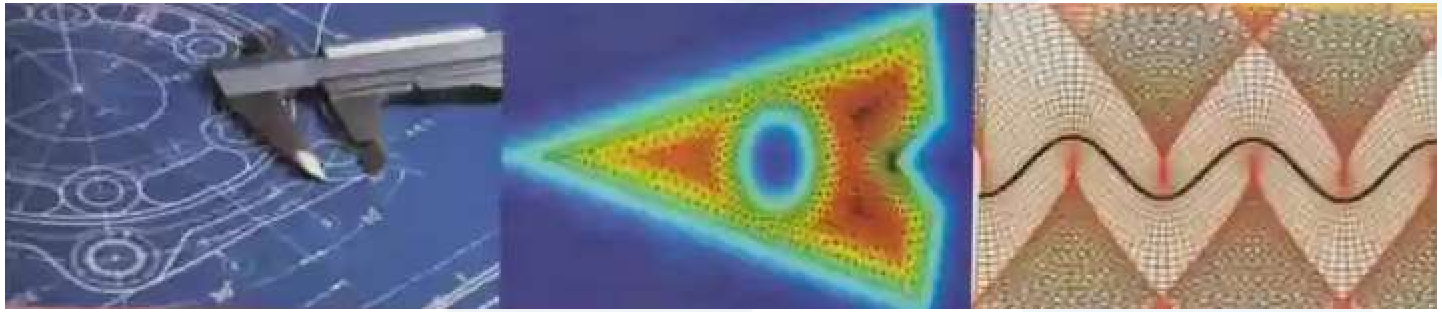
CEDC301: Engineering Mathematics

Lecture Notes 8: Fourier Analysis: Part A



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Chapter 5

Fourier Analysis

1. Orthogonal Functions
2. Fourier Series
3. Fourier Cosine and Sine Series
4. Complex Fourier Series
5. Fourier transform
6. Boundary-Value Problems in Rectangular Coordinates

1. Orthogonal Functions

Inner Product

- Let u, v, and w be vectors in a real vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $<u, v>$ with each pair of vectors u and v and satisfies the following axioms:

  (1) $<u, v> = <v, u>$

  (2) $<u, v + w> = <u, v> + <u, w>$

  (3) $c <u, v> = <cu, v>$

  (4) $<v, v> \geq 0$ and $<v, v> = 0$ if and only if $v = \mathbf{0}$

- Note: $<u, v> = u.v = \sum_{i=1}^{n} u_i v_i$ dot product (Euclidean inner product for R^n)

- Definition: The inner product of two piecewise-continuous functions f_1 and f_2 on an interval $[a, b]$ is the number:

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx$$

Orthogonal Functions

- Definition: Two functions f_1 and f_2 are said to be orthogonal on an interval $[a, b]$ if:

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx = 0$$

- Example 1: Orthogonal Functions

  The functions $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$.

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} x^2 x^3\, dx = \int_{-1}^{1} x^5\, dx = \frac{1}{6} x^6 \Big]_{-1}^{1} = 0$$

- Definition: The norm, or length, of a vector u is given by:

$$\|u\| = \sqrt{< u, u >}$$

Orthogonal Sets

- Definition: A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal on an interval $[a, b]$ if:

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n$$

Orthonormal Sets

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

- If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \ldots$, then $\{\phi_n(x)\}$ is said to be an orthonormal set on the interval.

- Example 2: Orthogonal Set of Functions

  Show that the set {1, $\cos x$, $\cos 2x$, …} is orthogonal on the interval $[-\pi, \pi]$

$$\langle 1, \cos nx\rangle = \int_{-\pi}^{\pi} \cos nx\, dx = \frac{1}{n}\sin nx\Big]_{-\pi}^{\pi} = \frac{1}{n}\big[\sin n\pi - \sin(-n\pi)\big] = 0$$

$$\langle \cos mx, \cos nx\rangle = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi}\big[\cos(m+n)x + \cos(m-n)x\big]dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \quad m \neq n$$

- Example 3: Norms

  Find the norms of each function in the orthogonal set given in Example 2

$$\|\phi_0(x)\| = \|1\| = \sqrt{\int_{-\pi}^{\pi} dx} = \sqrt{2\pi}$$

$$\|\phi_n(x)\| = \|\cos nx\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 nx\, dx} = \sqrt{\int_{-\pi}^{\pi} \tfrac{1}{2}\left[1+\cos 2nx\right] dx} = \sqrt{\pi}, \quad n > 0$$

- Note: Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, can be normalized—that is, made into an orthonormal set—by dividing each function by its norm.

