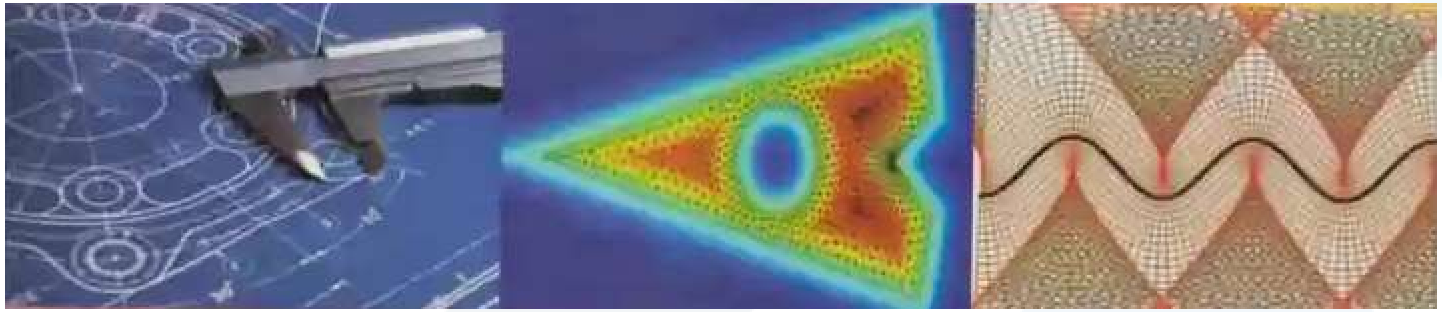
CEDC301: Engineering Mathematics

Lecture Notes 7: Laplace Transform: Part B



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Chapter 4

Laplace Transform

Transform of a Periodic Function

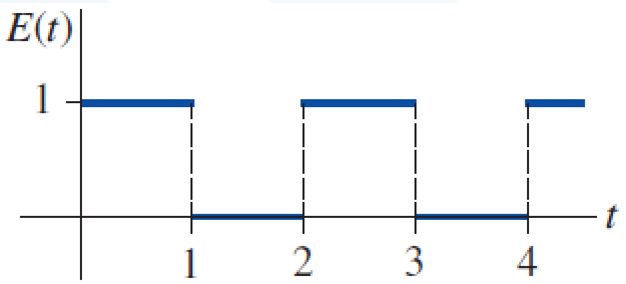
- **Theorem 9 (Transform of a Periodic Function):** If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_{0^-}^{T} e^{-st} f(t)\, dt$$

- **Example 15:** Transform of a Periodic Function

 Find the Laplace transform of the periodic function shown in the figure

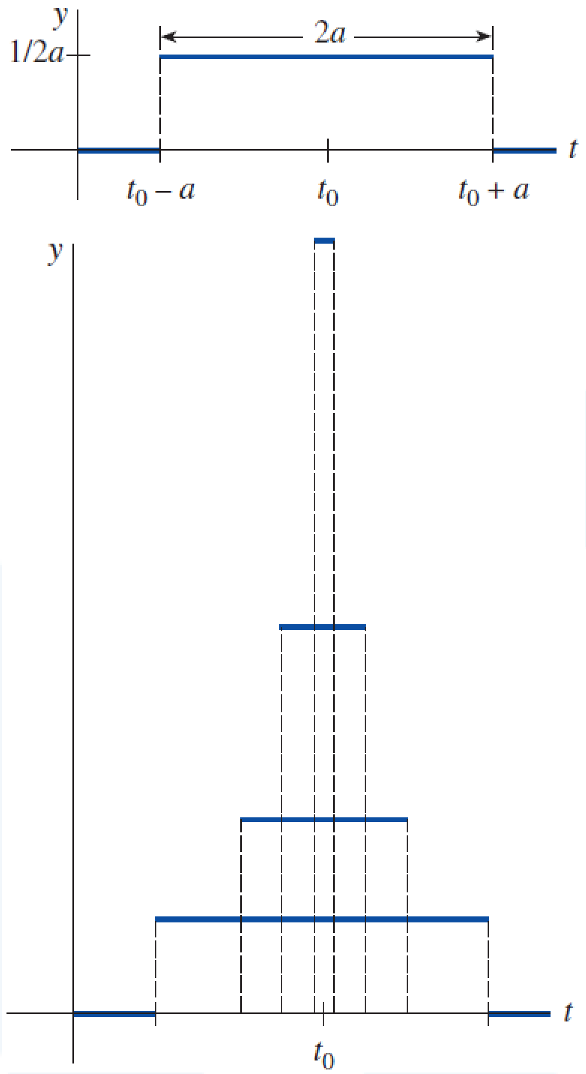
 The function $E(t)$ is called a square wave and has period $T = 2$. For $0 \leq t < 2$, $E(t)$ can be defined by:



$$E(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \end{cases}$$

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}}\int_{0^-}^{2} e^{-st}E(t)\,dt = \frac{1}{1-e^{-2s}}\int_{0^-}^{1} e^{-sT}\,dt$$

$$= \frac{1}{1-e^{-2s}}\,\frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$



5. The Dirac Delta Function

- We shall see that there does indeed exist a function, or more precisely a generalized function, whose Laplace transform is $F(s) = 1$.
- Mechanical systems are often acted on by an external force (or emf in an electrical circuit) of large magnitude that acts only for a very short period of time.

- The graph of the piecewise-defined function

$$\delta_a(t-t_0)=\begin{cases}0, & 0\le t<t_0-a\\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a\\ 0, & t\ge t_0+a\end{cases}$$

$a>0,\ t_0>0$, shown before could serve as a model for such a force.

