$$\Rightarrow Y(s) = \frac{s+2}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

$$y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

- Note: The next theorem indicates that not every arbitrary function of s is a Laplace transform of a piecewise-continuous function of exponential order.

- Theorem 4 (Behavior of $F(s)$ as $s \to \infty$): If f is piecewise continuous on $[0, \infty)$ and of exponential order, then:

$$\lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0$$

- The Laplace transform is well adapted to linear dynamical systems

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\} + \mathcal{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\} = y_0(t) + y_1(t)$$

- If the input is $g(t) = 0$, then the solution of the problem is $y_0(t) = \mathcal{L}^{-1}\{Q(s)/P(s)\}$. This solution is called the zero-input response of the system.

- The function $y_1(t) = \mathcal{L}^{-1}\{G(s)/P(s)\}$ is the output due to the input $g(t)$.
- If the initial state of the system is the zero state (all the initial conditions are zero), then $Q(s) = 0$, and so the only solution of the initial-value problem is $y_1(t)$, which is called the zero-state response of the system.
- $y_0(t)$ is a solution of the IVP consisting of the associated homogeneous equation with the given initial conditions, and $y_1(t)$ is a solution of the IVP consisting of the nonhomogeneous equation with zero initial conditions.
- In example 7:

$$y_0(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)(s-2)}\right\} = -3e^{t} + 4e^{2t}$$

$$y_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s+4)}\right\} = -\frac{1}{5}e^{t} + \frac{1}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

3. Translation Theorems

Translation on the s-axis

- **Theorem 5 (First Translation Theorem):** If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

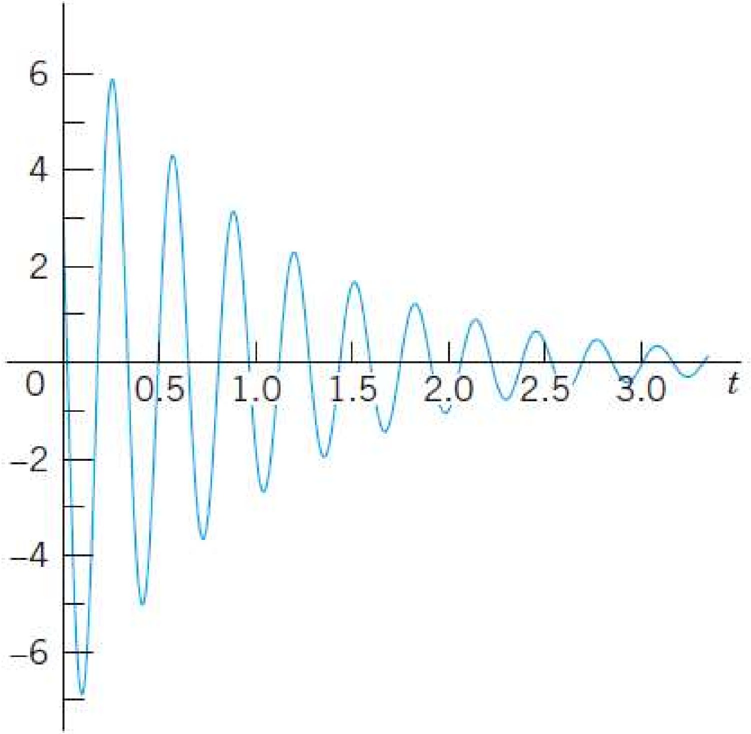
$$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \quad \text{or} \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

- **Example 8:** Damped Vibrations

Find the inverse of the transform $L\{f(t)\} = \dfrac{3s-137}{s^2+2s+401}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7\mathcal{L}^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}$$

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t)$$



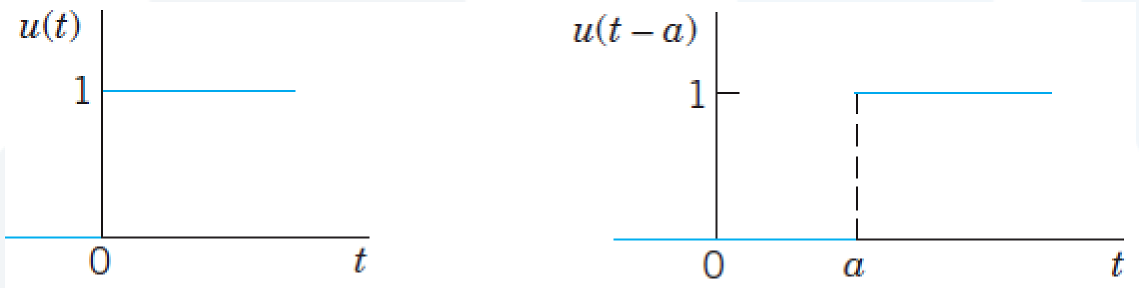
- We shall now reach the point where the Laplace transform shows its real power in applications. We shall introduce two auxiliary functions, the unit step function or Heaviside function $u(t-a)$ and Dirac's delta $\delta(t-a)$.

- These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant.

