CEDC301: Engineering Mathematics

Lecture Notes: Fourier Analysis: Part B

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Chapter 5
Fourier Analysis

1. Orthogonal Functions
2. Fourier Series
3. Fourier Cosine and Sine Series
4. Complex Fourier Series
5. Fourier transform
6. Boundary-Value Problems in Rectangular Coordinates

3. Fourier Cosine and Sine Series

Cosine and Sine Series

- **Definition:** Fourier Cosine and Sine Series

(i) The Fourier series of an even function on the interval $(-L, L)$ is the cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

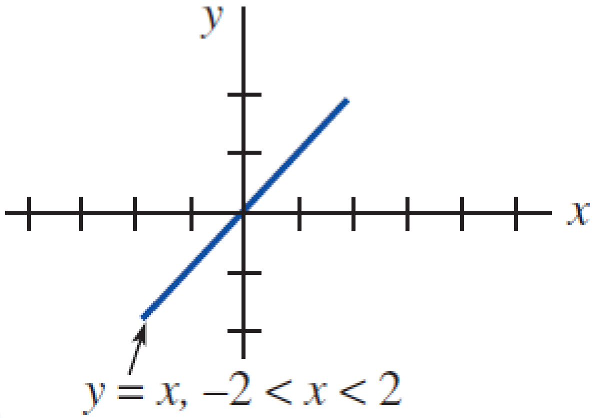
where $a_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x dx$

(ii) The Fourier series of an odd function on the interval $(-L, L)$ is the sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x dx$$

- Example 6: Expansion in a Sine Series

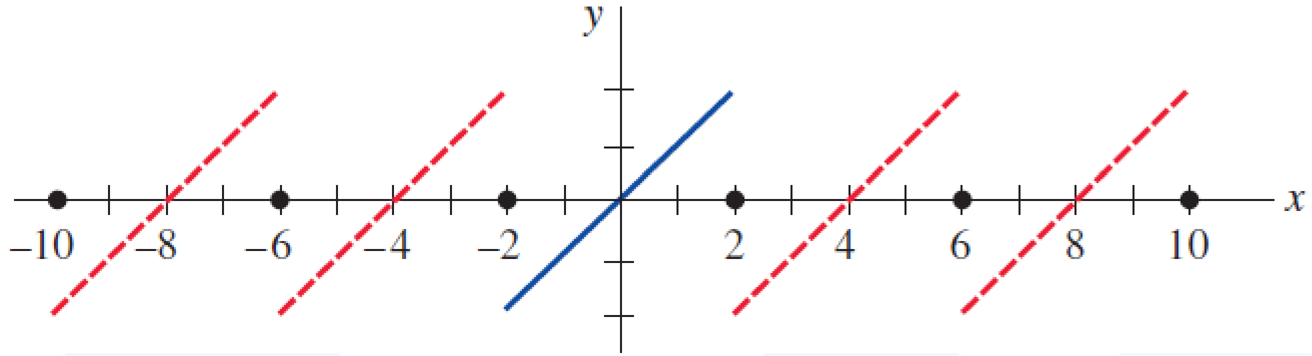
Expand $f(x) = x$, $-2 < x < 2$, in a Fourier series



The given function is odd on the interval (–2, 2), and so we expand f in a sine series. With the identification $2L = 4$, we have $L = 2$.

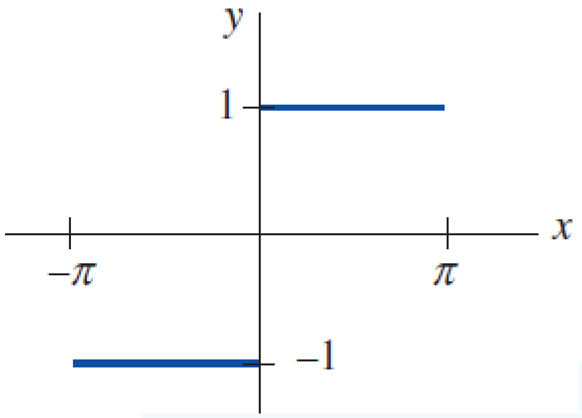
$$b_n = \int_0^2 x \sin \frac{n\pi}{L} x dx = \frac{4(-1)^{n+1}}{n\pi} \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin \frac{n\pi}{2} x$$

The series converges to the function on (–2, 2) and the periodic extension (of period 4).