- The function $\delta_a(t-t_0)$ is called a unit impulse since it possesses the integration property

$$\int_0^\infty \delta_a(t-t_0)\,dt=1$$

- The Dirac Delta Function $\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$

$$\delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-t_0)\,dt=1$$

$$\int_0^\infty f(t)\delta(t-t_0)\,dt = f(t_0)$$ sifting property

- **Theorem 10 (Transform of Dirac Delta Function):** For $t_0 > 0$,

$$\mathcal{L}\{\delta(t-t_0)\}=e^{-st_0}, \qquad \mathcal{L}\{\delta(t)\}=1$$

- **Example 16:** Two Initial-Value Problems

  Solve $y'' + y = 4\,\delta(t-2\pi)$ subject to

  (a) $y(0^-)=1$, $y'(0^-)=0$ (b) $y(0^-)=0$, $y'(0^-)=0$.

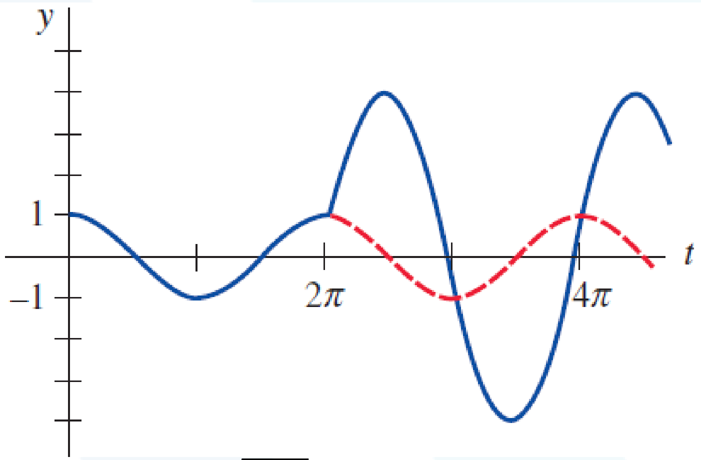
  The two initial-value problems could serve as models for describing the motion of a mass on a spring moving in a medium in which damping is negligible.

At $t = 2\pi$ the mass is given a sharp blow. In part (a) the mass is released from rest 1 unit below the equilibrium position. In part (b) the mass is at rest in the equilibrium position.

(a) $s^2Y(s) - s + Y(s) = 4e^{-2\pi s} \Rightarrow Y(s) = \dfrac{s}{s^2+1} + \dfrac{4e^{-2\pi s}}{s^2+1}$

$$y(t) = \cos t + 4\sin(t-2\pi)u(t-2\pi)$$

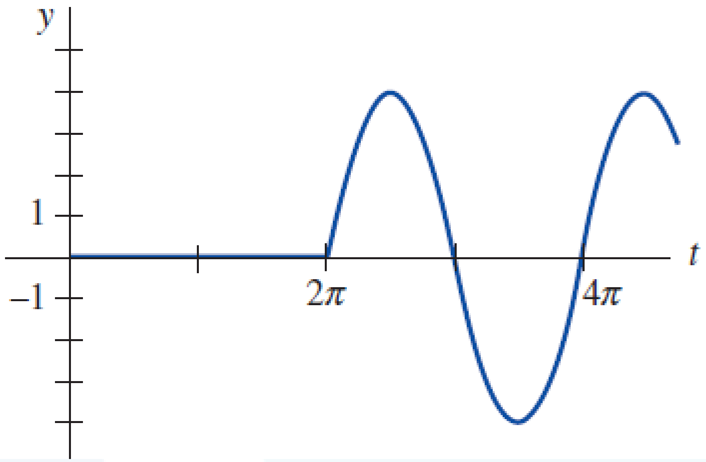
$$y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}$$



The mass is exhibiting simple harmonic motion until it is struck at $t = 2\pi$. The influence of the unit impulse is to increase the amplitude of vibration to $\sqrt{17}$ for $t > 2\pi$.

(b) $Y(s) = \dfrac{4e^{-2\pi s}}{s^2+1} \Rightarrow y(t) = 4\sin(t-2\pi)u(t-2\pi)$

$$y(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ 4\sin t, & t \ge 2\pi \end{cases}$$

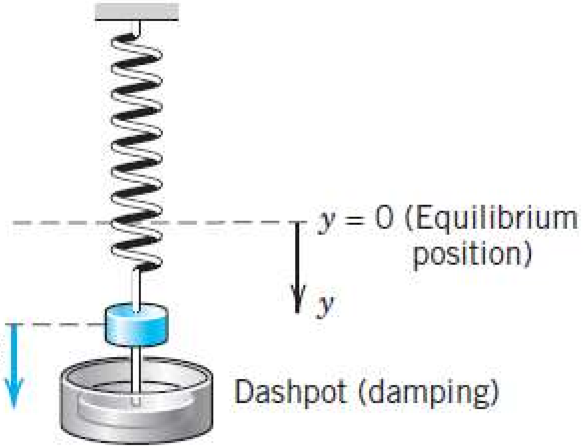


The mass exhibits no motion until it is struck at $t = 2\pi$.

- Example 17: Damped Forced Vibrations

  Solve the IVP for a damped mass-spring system acted upon by a sinusoidal force for some time interval.

  $y'' + 2y' + 2y = r(t)$, $r(t) = 10\sin 2t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0^-) = 1$, $y'(0^-) = -5$.