  For example the set $\left\{\dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos x}{\sqrt{\pi}}, \dfrac{\cos 2x}{\sqrt{\pi}}, \cdots\right\}$ is orthonormal on $[-\pi, \pi]$.

- Theorem 1 (Coordinates relative to a basis): If $B = \{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, is an orthogonal basis for an inner product space $V = C[a, b]$, and if f is any vector in V, then

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

where

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n(x)\|^2} = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}$$

$\{\phi_n(x)\}$ is an orthogonal basis: $f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n \rangle}{\|\phi_n(x)\|^2} \phi_n(x)$

$\{\phi_n(x)\}$ is an orthonormal basis: $f(x) = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n(x)$

- Definition: A set of real-valued functions $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a, b]$ if:

$$\langle \phi_m, \phi_n \rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n$$

where $w(x)$ is a positive continuous function

- Note: The inner product of two functions f_1 and f_2 on an interval $[a, b]$, used above is :

$$\langle f_1, f_2\rangle = \int_a^b f_1(x)f_2(x)w(x)dx$$

- The set $\{1, \cos x, \cos 2x, \ldots\}$ in Example 2 is orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-\pi, \pi]$.

- The series $f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$ is said to be an orthogonal series expansion of f or a generalized Fourier series.

Complete Sets

- To expand f in a series of orthogonal functions, it is certainly necessary that f not be orthogonal to each ϕ_n of the orthogonal set $\{\phi_n(x)\}$.

(If f were orthogonal to every ϕ_n, then $c_n = 0$, $n = 0, 1, 2, \ldots$.)

- To avoid the latter problem we shall assume that an orthogonal set is complete. This means that the only continuous function orthogonal to each member of the set is the zero function.
- Note: Suppose that $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is an infinite set of real-valued functions that are continuous on an interval $[a, b]$. If this set is linearly independent on $[a, b]$, then it can always be made into an orthogonal set using Gram-Schmidt process.

Orthogonal Polynomials

- Let P_∞ be the vector space of all polynomials and define the inner product of two polynomials P and Q, on P_∞ by:

$$\langle P, Q \rangle = \int_a^b P(x)Q(x)w(x)dx$$

- Let $P_0(x)$, $P_1(x)$, ... be a sequence of polynomials with deg $P_n(x) = n$ for each n. If $\langle P_m, P_n \rangle = 0$ whenever $m \neq n$, then $\{P_n(x)\}$ is said to be a sequence of orthogonal polynomials. If $\langle P_m, P_n \rangle = \delta_{mn}$, then $\{P_n(x)\}$ is said to be a sequence of orthonormal polynomials.

Legendre Polynomials

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)P_n(x)dx$$

$$\|P_n(x)\|^2 = \frac{2}{2n+1}, \; n = 0, 1, \dots$$

$P_n(1) = 1$ for each n, then $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$$

$$P_0(x)=1 \qquad P_1(x)=x \qquad P_2(x)=\tfrac{1}{2}(3x^2-1)$$

$$P_3(x)=\tfrac{1}{2}(5x^3-3x) \qquad P_4(x)=\tfrac{1}{8}(35x^4-30x^2+3)$$

Chebyshev Polynomials

$$\langle T_m,T_n\rangle=\int_{-1}^{1}T_m(x)T_n(x)(1-x)^{-1/2}dx$$

$$\|T_0(x)\|^2=\pi, \quad \|T_n(x)\|^2=\frac{\pi}{2},\ n=1,2,\ldots$$

$$a_0=1, \quad a_k=2^{k-1}\ (k=1,2,\ldots)$$

$T_n(\cos\theta)=\cos n\theta$ and $\cos(n+1)\theta=2\cos\theta\cos n\theta-\cos(n-1)\theta$ gives:

$$T_1(x)=xT_0(x), \qquad T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), \quad n\ge 1$$

$$(1-x^2)T_n''(x)-xT_n'(x)+n^2T_n(x)=0$$

$$T_0(x)=1 \qquad T_1(x)=x \qquad T_2(x)=2x^2-1$$

$$T_3(x)=4x^3-3x \qquad T_4(x)=8x^4-8x^2+1$$

Hermite Polynomials

$$\langle H_m, H_n\rangle=\int_{-\infty}^{\infty}H_m(x)H_n(x)e^{-x^2}dx$$

$$\|H_n(x)\|^2=\sqrt{\pi}\,2^n n!,\ n=0,1,\ldots$$

$$H_0(x)=1,\quad H_1(x)=2x$$

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x),\quad n\geq 1$$

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0$$

$$H_0(x)=1 \qquad H_1(x)=2x \qquad H_2(x)=4x^2-2$$

$$H_3(x)=8x^3-12x \qquad H_4(x)=16x^4-48x^2+12$$

Laguerre Polynomials

$$\langle L_m, L_n\rangle = \int_0^\infty L_m(x)L_n(x)e^{-x}dx$$

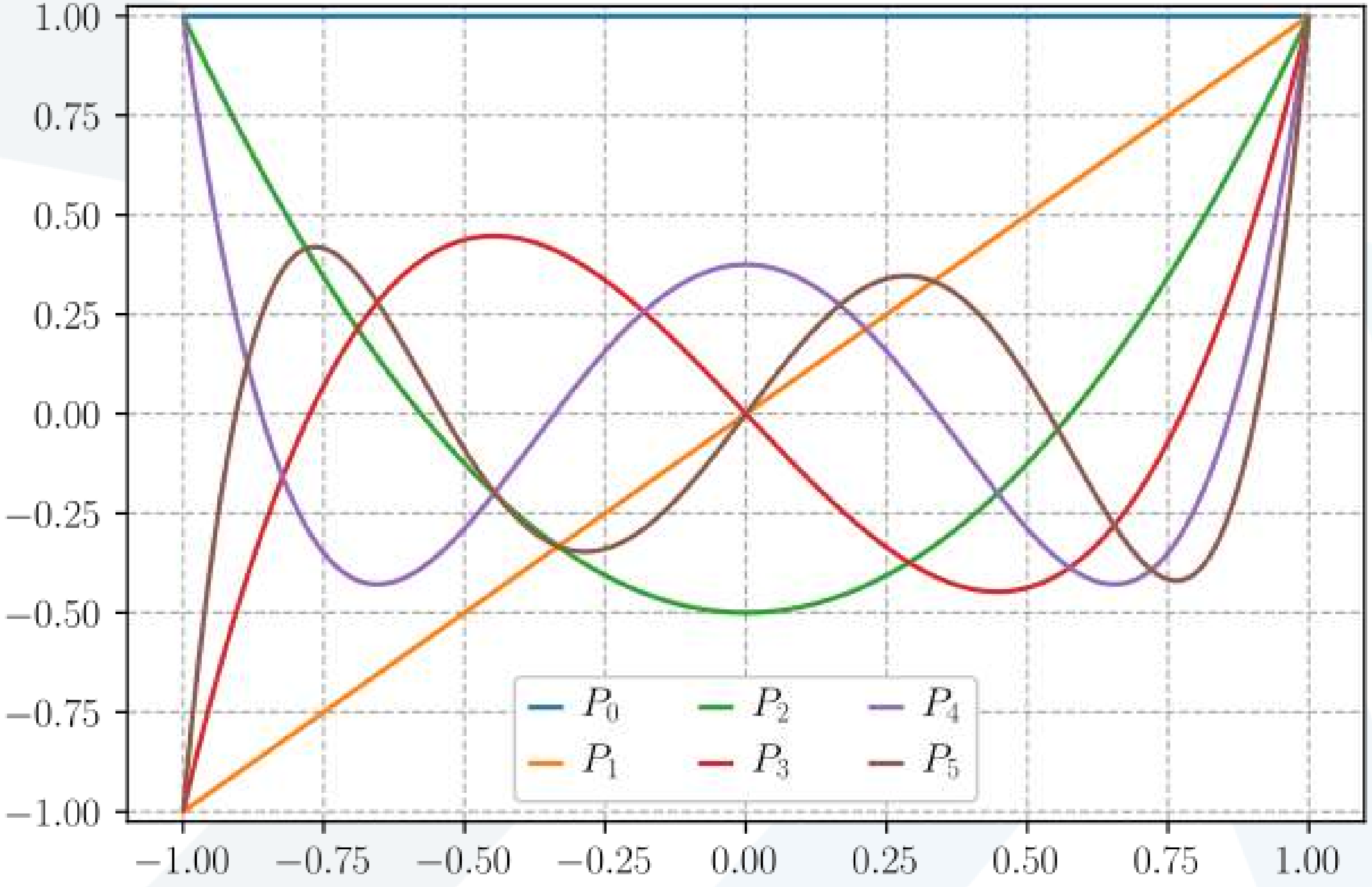
$$\left\|L_n(x)\right\|^2 = 1,\ n=0,1,\ldots$$