- Example 7: Expansion in a Sine Series

Expand $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$ in a Fourier series

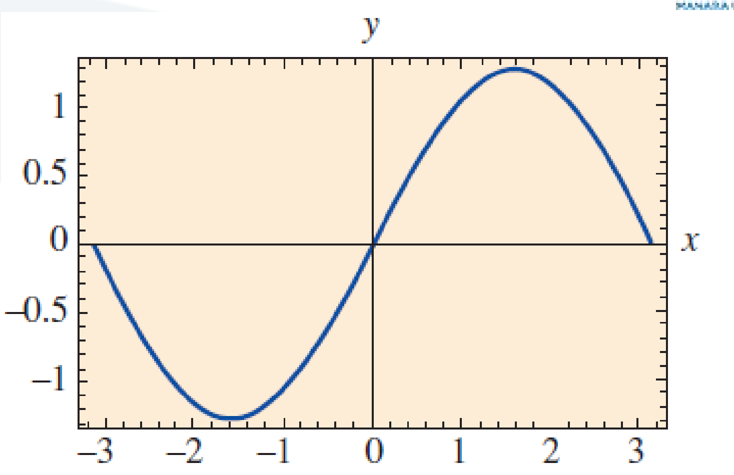


The given function is odd on the interval $(-\pi, \pi)$, and so we expand f in a sine series.

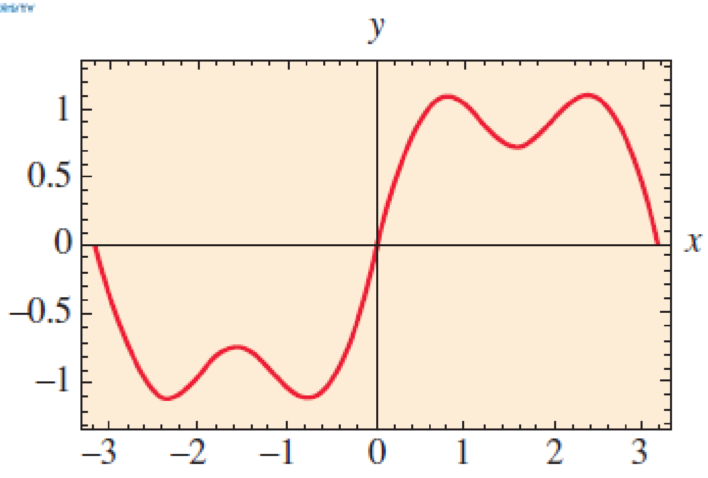
$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin\frac{n\pi}{L} x dx = \frac{2}{\pi}\frac{1-(-1)^n}{n} \Rightarrow f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

Gibbs Phenomenon

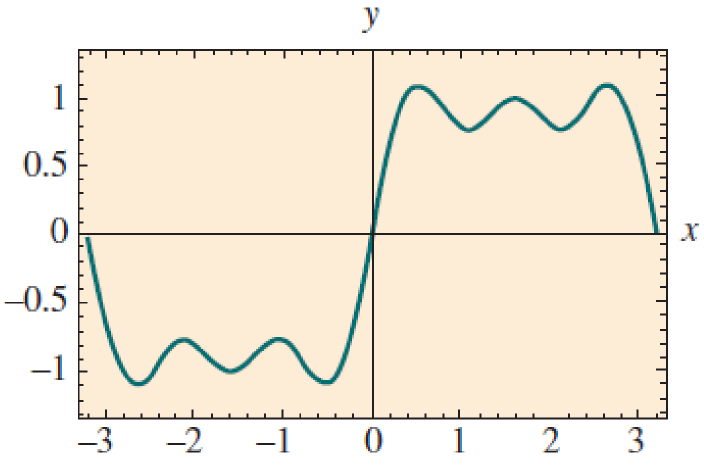
- The partial sums $\{S_N(x)\}$ of a Fourier series shows oscillations (spikes) near the points of discontinuity of $f(x)$. these oscillations don't disappear as the value of N gets larger. With increasing N, they are shifted closer to the points of discontinuity of $f(x)$. This behavior is known as the Gibbs phenomenon.



