CEDC301: Engineering Mathematics

Lecture Notes: Fourier Analysis: Part C

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Chapter 5

Fourier Analysis

5. Fourier transform

- Fourier series enable functions and solutions of linear systems defined over a finite interval to be represented as an infinite series of sines and cosines. This suffices for many physical problems.
- When working with partial differential equations that describe heat conduction and diffusion in a half-space, Fourier series cannot be used.
- If a nonperiodic function is to be represented over an arbitrarily large interval, some generalization of a Fourier series is required.
- Letting $L \to \infty$ in a Fourier series leads to the introduction of a different type of representation called a Fourier integral representation, where the function $f(x)$ is defined for all x and need not be periodic. This representation forms the basis of an integral transform called the Fourier transform.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} e^{in\pi x/L} \frac{1}{2L}\int_{-L}^{L} f(u) e^{-in\pi u/L} du$$

Let $\omega_n = n\pi/L$ and $\Delta\omega_n = \omega_{n+1} - \omega_n = \pi/L$

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega_n) e^{i\omega_n x}\Delta\omega_n, \qquad F(\omega_n) = \int_{-L}^{L} f(u) e^{-in\pi u/L} du$$

As $L \to \infty$ $\omega_n \to \omega$ and $\Delta\omega_n \to d\omega$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega, \qquad F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Fourier Transform and Its Inverse

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$ Fourier transform

$$f(x) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$ Inverse Fourier transform

- **Theorem 5 (Existence of the Fourier Transform):** If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $F(\omega)$ exists.

- **Example 17:** Fourier Transform

  Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

$$F(\omega) = \int_{-1}^{1} 1 \cdot e^{-i\omega x} dx = \left.\frac{e^{-i\omega x}}{-i\omega}\right|_{-1}^{1} = \frac{1}{-i\omega}\left(e^{-i\omega} - e^{i\omega}\right) = 2\frac{\sin\omega}{\omega} = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

- **Example 18:** Fourier Transform

Find the Fourier transform of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0,\ a > 0$

$$F(\omega) = \int_0^\infty e^{-ax} \cdot e^{-i\omega x} dx = \left. \frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right|_0^\infty = \frac{1}{a+i\omega}$$

- **Example 19:** Fourier Transform for the Delta Dirac Function

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad \int_{-\infty}^{\infty} f(x)\delta(x-a) dx = f(a)$$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-i\omega x} dx = \left. e^{-i\omega x} \right|_{x=0} = 1$$

$$\mathcal{F}\{\delta(x-a)\} = \int_{-\infty}^{\infty} \delta(x-a) e^{-i\omega x} dx = \left. e^{-i\omega x} \right|_{x=a} = e^{-i\omega a}$$

properties of the Fourier transform

- **Theorem 6 (Linearity of the Fourier Transform):** The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b, the Fourier transform of $af + bg$ exists, and

$$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

- **Theorem 7 (Differentiation in the time domain):** Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n\mathcal{F}\{f(x)\}$$

- **Theorem 8 (Differentiation in the frequency domain):** Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

$$\mathcal{F}\{xf(x)\} = i\frac{d}{d\omega}[F(\omega)]$$

$$\mathcal{F}\{x^n f(x)\} = i^n \frac{d^n}{d\omega^n}[F(\omega)]$$

for all n such that $F^{(n)}(\omega) \to 0$ as $|\omega| \to \infty$

- **Example 20:** Fourier Transform

Find the Fourier transform of $f(x) = e^{-a^2x^2}$, $a > 0$

The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty}\left|e^{-a^2x^2}\right|dx = \int_{-\infty}^{\infty} e^{-a^2x^2}dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{a}$$

absolutely integrable over the interval $(-\infty, \infty)$. $f(x)$ satisfies the differential equation: $f' + 2a^2 x f = 0$.

$$\mathcal{F}\{f'(x)\} + 2a^2 \mathcal{F}\{xf(x)\} = 0 \Rightarrow 2a^2 F'(\omega) + \omega F(\omega) = 0$$

$$\int \frac{F'}{F} d\omega = -\frac{1}{2a^2} \int \omega d\omega \Rightarrow \ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A \Rightarrow F(\omega) = A e^{-\frac{\omega^2}{4a^2}}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \Rightarrow F(0) = A = \int_{-\infty}^{\infty} f(x) dx = \frac{\sqrt{\pi}}{a}$$

$$F(\omega) = F\{f(x)\} = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$$

Fourier transform of a Gaussian

$f(x) = e^{-\pi x^2}$ Normalized Gaussian function $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

$$F(\omega) = F\{f(x)\} = e^{-\frac{\omega^2}{4a^2}}$$

$$\omega = 2\pi f \Rightarrow F(f) = e^{-\pi f^2}$$

The Gaussian $f(x) = e^{-\pi x^2}$ is its own Fourier transform.