$$(n+1)L_{n+1}(x)=(2n+1-x)L_n(x)-nL_{n-1}(x)$$

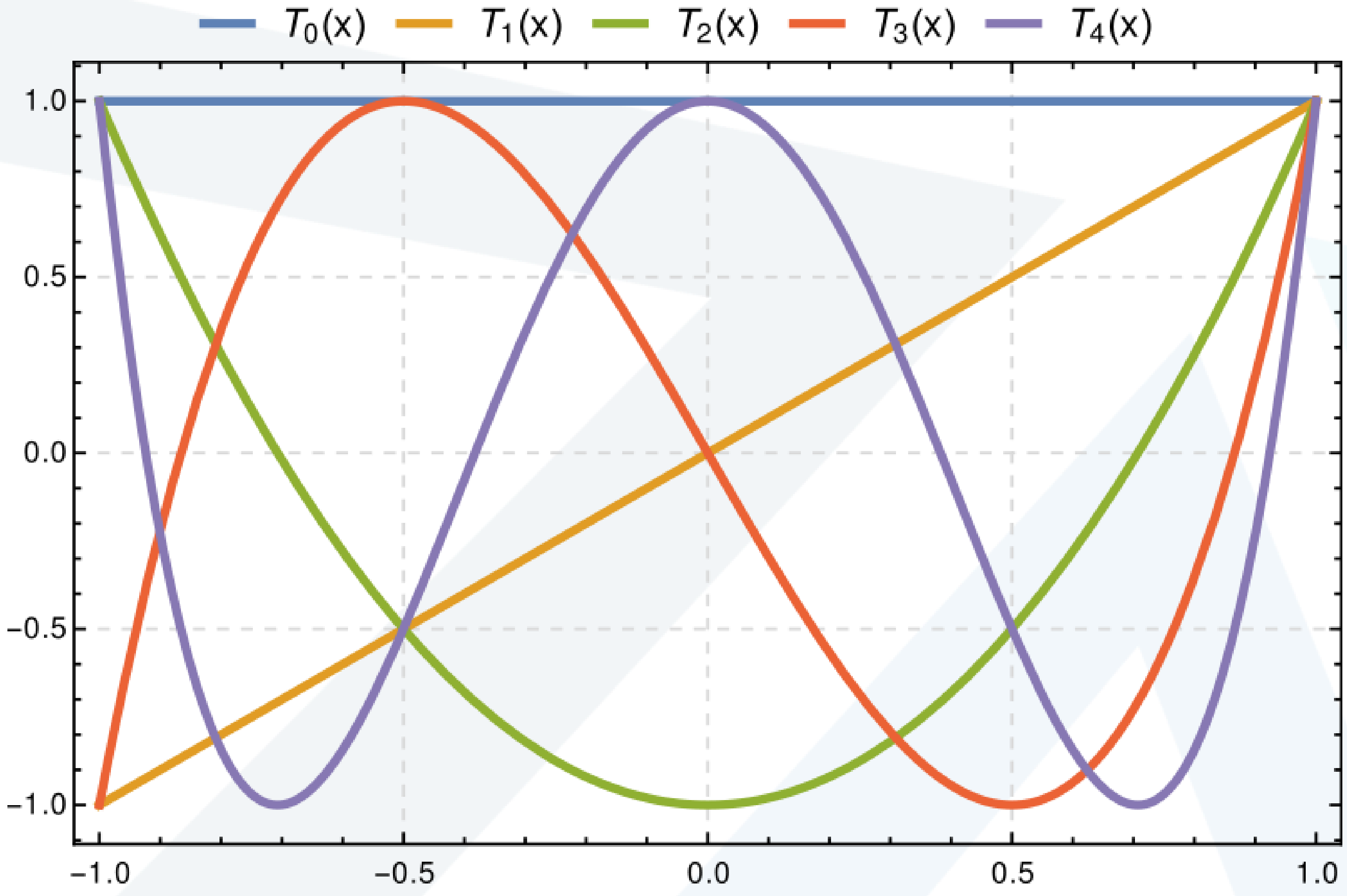
$$xy''+(1-x)y'+ny=0$$

$$L_0(x)=1 \qquad L_1(x)=-x+1 \qquad L_2(x)=\tfrac{1}{2}(x^2-4x+2)$$

$$L_3(x)=\tfrac{1}{6}(-x^3+9x^2-18x+6) \qquad L_4(x)=\tfrac{1}{24}(x^4-16x^3+72x^2-96x+24)$$



Legendre Polynomials



Chebyshev Polynomials

2. Fourier Series

- If $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is a set of real-valued functions that is orthogonal on an interval $[a,\ b]$ and if f is a function defined on the same interval, then we can formally expand f in an orthogonal series $c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$.

- In this section we shall expand functions in terms of a special orthogonal set of trigonometric functions.

Trigonometric Series

- The set of trigonometric functions

$$\left\{1,\ \cos\frac{\pi}{L}x,\ \cos\frac{2\pi}{L}x,\ \cos\frac{3\pi}{L}x,\ \cdots,\ \sin\frac{\pi}{L}x,\ \sin\frac{2\pi}{L}x,\ \sin\frac{3\pi}{L}x,\ \cdots\right\}$$

is orthogonal on the interval $[-L,\ L]$.