Mechanical system

$$(s^2Y(s)-s+5)+2(sY(s)-1)+2Y(s)=10\frac{2}{s^2+4}(1-e^{-\pi s})$$

$$Y(s)=\frac{20}{(s^2+4)(s^2+2s+2)}-\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}+\frac{s-3}{s^2+2s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{s-3}{s^2+2s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\}=e^{-t}(\cos t-4\sin t)$$

$$\frac{20}{(s^2+4)(s^2+2s+2)}=\frac{-2s-2}{s^2+4}+\frac{2(s-1)+6-2}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{20}{(s^2+4)(s^2+2s+2)}\right\}=-2\cos 2t-\sin 2t+e^{-t}(2\cos t+4\sin t)$$

$$y(t) = e^{-t}(\cos t - 4\sin t) - 2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t)$$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t, \qquad 0 < t < \pi$$

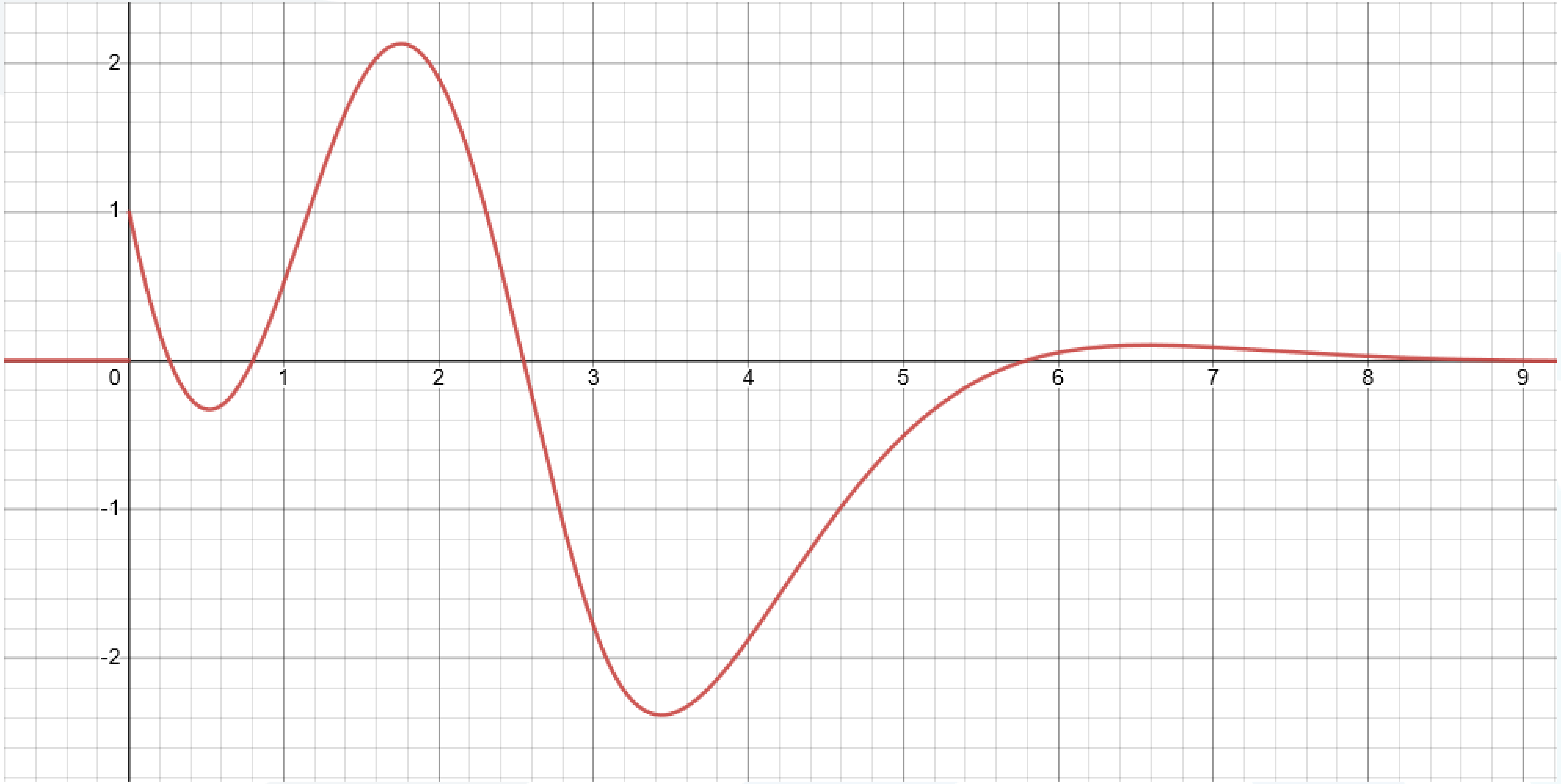
$$\mathcal{L}^{-1}\left\{-\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right\} = 2\cos(2t-2\pi) + \sin(2t-2\pi)$$

$$- e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)]$$

$$\mathcal{L}^{-1}\left\{-\frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)}\right\} = 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t)$$

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t + 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t)$$

$$y(t) = e^{-t}[(3+2e^{\pi})\cos t + 4e^{\pi}\sin t], \qquad t > \pi$$



2
1
-1
-2
0
1
2
3
4
5
6
7
8
9

- Theorem 11 (Initial Value Theorem): If $\mathcal{L}\{f(t)\} = F(s)$ and if the indicated limits exist, then:

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

- When F is known but f is not, the initial value theorem eliminates the need to explicitly find f in order to evaluate $f(0^+)$.

- Example 18: Initial value theorem

  Calculate the initial value of the function $f(t)$, whose Laplace transform is:

$$F(s) = \frac{2(s+1)}{(s+1)^2 + 5^2}$$

$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{2s(s+1)}{(s+1)^2 + 5^2} = 2$$

Verification:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+5^2}\right\} = 2e^{-t}\cos(5t) \Rightarrow$$

$$f(0^+) = f(0) = 2$$

- Theorem 12 (Final Value Theorem): If $\mathcal{L}\{f(t)\} = F(s)$ and if the indicated limits exist, then:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- When F is known but f is not, the final value theorem eliminates the need to explicitly find f in order to evaluate limit $_{t\to\infty} f(t)$.