Convolution property

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

- **Theorem 9 (The convolution theorem for Fourier transforms):** Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective Fourier transforms $F(\omega)$ and $G(\omega)$. Then

$$\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\} = F(\omega)G(\omega)$$

and, conversely,

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega$$

- Example 21: Fourier Transform

  It was shown in Example 17 that the function $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise, has the Fourier transform $F(\omega) = 2\mathrm{sinc}(\omega/\pi)$, so by the convolution theorem it follows that $\mathcal{F}\{(f * f)(x)\} = F(\omega)F(\omega) = 4\mathrm{sinc}^2(\omega/\pi)$.
  Confirm this result by calculating $(f * f)(x)$ and finding its Fourier transform.

$$f(t)f(x-t) = \begin{cases} 1, & -1 < t < x+1, \ (-2 < x < 0) \\ 0, & \text{otherwise} \end{cases}$$

$$f(t)f(x-t) = \begin{cases} 1, & x-1 < t < 1, \ (0 < x < 2) \\ 0, & \text{otherwise} \end{cases}$$

$$(f*f)(x)=\begin{cases}\int_{-1}^{x+1} dt = 2+x, & (-2<x<0)\\ \int_{x-1}^{1} dt = 2-x, & (0<x<2)\end{cases} \quad \text{and } (f*f)(x)=0 \text{ otherwise}$$

$$\mathcal{F}\{(f*f)(x)\} = \int_{-2}^{0}(2+x)e^{-i\omega x}dx + \int_{0}^{2}(2-x)e^{-i\omega x}dx$$

$$= 2\frac{1-\cos 2\omega}{\omega^2} = 4\frac{\sin^2\omega}{\omega^2} = 4\text{sinc}^2\left(\frac{\pi}{\omega}\right)$$

Parseval formula

- **Theorem 10 (The Parseval relation for the Fourier transforms):** If $f(x)$ has the Fourier transforms $F(\omega)$, Then

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$$

- Example 22: Using Parseval formula

  Using the result of Example 17 and the Parseval relation, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi$$

  The Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise is

$$F(\omega) = 2\frac{\sin\omega}{\omega}$$

$$\int_{-1}^{1} 1^2 dx = 2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\frac{\sin^2\omega}{\omega^2} d\omega \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2} d\omega = \pi$$

- Theorem 11 (Fourier transforms involving scaling x by a, shifting x by a, and shifting ω by ω_0): If $f(x)$ has the Fourier transforms $F(\omega)$, Then:

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}F(\omega/a),\ a > 0$$

$$\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega)$$

$$\mathcal{F}\{e^{i\omega_0 x}f(x)\} = F(\omega-\omega_0)$$

Duality property

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \Rightarrow 2\pi f(x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}d\lambda$$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(\lambda)e^{-i\omega\lambda}d\lambda = \int_{-\infty}^{\infty} F(x)e^{-i\omega x}dx = \mathcal{F}\{F(x)\}$$

$$\mathcal{F}\{F(x)\} = 2\pi f(-\omega)$$

- Example 23: Fourier transform of $f(x) = 1$

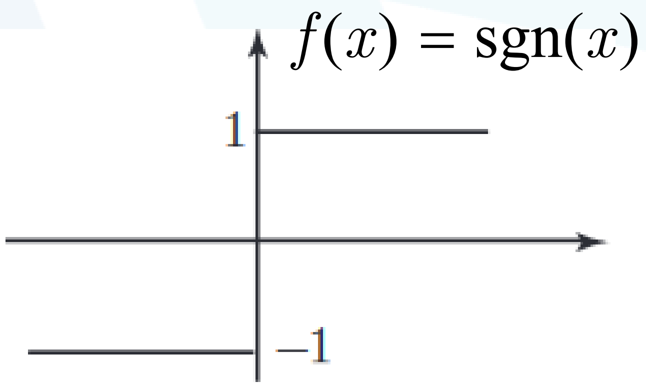
  Find the Fourier transform of $f(x) = 1$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx = \int_{-\infty}^{\infty} e^{-i\omega x} dx$$ could not be evaluated