- Expand a function f defined on $[-L, L]$ in an orthogonal series consisting of the trigonometric functions.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots,$ can be determined using

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n(x)\|^2} = \frac{\int_{-L}^{L} f(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$$\|\phi_0(x)\|^2 = \|1\|^2 = \int_{-L}^{L} dx = 2L, \quad \|\phi_n(x)\|^2 = \left\|\cos\frac{n\pi}{L}x\right\|^2 = \int_{-L}^{L} \cos^2\frac{n\pi}{L}x\, dx = L, \quad n > 0$$

$$\|\phi_n(x)\|^2 = \left\|\sin\frac{n\pi}{L}x\right\|^2 = \int_{-L}^{L} \sin^2\frac{n\pi}{L}x\, dx = L, \quad n > 0$$

$$\frac{\int_{-L}^{L} f(x)\phi_n(x)\,dx}{\left\|\phi_n(x)\right\|^2} = \begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\dfrac{n\pi}{L}x\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\dfrac{n\pi}{L}x\,dx \end{cases}$$

Fourier coefficients of f

- Definition: The Fourier series of a function f defined on the interval $(-L, L)$ is given by:

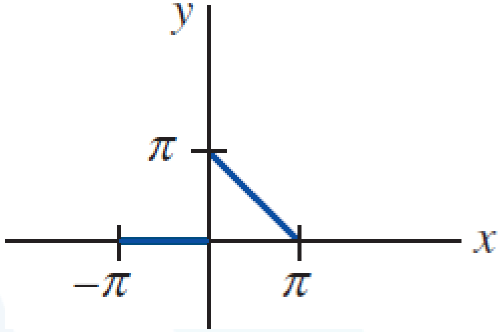
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