- Example 19: Final value theorem

Calculate the final value of the function $f(t)$, whose Laplace transform is:

$$F(s) = \frac{s+2}{s(s+1)}$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s(s+2)}{s(s+1)} = \lim_{s\to 0} \frac{s+2}{s+1} = 2$$

Verification:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{s(s+1)}\right\} = (2 - e^{-t})$$

$$\Rightarrow \lim_{t\to\infty} f(t) = 2$$

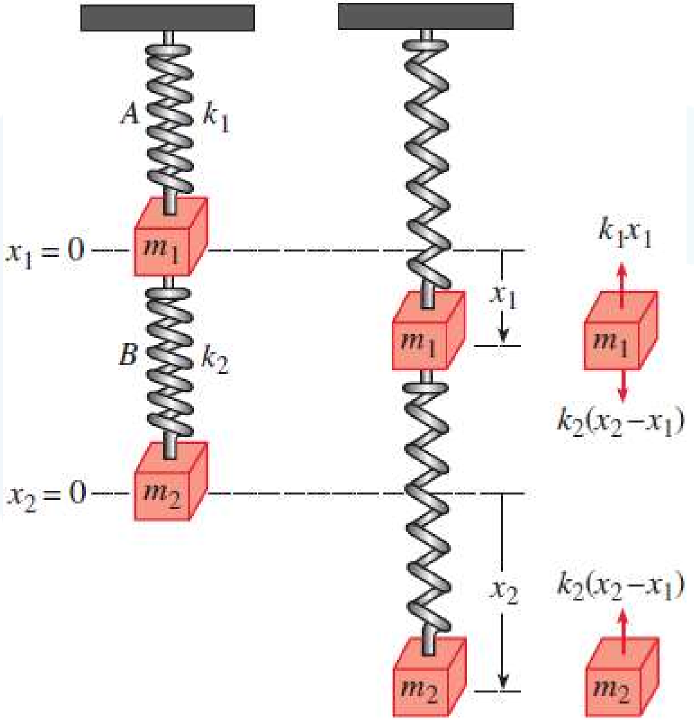
6. Systems of Linear Differential Equations

- When initial conditions are specified, the LT reduces a system of linear DEs with constant coefficients to a set of simultaneous algebraic equations in the transformed functions.

Coupled Springs

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1)$$

that describes the motion of two masses m_1 and m_2 in the coupled spring/mass system shown in the figure beside.



Equilibrium *Motion* *Forces*

- Example 20: Coupled Springs

Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$-4x_1 + x_2'' + 4x_2 = 0$$

$(k_1 = 6,\ k_2 = 4, m_1 = 1,$ and $m_2 = 1)$

subject to $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$

$$s^2X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) = 0$$
$$-4X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) = 0$$

$$(s^2 + 10)X_1(s) - 4X_2(s) = 1$$
$$-4X_1(s) + (s^2 + 4)X_2(s) = -1$$

Solving for $X_1(s)$ and using partial fractions on the result yields:

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

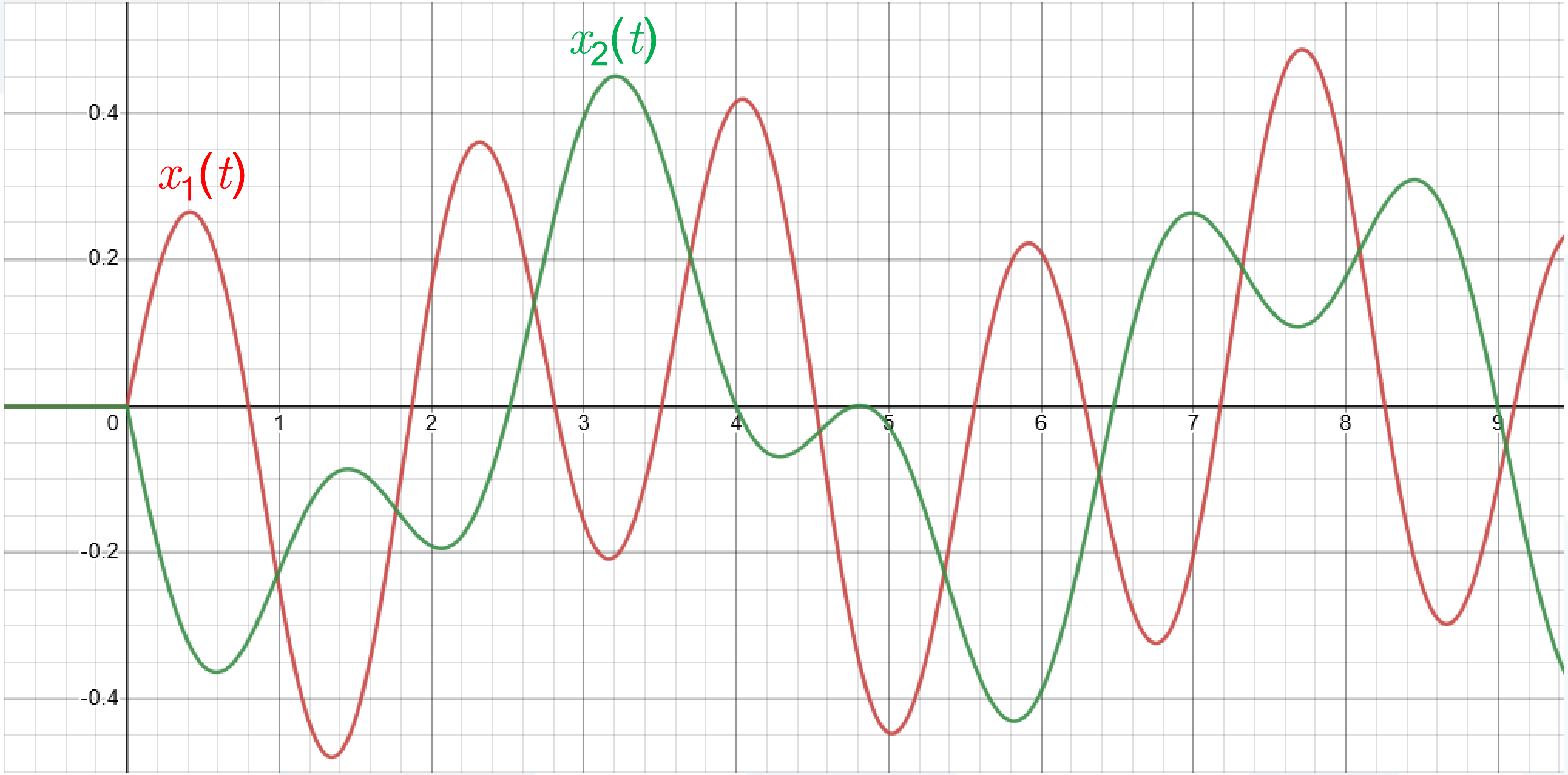
$$x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