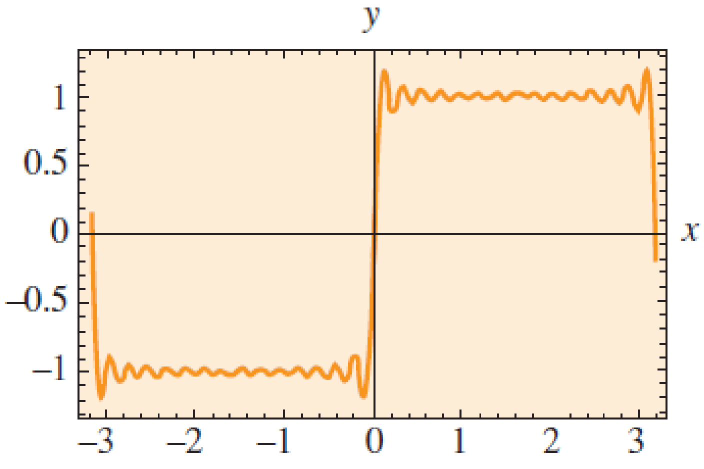
(a) $S_1(x)$



(b) $S_2(x)$



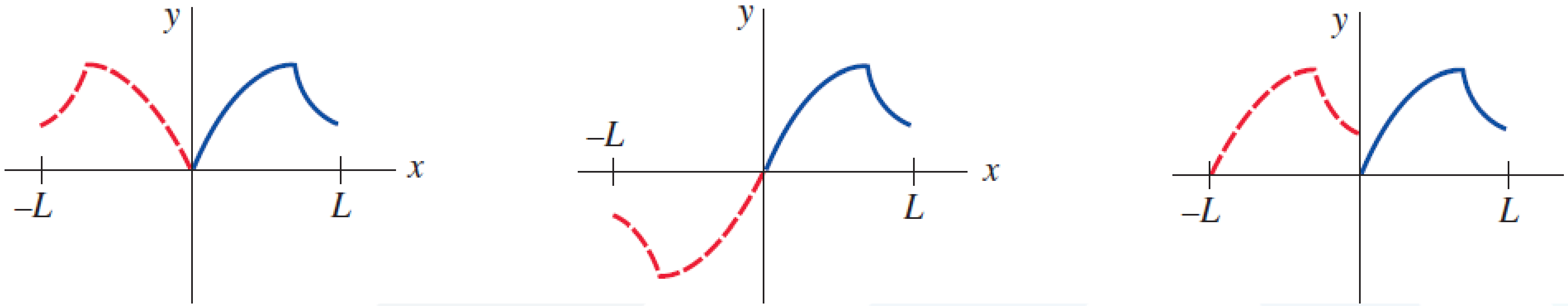
(c) $S_3(x)$



(d) $S_{15}(x)$

Half-Range Expansions

- When we are interested in representing a function that is defined on an interval $(0, L)$ by a trigonometric series.
- This can be done in many different ways by supplying an arbitrary definition of the function on the interval $(-L, 0)$. Three most important cases:



Even reflection *Odd reflection* *Identity reflection*

Even reflection: The function is even on the interval $(-L, L)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx$$

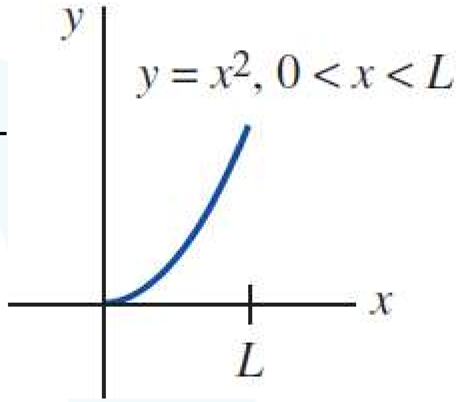
Odd reflection: The function is odd on the interval $(-L, L)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx$$

Identity reflection: The function values on the interval $(-L, 0)$ are the same as the values on $(0, L)$. We identify $L \to L/2$ and The resulting Fourier series will give the periodic extension of the function with period L.