- **Example 4:** Expansion in a Fourier Series



Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$ in a Fourier series

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)\,dx = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx\,dx = \frac{1-\cos n\pi}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,dx = \frac{1}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right)$$

Convergence of a Fourier Series

- **Theorem 2 (Conditions for Convergence):** Let f and f' be piecewise continuous on the interval $[-L, L]$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then for all x in the interval $(-L, L)$ the Fourier series of f converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average $\dfrac{f(x^+) + f(x^-)}{2}$

- Example 5: Convergence of a Point of Discontinuity

The function in Example 4 satisfies the conditions of Theorem 2. Thus for every x in the interval $(-L, L)$, except at $x = 0$, the series will converge to $f(x)$. At $x = 0$ the function is discontinuous, and so the series will converge to

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

at $x = \pi/2$ the series converge to $f(\pi/2) = \pi/2$.

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos n\frac{\pi}{2} + \frac{1}{n}\sin n\frac{\pi}{2}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

at $x = 0$ the series converge to $\pi/2$.

$$\frac{\pi}{2}=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos n0+\frac{1}{n}\sin n0\right)$$

$$\frac{\pi}{4}=\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Periodic Extension

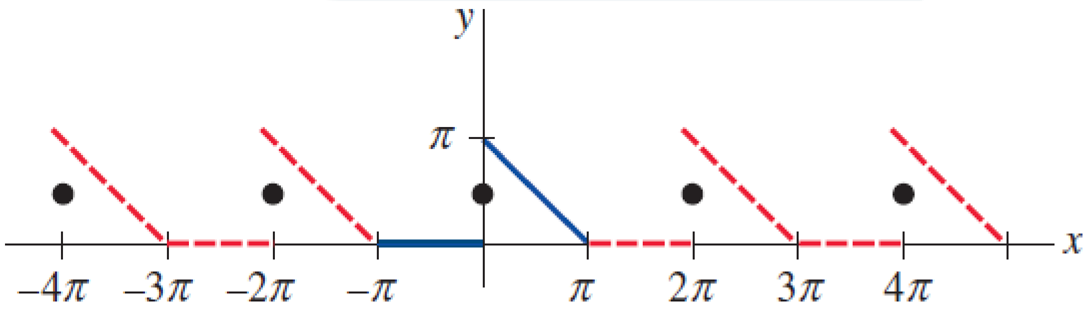
$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)$$

- The right hand side of equation above is $2L$-periodic; indeed, $2L$ is the fundamental period of the sum.

- We conclude that a Fourier series not only represents the function on the interval $(-L, L)$ but also gives the periodic extension of f outside this interval.
- We may assume from the outset that the given function is periodic with period $T = 2L$; that is, $f(x + T) = f(x)$.
- When f is piecewise continuous and the right-and left-hand derivatives exist at $x = -L$ and $x = L$, respectively, then the series converges to $[f(L^-) + f(-L^+)]/2$ at these endpoints and to this value extended periodically to $\pm 3L$, $\pm 5L$, $\pm 7L$, and so on.
- Fourier series in example 4 converges to the periodic extension of $f(x)$ on the entire x-axis.
- At 0, $\pm 2\pi$, $\pm 4\pi$, …, and at $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, …, the series converges to the values:

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi^-) + f(-\pi^+)}{2} = 0$$

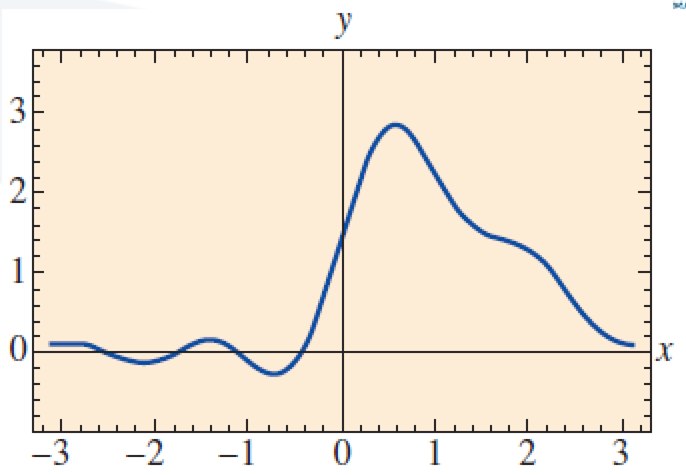
respectively. The solid dots in figure below represent the value $\pi/2$.