Substituting the expression for $X_1(s)$ into the first equation

$$X_2(s) = \frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t$$

The motion of each mass is harmonic (the system is undamped!), being the superposition of a slow oscillation and a rapid oscillation.

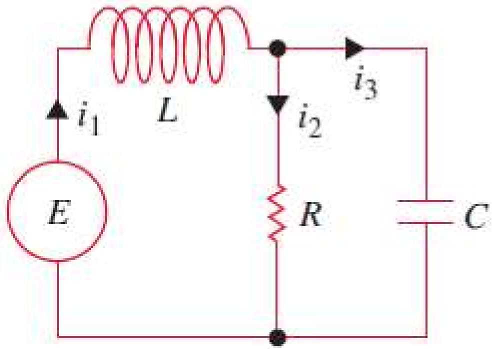


$x_1(t)$
$x_2(t)$
0.4
0.2
-0.2
-0.4
0
1
2
3
4
5
6
7
8
9

Networks

$$L\frac{di_1(t)}{dt} + Ri_2(t) = E(t)$$

$$RC\frac{di_2(t)}{dt} + i_2(t) - i_1(t) = 0$$



- Example 21: An Electrical Network

  Solve the system before under the conditions $E(t) = 60$ V, $L = 1$ H, $R = 50\ \Omega$, $C = 10^{-4}$ F, and the currents i_1 and i_2 are initially zero.