- Example 8: Half Range Expansion

Expand $f(x) = x^2$, $0 < x < L$, (a) in a cosine series, (b) in a sine series, (c) in a Fourier series.



(a) $a_0 = \dfrac{1}{L}\displaystyle\int_0^L x^2 dx = \dfrac{1}{3}L^2, \qquad a_n = \dfrac{2}{L}\int_0^L x^2 \cos\dfrac{n\pi}{L}x\,dx = \dfrac{4L^2(-1)^n}{n^2\pi^2}$

$$f(x) = \frac{1}{3}L^2 + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi}{L}x$$

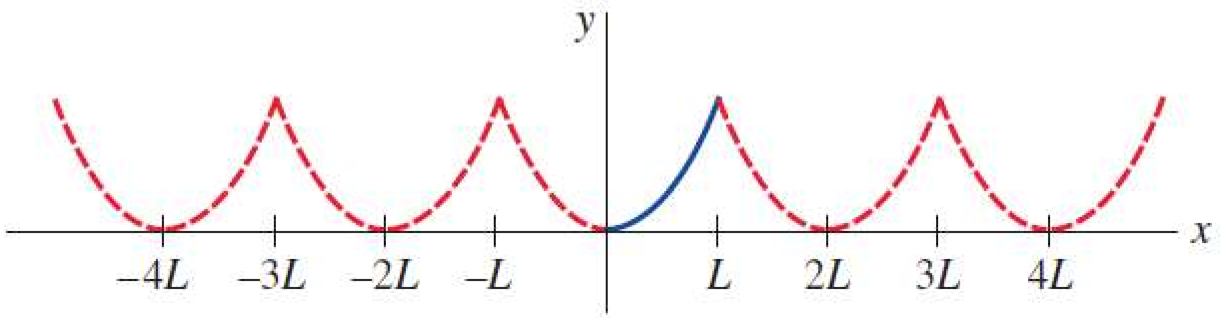
(b) $b_n = \dfrac{2}{L}\displaystyle\int_0^L x^2 \sin\dfrac{n\pi}{L}x\,dx = \dfrac{2L^2(-1)^{n+1}}{n\pi} + \dfrac{4L^2}{n^3\pi^3}[(-1)^n - 1]$

$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^3}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x$$

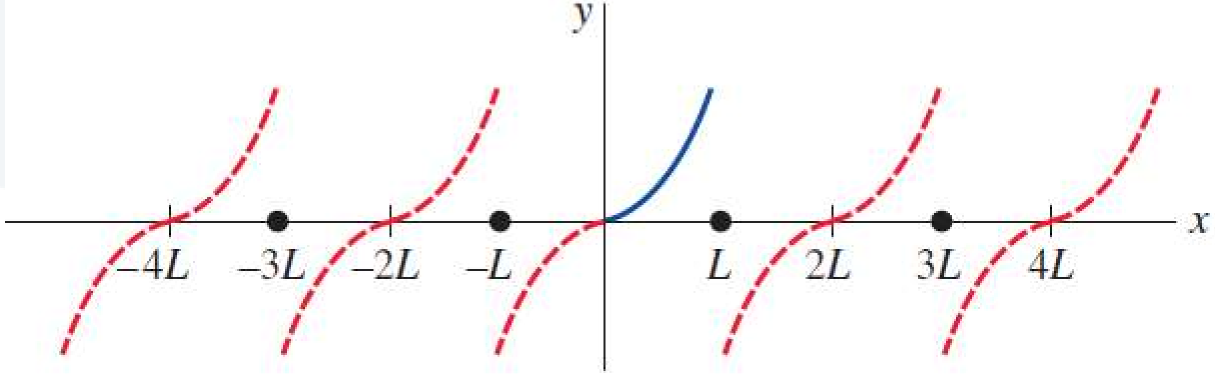
(c) $a_0 = \frac{1}{L}\int_0^L x^2 dx = \frac{1}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{2n\pi}{L}x dx = \frac{L^2}{n^2\pi^2}$