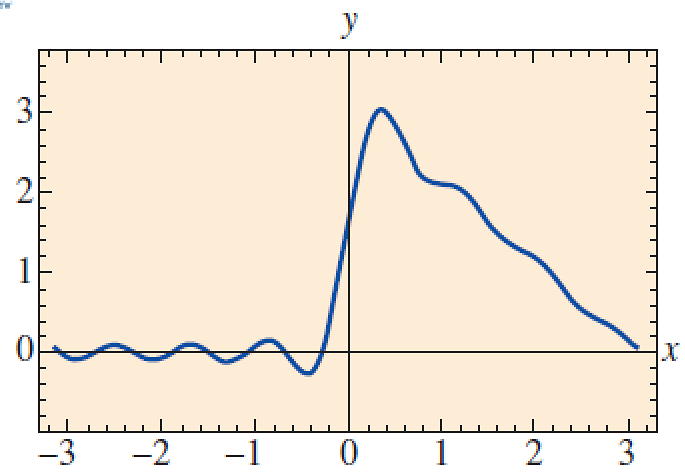
Sequence of Partial Sums

- It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. In example 4, the first three partial sums are:

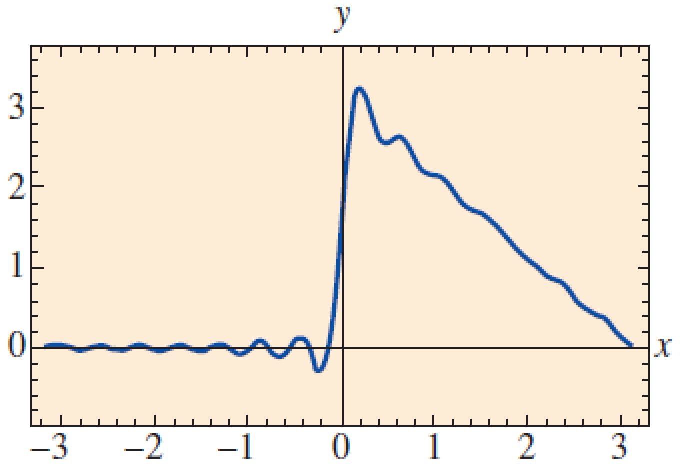
$$S_0(x) = \frac{\pi}{4}, \quad S_1(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$



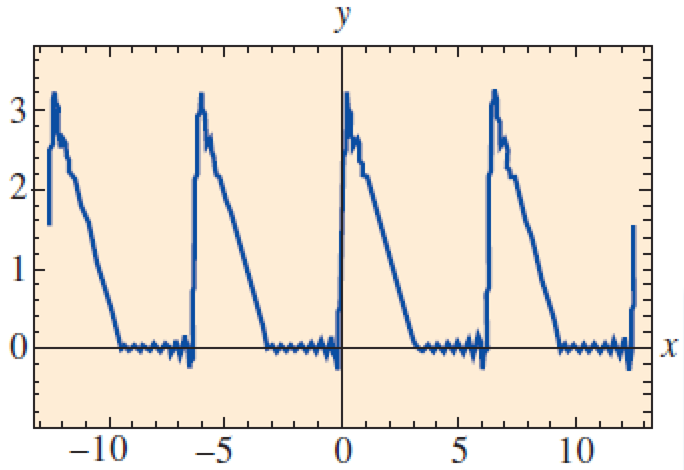
(a) $S_5(x)$ on $(-\pi, \pi)$



(b) $S_8(x)$ on $(-\pi, \pi)$



(c) $S_{15}(x)$ on $(-\pi, \pi)$



(d) $S_{15}(x)$ on $(-4\pi, 4\pi)$