$$\frac{di_1(t)}{dt} + 50i_2(t) = 60$$

$$5\times 10^{-3}\frac{di_2(t)}{dt} + i_2(t) - i_1(t) = 0$$

$$i_1(0^-) = i_2(0^-) = 0$$

$$sI_1(s) + 50I_2(s) = \frac{60}{s}$$

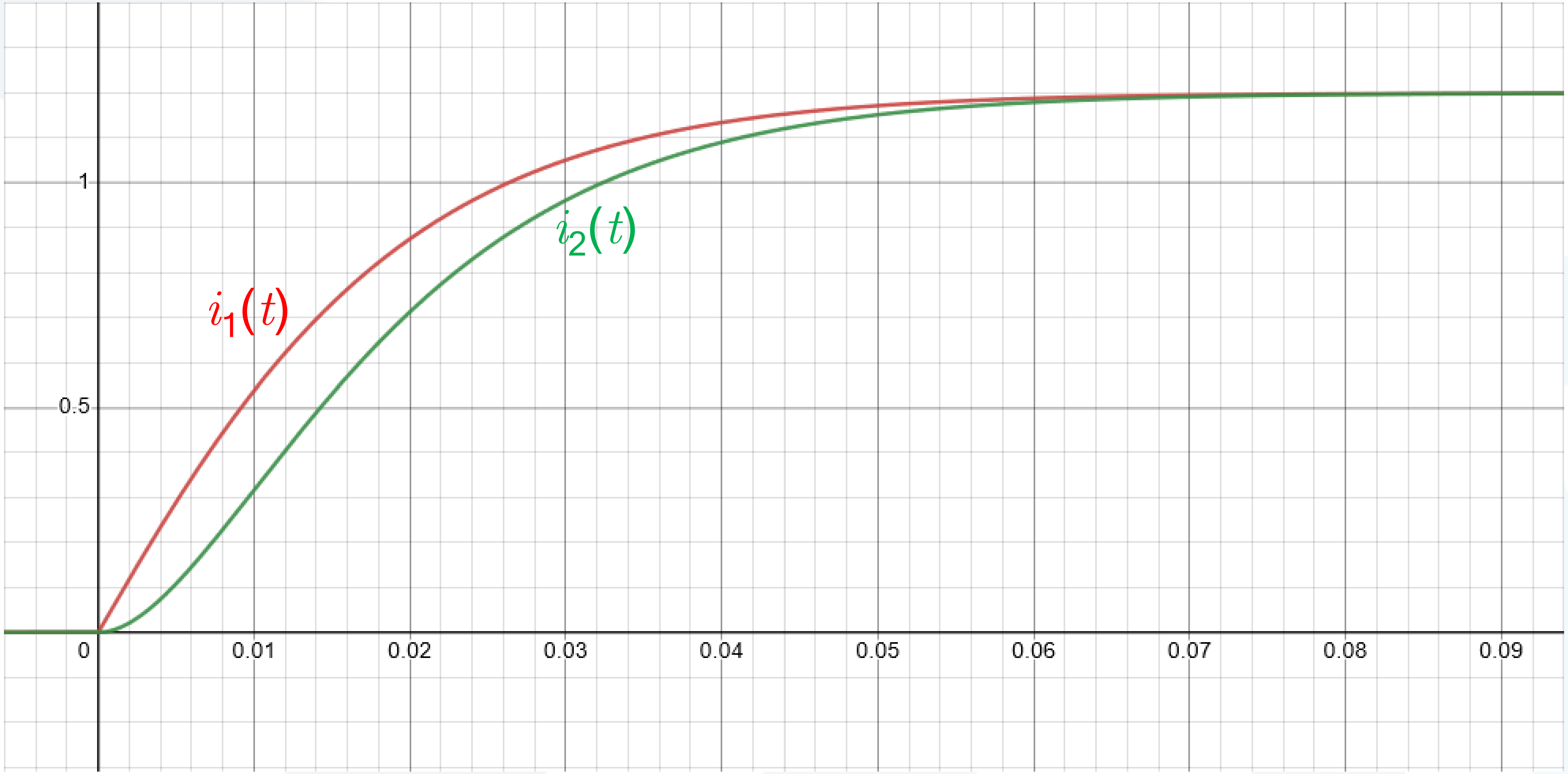
$$-200I_1(s) + (s+200)I_2(s) = 0$$

$$I_1(s) = \frac{60s + 12000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{60}{(s+100)^2}$$

$$i_1(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}$$

$$I_2(s) = \frac{12000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}$$

$$i_2(t) = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}$$

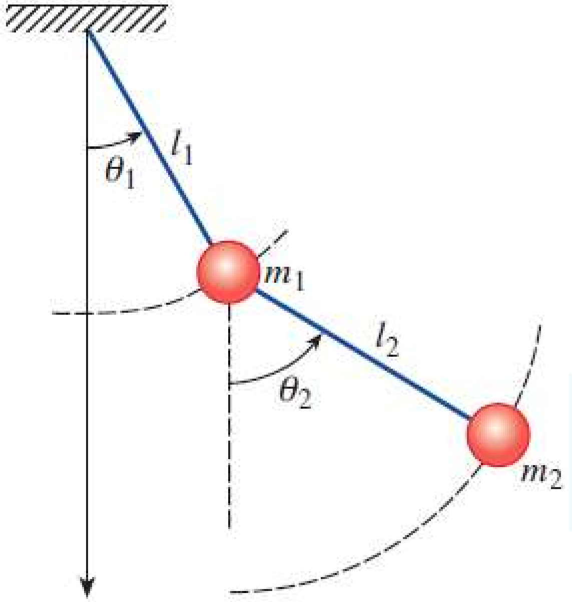


1
0.5
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
$i_1(t)$
$i_2(t)$

Double Pendulum

$$(m_1 + m_2)l_1^2\theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2)l_1 g\theta_1 = 0$$

$$m_2 l_2^2\theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 l_2 g\theta_2 = 0$$