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{2n\pi}{L}x dx = \frac{-L^2}{n\pi}$$

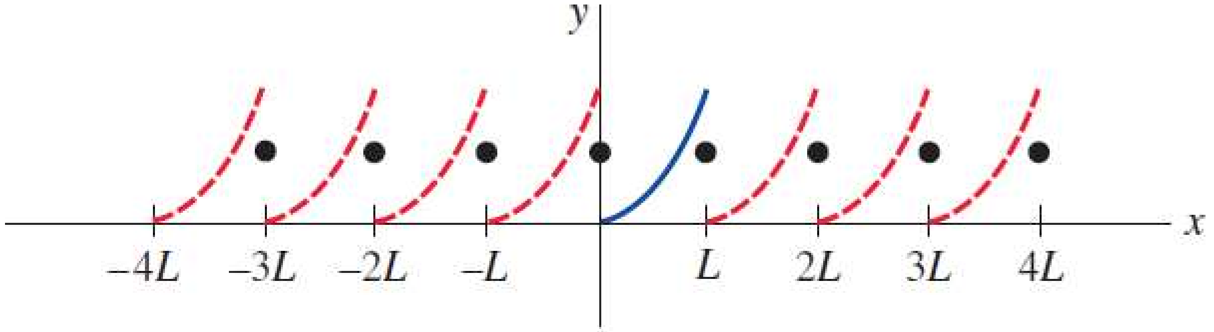
$$f(x) = \frac{1}{3}L^2 + \frac{L^2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{L^2}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}$$



(a) Cosine series



(b) Sine series



(c) Fourier series

Parseval formula

For a full Fourier Series on $[-L, L]$: $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

- Example 9: Expansion in a Sine Series

  The Fourier series for the function $f(x) = x\ (-\pi < x < \pi)$: $f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2 \Rightarrow \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Differentiation of Fourier Series

- **Theorem 3 (Differentiation of Fourier series):** Let f be a continuous function on the interval $[-L, L]$ such that $f(-L) = f(L)$, and suppose also that f' is piecewise continuous on the interval $(-L, L)$. Then for any x strictly inside the interval at which $f''(x)$ exists, the derivative of $f(x)$ can be obtained by term-by-term differentiation of the Fourier series representation of f. So, if f has the Fourier series representation:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

then

$$f'(x) = \frac{\pi}{L}\sum_{n=1}^{\infty}\left(-na_n \sin\frac{n\pi}{L}x + nb_n \cos\frac{n\pi}{L}x\right) \qquad \text{for } -L < x < L$$

except for points at where $f'(x)$ and $f''(x)$ are not defined.

- Note: Not all Fourier series are differentiable.
- Example 10: a Series is not differentiable

The Fourier series for the function $f(x) = x\ (-\pi < x < \pi)$ converges to $f(x)$ at each point in the interval $-\pi < x < \pi$:

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

But the differentiated series

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$$

does not converge since its nth term fails to approach zero as n tends to infinity.

- Example 11: a Series is differentiable

The Fourier series for the function $f(x) = \cosh ax\ (-\pi \le x \le \pi)\ a \ne 0$

$$\cosh ax = \frac{\sinh a\pi}{a\pi}\left[1 + 2a^2\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx\right]$$

This series converges to $\cosh ax$ on the interval $-\pi \le x \le \pi$. The hypothesis of the theorem is satisfied when, it follows that:

$$\sinh ax = \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{a^2+n^2}\sin nx \qquad -\pi < x < \pi$$

- Note: The equation above is valid when the condition $a = 0$ is dropped.