Unit Step Function

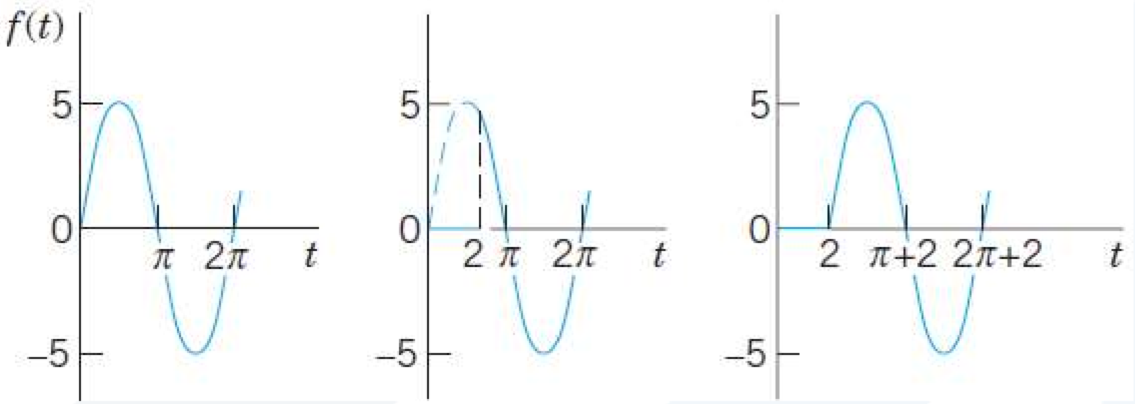
- Definition: The unit step function $u(t-a)$ is defined to be

$$u(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}, \qquad a\geq 0$$

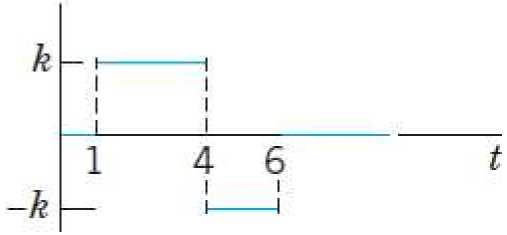


$$\mathcal{L}\{u(t-a)\} = \int_{0^-}^{\infty} u(t-a)e^{-st}dt = \int_{a}^{\infty} e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_a^{\infty} = \frac{e^{-as}}{s}, \qquad Re\{s\} > 0$$

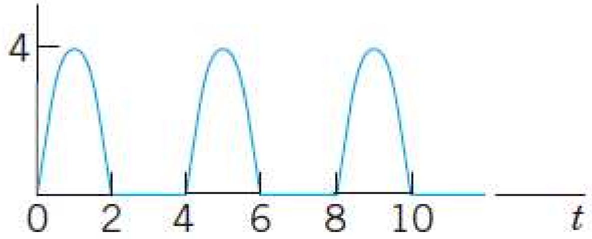
- Note: Let $f(t) = 0$ for all negative t. Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.



$f(t) = 5 \sin t$ $\quad f(t)u(t-2)$ $\quad f(t-2)u(t-2)$



$k[u(t-1)-2u(t-4)+u(t-6)]$



$4\sin(\frac{1}{2}\pi t)[u(t)-u(t-2)+u(t-4)-u(t-6)+\cdots]$

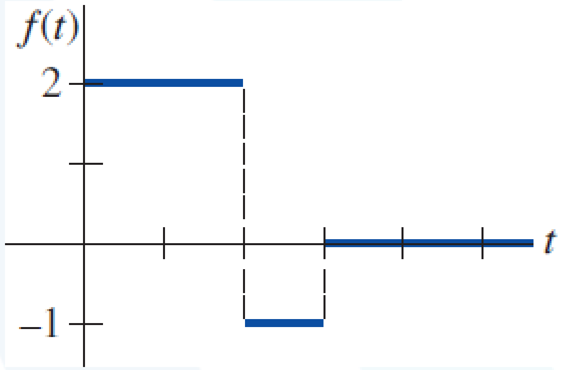
Translation on the t-axis

- **Theorem 6 (Second Translation Theorem):** If $\mathcal{L}\{f(t)\}=F(s)$ and $a>0$, then

$$\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s) \quad \text{or} \quad f(t-a)u(t-a)=\mathcal{L}^{-1}\{e^{-as}F(s)\}$$

- **Example 9:** Second Translation Theorem

  Find the Laplace transform of the function f whose graph is given in the figure.



$$f(t) = 2 - 3u(t-2) + u(t-3)$$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 3\mathcal{L}\{u(t-2)\} + \mathcal{L}\{u(t-3)\}$$

$$= \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

- Example 10: An Initial-Value Problem

Solve $y' + y = f(t)$, $y(0^-) = 5$, where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 3\mathcal{L}\{\cos t\, u(t-\pi)\} = -3\mathcal{L}\{\cos(t-\pi)\, u(t-\pi)\}$$