The signal $f(x) = 1$ does not satisfy the existence conditions; it is neither absolute integrable nor square integrable. Its FT does not converge.

$$\mathcal{F}\{\delta(t)\} = 1 \quad \Rightarrow \quad \mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

- Example 24: Fourier transform of the signum function

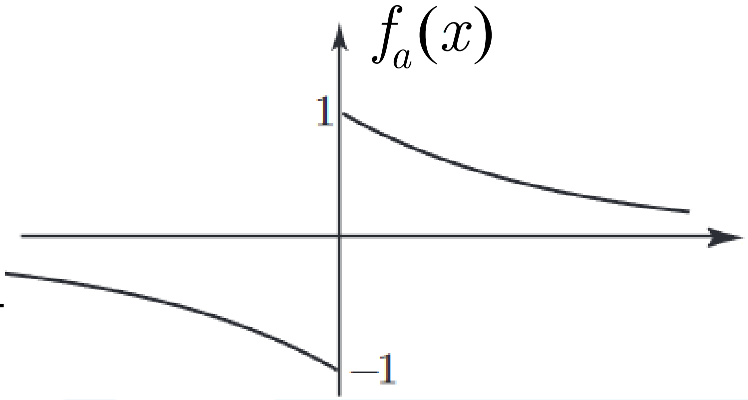


$$f(x) = \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{0} (-1)\, e^{-i\omega x} dx + \int_{0}^{\infty} (1)\, e^{-i\omega x} dx$$

The two integrals cannot be evaluated. Instead, we will define an intermediate signal $f_a(x)$ as:

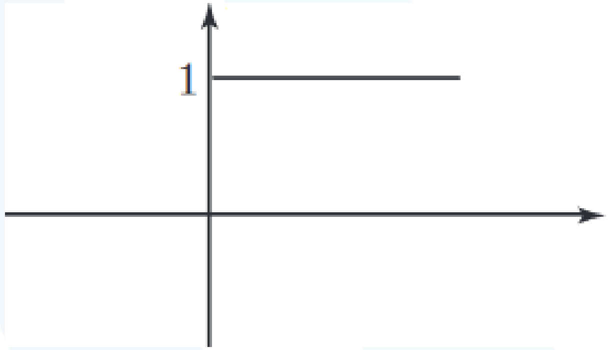
$$f_a(x) = \begin{cases} -e^{ax}, & x < 0 \\ e^{-ax}, & x > 0 \end{cases}, \text{ where } a \geq 0$$



$$F_a(\omega) = \int_{-\infty}^{0} (-e^{at})\, e^{-i\omega t} dt + \int_{0}^{\infty} (e^{-at})\, e^{-i\omega t} dt = -\frac{i2\omega}{a^2 + \omega^2}$$

$$F(\omega) = \mathcal{F}\{\mathrm{sgn}(x)\} = \lim_{a \to 0} \left[-\frac{i2\omega}{a^2 + \omega^2} \right] = \frac{2}{i\omega}$$

- Example 25: Fourier transform of the unit step function



$$f(x) = H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$\mathcal{F}\{H(x)\} = \int_{-\infty}^{\infty} H(x)\, e^{-i\omega x} dx = \int_{0}^{\infty} e^{-i\omega x} dx$$ could not be evaluated

$$H(x) = ½ + ½ \operatorname{sgn}(x) \Rightarrow \mathcal{F}\{H(x)\} = \mathcal{F}\{½ + ½ \operatorname{sgn}(x)\}$$

$$\mathcal{F}\{H(x)\} = \pi\delta(\omega) + \frac{1}{i\omega}$$

6. Boundary-Value Problems in Rectangular Coordinates

Separable Partial Differential Equations PDEs

- A PDE is an equation that contains one or more partial derivatives of an unknown function, call it u, that depends on at least two variables. Usually one of these deals with time t and the remaining with space (spatial variable(s)).
- The most important PDEs are the wave equations that can model the vibrating string and the vibrating membrane, the heat equation for temperature in a bar or wire, and the Laplace equation for electrostatic potentials.

- PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics.
- PDEs, like ordinary differential equations (ODEs), are classified as either linear or nonlinear.
- The dependent variable u and its partial derivatives in a linear PDE are only to the first power. We shall be interested in linear second-order PDEs.
- Example 26: Important Second-Order PDEs

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional wave equation}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional heat equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Two-dimensional wave equation

- the general form of a linear second-order PDE is given by:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

  where the coefficients A, B, C, ..., G are functions of x and y. When $G(x, y) = 0$, the equation is said to be homogeneous; otherwise, it is nonhomogeneous.

- A solution of a linear PDE is a function $u(x, y)$ of two independent variables that possesses all partial derivatives occurring in the equation and that satisfies the equation in some region of the xy-plane.

- It is often difficult to obtain a general solution of a linear second-order PDE. In general, the totality of solutions of a PDE is very large. For example, the functions: $u = x^2 - y^2, \quad u = e^x \cos y, \quad u = \sin x \cosh y, \quad u = \ln(x^2 + y^2)$ which are entirely different from each other, are solutions of 2D Laplace equation.
- Thus our focus throughout will be on finding particular solutions of some of the more important linear PDEs.
- There are several methods that can be tried to find particular solutions of a linear PDE, the one we are interested is called the method of separation of variables. In this method we seek a particular solution of the form of a product of a function of x and a function of y: $u(x, y) = X(x)Y(y)$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

- Example 27: Separation of Variables

Find product solutions of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$

Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation

$$X''Y = 4XY' \Rightarrow \frac{X''}{4X} = \frac{Y'}{Y}$$

Since the left-hand side of the last equation is independent of y and is equal to the right-hand side, which is independent of x, we conclude that both sides of the equation are independent of x and y.

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

$$X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0$$

Case I If $\lambda = 0$, then the two ODEs are: $X'' = 0$ and $Y' = 0$

$$X = c_1 + c_2 x \text{ and } Y = c_1 \Rightarrow u = XY = A_1 + B_1 x$$

Case II If $\lambda = -\alpha^2 < 0$, then the two ODEs are:

$$X'' - 4\alpha^2 X = 0 \text{ and } Y' - \alpha^2 Y = 0$$

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \text{ and } Y = c_6 e^{\alpha^2 y}$$

$$\Rightarrow u = XY = A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x$$

Case III If $\lambda = \alpha^2 > 0$, then the two ODEs are:

$$X'' + 4\alpha^2 X = 0 \text{ and } Y' + \alpha^2 Y = 0$$

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \text{ and } Y = c_9 e^{-\alpha^2 y}$$

$$\Rightarrow u = XY = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_2 e^{-\alpha^2 y} \sin 2\alpha x$$

- **Theorem 12 (Superposition principle):** If u_1, u_2, ..., u_k are solutions of a homogeneous linear partial differential equation, then the linear combination $u = c_1u_1 + c_2u_2 + ... + c_ku_k$, where the c_i, $i = 1, 2, ..., k$, are constants, is also a solution.

- **Definition:** classification of equations

  The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

  where the coefficients $A, B, C, ..., G$ are real constants, is said to be

  hyperbolic if $B^2 - 4AC > 0$,

  parabolic if $B^2 - 4AC = 0$,

  elliptic if $B^2 - 4AC < 0$.

Heat Equation: Solution by Fourier Series

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$$ boundary conditions

$$u(x, 0) = f(x), \quad 0 < x < L$$ initial condition

$$u(x,t) = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{c^2 T} = -\lambda$$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + c^2 \lambda T = 0$$

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0$$

$T(t) \neq 0$ for all $t \Rightarrow X(0) = 0$ and $X(L) = 0$

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0$$

$$X(x) = c_1 + c_2 x, \qquad \lambda = 0$$
$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \qquad \lambda = -\alpha^2 < 0$$
$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \qquad \lambda = \alpha^2 > 0$$