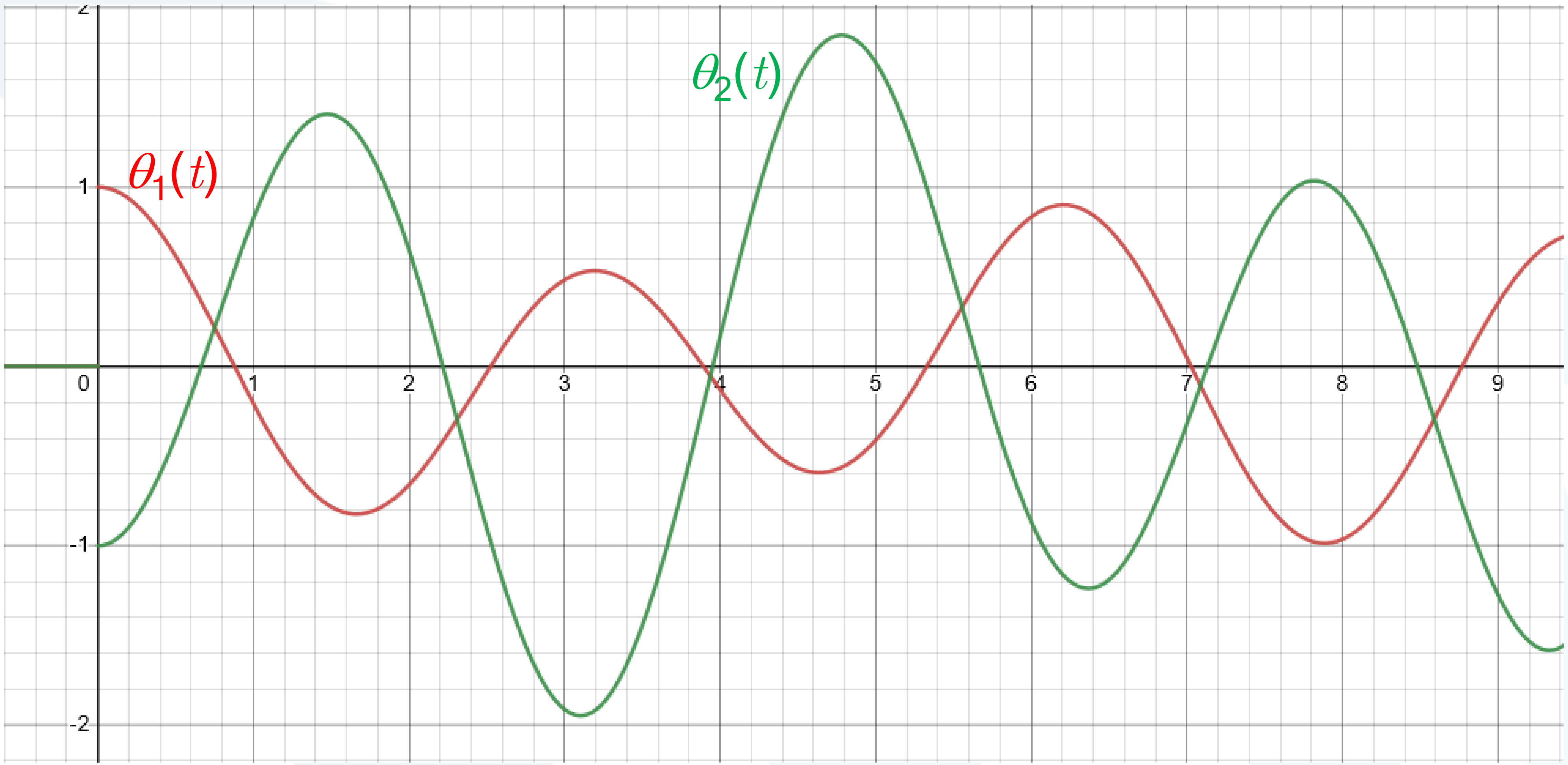


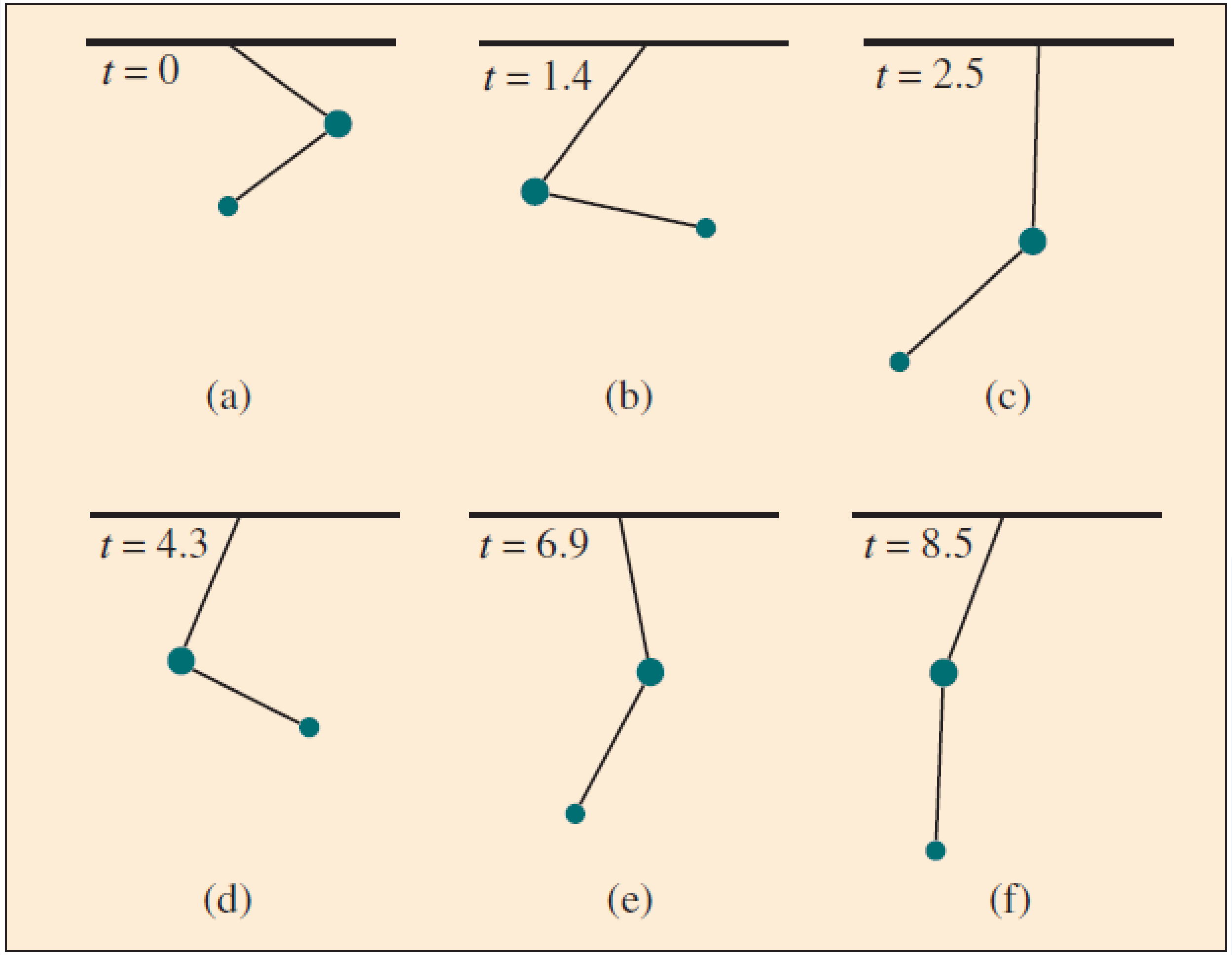
- Example 22: Double Pendulum

$$m_1 = 3,\ m_2 = 1,\ l_1 = l_2 = 16,\ \theta_1(0) = 1,\ \theta_2(0) = -1,\ \theta_1'(0) = \theta_2'(0) = 0$$