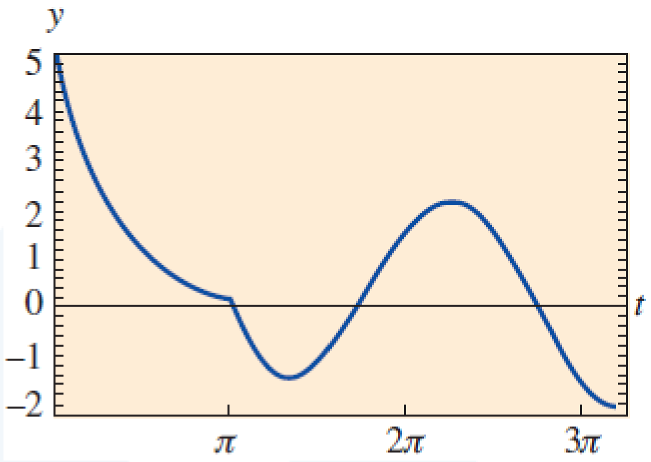
$$sY(s) - y(0^-) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s} \Rightarrow Y(s) = \frac{5}{s+1} - \frac{3s}{(s+1)(s^2+1)}e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]$$

$$y(t) = 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}u(t-\pi) - \frac{3}{2}\sin(t-\pi)u(t-\pi) - \frac{3}{2}\cos(t-\pi)u(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]u(t-\pi)$$

$$= \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}[e^{-(t-\pi)} + \sin t + \cos t], & t \ge \pi \end{cases}$$



4. Additional Operational Properties

Derivatives of Transforms

- **Theorem 7 (Derivatives of Transforms):** If $\mathcal{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, \ldots$, then:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

- Example 11: Using Theorem

  Evaluate $\mathcal{L}\{t \sin kt\}$

$$\mathcal{L}\{t \sin kt\} = -\frac{d}{ds}\mathcal{L}\{\sin kt\} = -\frac{d}{ds}\frac{k}{s^2+k^2} = \frac{2ks}{(s^2+k^2)^2}$$

Convolution

If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the convolution of f and g, denoted by the symbol $f * g$, is a function defined by the integral:

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\sin t * \sin t = \int_0^t \sin(\tau)\sin(t-\tau)\,d\tau = \frac{1}{2}\int_0^t [\cos(2\tau - t) - \cos(t)]\,d\tau$$

$$= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$$

- Example 12: Convolution of Two Functions

  Evaluate (a) $e^t * \sin t$ (b) $\mathcal{L}\{e^t * \sin t\}$

  (a) $e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)d\tau = \frac{1}{2}(-\sin t - \cos t + e^t)$

  (b) $\mathcal{L}\{e^t * \sin t\} = -\frac{1}{2}\frac{1}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s-1} = \frac{1}{(s-1)(s^2+1)}$

Properties of Convolution

- Is commutative. For any two functions f and g, $f * g = g * f$.
- Is associative. For any functions f, g, and h, $(f * g) * h = f * (g * h)$.
- Is distributive with respect to addition. For any functions f, g, and h, $f * (g + h) = f * g + f * h$.

- Theorem 8 (Convolution Theorem): If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

Transform of an Integral

- When $g(t) = 1$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

- Example 13: An Integral Equation

Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau$

$$\int_0^t f(\tau)e^{t-\tau}d\tau = f(t) * e^t$$

$$F(s) = 3\frac{2}{s^3} - \frac{1}{s+1} - F(s)\frac{1}{s-1} \Rightarrow F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

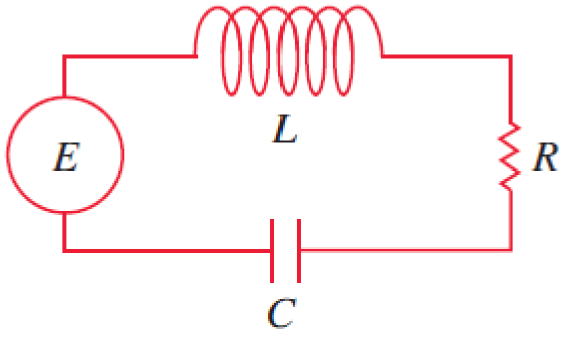
$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}$$

Series Circuits

- The voltage drops across an inductor, resistor, and capacitor are, respectively,

$$v_L = L\frac{di(t)}{dt}, \quad v_R = Ri(t), \quad v_C = \frac{1}{C}\int_0^t i(\tau)d\tau$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$



- Example 14: An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit when $L = 0.1$ H, $R = 2\ \Omega$, $C = 0.1$ F, $i(0^-) = 0$, and the impressed voltage is $E(t) = 120t - 120t\,u(t-1)$.

$$0.1\frac{di(t)}{dt} + 2i(t) + 10\int_0^t i(\tau)d\tau = 120t - 120t\,u(t-1)$$

$$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right]$$

$$I(s) = 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right]$$

$$I(s) = 1200\left[\frac{1/100}{s} - \frac{1/100}{s+10} - \frac{1/10}{(s+10)^2} - \frac{1/100}{s}e^{-s}\right.$$

$$+\frac{1/100}{s+10}e^{-s}+\frac{1/10}{(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\Bigg]$$

$$i(t)=12[1-u(t-1)]-12[e^{-10t}-e^{-10(t-1)}u(t-1)]$$

$$-120te^{-10t}-1080(t-1)e^{-10(t-1)}u(t-1)$$