Integration of Fourier Series

- Theorem 4 (Integration of Fourier series): A Fourier series of a piecewise smooth function f can always be integrated term by term and the result is a convergent infinite series that always converges to the integral of f on $[-L, L]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

the equation

$$\int_{-L}^{x} f(u)du = a_0(x+L) + \frac{L}{\pi}\sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin\frac{n\pi}{L}x - \frac{b_n}{n}\left(\cos\frac{n\pi}{L}x + (-1)^{n+1}\right)\right]$$

is valid when $-L \le x \le L$.

- Example 12: Integration of Fourier Series

Use the Fourier series representation of the function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ to find a Fourier series representation of $F(x) = \int_{-\pi}^{x} f(t)dt$ in the interval $-\pi < x < \pi$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

$$F(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{x}\frac{\sin(2n-1)t}{2n-1}dt=-\frac{4}{\pi}\left[\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}-\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2}\right]$$

$$F(x)=-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

applying the Parseval formula to the Fourier series representation of $f(x)$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}1\cdot dx=\sum_{n=1}^{\infty}\left(\frac{4}{\pi}\frac{1}{2n-1}\right)^2\Rightarrow 2=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

$$F(x)=-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}-\frac{4}{\pi}\frac{\pi^2}{8}=-\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

$$F(x)=\int_{-\pi}^{x} f(t)dt=\begin{cases}\int_{-\pi}^{x} -1\,dt=-(x+\pi), & -\pi<x<0\\ \int_{-\pi}^{0} -1\,dt+\int_{0}^{x} 1\,dt=x-\pi, & 0<x<\pi\end{cases}$$

$$F(x)=\int_{-\pi}^{x} f(t)dt=|x|-\pi$$

$$F(x)=|x|-\pi=-\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

$$|x|=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}\qquad -\pi\le x\le\pi$$

- Example 13: A Fourier-Legendre expansion

The Fourier-Legendre expansion of the discontinuous function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) = a_0 + a_1 P_1(x) + \cdots$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x)dx$$

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{3}{4}, \quad a_2 = 0, \quad a_3 = -\frac{7}{16}, \cdots$$

$$f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \cdots$$

4. Complex Fourier Series

- In certain applications, for example, the analysis of periodic signals in electrical engineering, it is actually more convenient to represent a function f in an infinite series of complex-valued functions of a real variable x such as the exponential functions e^{inx}, $n = 0, 1, 2, \ldots$, and where i is the imaginary unit.

Complex Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$\cos\frac{n\pi}{L}x = \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2}, \qquad \sin\frac{n\pi}{L}x = \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} + b_k \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}\right)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(\frac{1}{2}(a_n - ib_n)e^{in\pi x/L} + \frac{1}{2}(a_n + ib_n)e^{-in\pi x/L}\right)$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=1}^{\infty} c_{-n} e^{-in\pi x/L}$$

where

$$c_0 = a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx$$

- **Definition:** The complex Fourier series of a function f defined on the interval $(-L, L)$ is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \qquad n = 0, \pm 1, \pm 2, \ldots$$

- **Note:** When the function f is real, c_n and c_{-n} are complex conjugates: $c_{-n} = \overline{c_n}$

- **Note:** The functions $e^{im\pi x/L}$ and $e^{-in\pi x/L}$ are orthogonal over the interval $[-L, L]$.

$$\int_{-L}^{L} e^{im\pi x/L}e^{-in\pi x/L}dx = \begin{cases} 0, & m \neq n \\ 2L, & m = n \end{cases}$$

- If f satisfies the hypotheses of Theorem 2, a complex Fourier series converges to $f(x)$ at a point of continuity and to the average $\frac{1}{2}[f(x^+) + f(x^-)]$ at a point of discontinuity.