$$\theta_1(t) = \frac{1}{4}\cos\frac{2}{\sqrt{3}}t + \frac{3}{4}\cos 2t$$

$$\theta_2(t) = \frac{1}{2}\cos\frac{2}{\sqrt{3}}t - \frac{3}{2}\cos 2t$$


θ₁(t)
θ₂(t)
0
1
2
3
4
5
6
7
8
9
1
-1
-2



Positions of masses at various times

Solving Systems of Equations in Matrix Form

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}(t)$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad \boldsymbol{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

subject to the initial conditions $x_1(0^-) = x_1$, $x_2(0^-) = x_2$, ..., $x_n(0^-) = x_n$

$$\boldsymbol{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix}, \quad \boldsymbol{B}(s) = \begin{bmatrix} B_1(s) \\ B_2(s) \\ \vdots \\ B_n(s) \end{bmatrix}, \quad \boldsymbol{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$s\boldsymbol{X}(s) - \boldsymbol{x}_0 = A\boldsymbol{X}(s) + \boldsymbol{B}(s)$$

$$[sI - A]\boldsymbol{X}(s) = \boldsymbol{x}_0 + \boldsymbol{B}(s)$$

$$\boldsymbol{X}(s) = [sI - A]^{-1}[\boldsymbol{x}_0 + \boldsymbol{B}(s)]$$

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\{[sI - A]^{-1}[\boldsymbol{x}_0 + \boldsymbol{B}(s)]\}$$

- Example 23: Initial value problem

  Solve the initial value problem

$$x' - 2x + y = \sin t$$
$$y' + 2x - y = 1$$

subject to $x(0) = 1,\ y(0) = -1$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}, \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{B}(s) = \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix}$$

$$\boldsymbol{X}(s)=\left[s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\right]^{-1}\left[\begin{bmatrix}1\\ -1\end{bmatrix}+\begin{bmatrix}1/(s^2+1)\\ 1/s\end{bmatrix}\right]$$

$$\boldsymbol{X}(s)=\begin{bmatrix}s-2 & 1\\ 2 & s-1\end{bmatrix}^{-1}\begin{bmatrix}(s^2+2)/(s^2+1)\\ (1-s)/s\end{bmatrix}=$$

$$\boldsymbol{X}(s)=\begin{bmatrix}\dfrac{s-1}{s(s-3)} & \dfrac{-1}{s(s-3)}\\ \dfrac{-2}{s(s-3)} & \dfrac{s-2}{s(s-3)}\end{bmatrix}\begin{bmatrix}\dfrac{s^2+2}{(s^2+1)}\\ \dfrac{1-s}{s}\end{bmatrix}=\begin{bmatrix}\dfrac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)}\\ \dfrac{-(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)}\end{bmatrix}$$

$$x(t)=\frac{4}{9}+\frac{1}{3}t-\frac{1}{5}\sin t-\frac{2}{5}\cos t+\frac{43}{45}e^{3t}$$

$$y(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t}$$

Determination of e^{tA} by Means of the Laplace Transform

- The solution of the homogeneous system of equations: $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$ subject to the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, can be written

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}_0$$

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\{[\mathrm{s}I - A]^{-1}\}\boldsymbol{x}_0$$

$$\Rightarrow e^{tA} = \mathcal{L}^{-1}\{[\mathrm{s}I - A]^{-1}\}$$

- **Theorem 13 (Determination of e^{tA} by Means of the Laplace Transform):** Let A be a real $n{\times}n$ matrix with constant elements. Then the exponential matrix

$$e^{tA} = \mathcal{L}^{-1}\{[\mathrm{s}I - A]^{-1}\}$$

- Note: Theorem 13 determines e^{tA} in the cases when A is diagonalizable with real eigenvalues, when it is diagonalizable with complex conjugate eigenvalues, and also when it is not diagonalizable.

- Example 24: Determination of e^{tA}

$$A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$$

Matrix A has the distinct eigenvalues 1 and 2, and so is diagonalizable.

$$[sI - A] = \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix} \Rightarrow [sI - A]^{-1} = \begin{bmatrix} \dfrac{s-5}{s^2-3s+2} & \dfrac{6}{s^2-3s+2} \\ \dfrac{-2}{s^2-3s+2} & \dfrac{s+2}{s^2-3s+2} \end{bmatrix}$$

$$e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} 4e^{t} - 3e^{2t} & -6e^{t} + 6e^{2t} \\ 2e^{t} - 2e^{2t} & -3e^{t} + 4e^{2t} \end{bmatrix}$$

- **Example 25:** Determination of e^{tA}

$$A = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}$$

Matrix A has the complex conjugate eigenvalues $-1 \pm 2i$.

$$[sI - A] = \begin{bmatrix} s+3 & 4 \\ -2 & s-1 \end{bmatrix} \Rightarrow [sI - A]^{-1} = \begin{bmatrix} \dfrac{s-1}{s^2+2s+5} & \dfrac{-4}{s^2+2s+5} \\ \dfrac{2}{s^2+2s+5} & \dfrac{s+3}{s^2+2s+5} \end{bmatrix}$$

$$e^{tA} = \mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix}$$

- Example 26: Determination of e^{tA}

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Matrix A has the repeated eigenvalue 4 and is not diagonalizable.

$$[sI - A] = \begin{bmatrix} s-4 & -1 \\ 0 & s-4 \end{bmatrix} \Rightarrow [sI - A]^{-1} = \begin{bmatrix} \dfrac{1}{s-4} & \dfrac{1}{(s-4)^2} \\ 0 & \dfrac{1}{s-4} \end{bmatrix} \Rightarrow e^{tA} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}$$