- When the boundary conditions $X(0) = 0$ and $X(L) = 0$ are applied to the first and second equations, these solutions yield only $X(x) = 0$, so $u = 0$.
- But when $X(0) = 0$ is applied to the third equation, we find that $c_1 = 0$ and $X(x) = c_2 \sin \alpha x$. The second boundary condition then implies that $X(L) = c_2 \sin \alpha L = 0$.
- To obtain a nontrivial solution, we must have $c_2 \neq 0$ and $\sin \alpha L = 0$. So $\alpha L = n\pi$ or $\alpha = n\pi/L$.
- Hence $X'' + \lambda X = 0$ possesses nontrivial solutions when:

$$\lambda_n = \alpha_n^2 = n^2\pi^2/L^2, \qquad n = 1, 2, 3, \dots$$

- These values of λ are the eigenvalues of the problem; the eigenfunctions are:

$$X_n(x) = c_2 \sin \frac{n\pi}{L} x$$

$$T' + c^2 \lambda T = 0 \Rightarrow T_n(t) = c_3 e^{-c^2 (n^2 \pi^2 / L^2) t}$$

$$u_n(x,t) = X_n(x) T_n(t) = c_2 \sin \frac{n\pi}{L} x \, c_3 e^{-c^2 (n^2 \pi^2 / L^2) t} = A_n e^{-c^2 (n^2 \pi^2 / L^2) t} \sin \frac{n\pi}{L} x$$

- Each of the product functions $u_n(x,t)$ is a particular solution of the partial differential equation, and each $u_n(x,t)$ satisfies both boundary conditions as well.

- The solution of the entire problem: by the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-c^2 (n^2 \pi^2 / L^2) t} \sin \frac{n\pi}{L} x$$

- To satisfy the initial condition, we would have to choose the coefficient A_n in such a manner that:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

- Hence A_n must be the coefficients of the Fourier sine series (half-range expansion of f in a sine series), thus

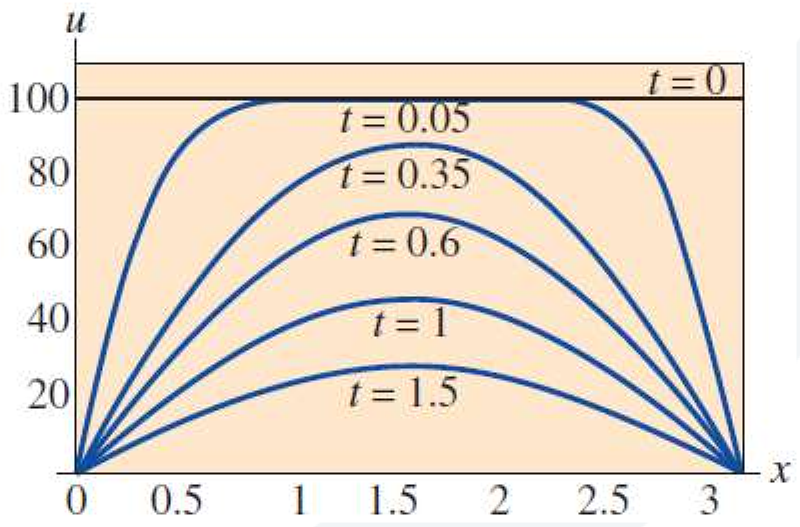
$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx\right) e^{-c^2 (n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x$$

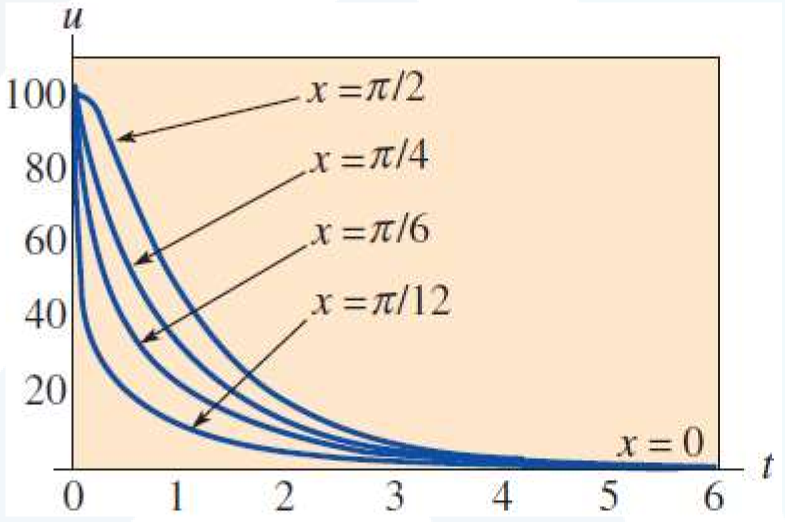
In the special case when the initial temperature is $u(x, 0) = 100$, $L = \pi$, and $c^2 = 1$,

$$A_n = \frac{200}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]e^{-n^2 t}\sin nx$$



u(x, t) graphed as a function of x



u(x, t) graphed as a function of t