- Example 14: Complex Fourier Series

Expand $f(x) = e^{-x}$, $-\pi < x < \pi$, (a) in a complex Fourier series.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}dx = \frac{1}{2\pi(in+1)}\left[e^{-(in+1)\pi} - e^{(in+1)\pi}\right]$$

$$c_n = (-1)^n \frac{(e^{\pi} - e^{-\pi})}{2(in+1)\pi} = (-1)^n \frac{\sinh \pi}{\pi}\frac{1-ni}{n^2+1}$$

$$f(x) = \frac{\sinh \pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{1-ni}{n^2+1}e^{inx}$$

The series converges to the 2π-periodic extension of f.

Fundamental Frequency

- The Fourier series define a periodic function and the fundamental period of that function (that is, the periodic extension of f) is $T = 2L$.

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega x + b_k \sin n\omega x\right) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

where number $\omega = 2\pi / T$ is called the fundamental angular frequency.

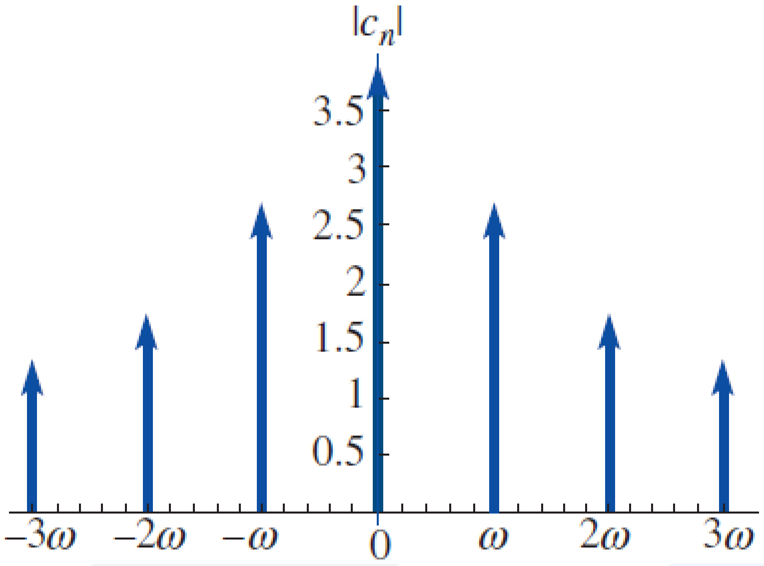
Frequency Spectrum

- If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$, where ω is the fundamental angular frequency and the c_n are the Fourier coefficients, is called the frequency spectrum of f.

- Example 15: Frequency Spectrum

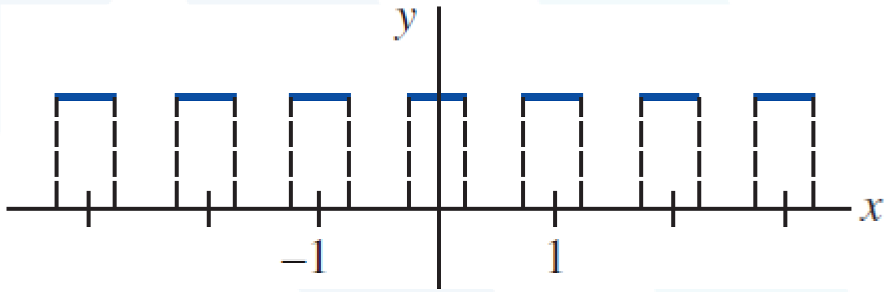
  From example 14:

$$\left|c_n\right| = \frac{\sinh \pi}{\pi} \frac{1}{\sqrt{n^2+1}}$$



- Example 16: Frequency Spectrum

  Find the frequency spectrum of the periodic square wave or periodic pulse. The wave is the periodic extension of the function $y = f(x)$:



$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < -\frac{1}{4} \\ 1, & -\frac{1}{4} < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$$

$$c_n = \int_{-1/4}^{1/4} 1 \cdot e^{-2in\pi x} dx = -\frac{e^{-2in\pi x}}{2in\pi}\Bigg]_{-1/4}^{1/4} = \frac{1}{n\pi}\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i} = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$|c_n| = \frac{1}{n\pi}\left|\sin\frac{n\pi}{2}\right|$$