CEDC301: Engineering Mathematics

Lecture Notes: Z-Transform: Part A

Ramez Koudsieh, Ph.D.

Faculty of Engineering
Department of Robotics and Intelligent Systems
Manara University

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Chapter 6

Z-Transform

1. Definition of the Z-Transform
2. Properties of the Z-Transform
3. Inverse Z-Transform
4. Solution of Difference Equations
5. Summation of Infinite Series

1. Definition of the Z-Transform

- The z-transform plays the same role in the discrete analysis as Laplace and Fourier transforms in a continuous system.
- Difference equations are formed in the discrete system, and their solution and analysis are carried out by z-transform, similar to the method of Laplace transformation in connection with differential equations.
- Definition: The z-transform of a sequence $f(n)$ is defined as:

$$Z\{f(n)\} = F(z) = \sum f(n) z^{-n}$$

provided that the power series converges.

where z, the independent variable of the transform is a complex number.

- There are two important variants:

Unilateral (or one-sided): $F(z) = \sum_{n=0}^{\infty} f(n) z^{-n};$

Bilateral (or two sided): $F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n};$

- When we refer to z-transform (ZT) without the qualifier word "bilateral" or "unilateral", we will always imply the unilateral ZT.
- Example 1: A simple z-transform example

$$f(n) = \{1.5, 1.2, -1.5, 3.6, 5.1\}$$

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} = 1.5 + 1.2z^{-1} - 1.5z^{-2} + 3.6z^{-3} + 5.1z^{-4}$$

The transform converges at all points in the complex z-plane except of $z = 0$.

Regions of Convergence (ROC)

- For the z-transform $F(z)$ of $f(n)$ to exist we need that:

$$|F(z)| = \left|\sum_{n=-\infty}^{\infty} f(n) z^{-n}\right| \le \sum_{n=-\infty}^{\infty} |f(n)|\left|r^{-n} e^{-i\theta n}\right| = \sum_{n=-\infty}^{\infty} |f(n)| r^{-n} < \infty$$

Thus, the regions of Convergence depends only on r and not on θ.

- Example 2: Z-Transform of an exponential sequence

$$f(n) = a^n, \; n \ge 0$$

$$F(z) = Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

converge if: $|az^{-1}| < 1 \Rightarrow |z| > |a|$

When $a = 1$, we obtain

$$Z\{1\} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}, \quad |z| > 1$$

- Example 3: Z-Transform of the unit-impulse

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$F(z) = Z\{\delta(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = f(0) z^{0} = 1$$

It converges at every point in the z-plane

- Example 4: Z-Transform of the unit-step

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

$$F(z) = \mathcal{Z}\{u(n)\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

converge if: $|z^{-1}| < 1 \Rightarrow |z| > 1$

- Example 5: Z-Transform of a discrete-time pulse

$$f(n) = \begin{cases} 1, & 0 \le n < N \\ 0, & \text{otherwise} \end{cases}$$

$$F(z) = \sum_{n=0}^{N-1} (1) z^{-n} = \frac{1-z^{-N}}{1-z^{-1}}, \quad |z| > 0$$

$$F(z) = \frac{z^N - 1}{z^{N-1}(z-1)}$$ It seems as though $F(z)$ might have a pole at $z = 1$

Zeros: $z_k = e^{i2\pi k/N}, \quad k = 0, \cdots, N-1$

Poles: $z = 1$ and $p_k = 0, \quad k = 1, \cdots, N-1$

The factors $(z-1)$ in numerator and denominator polynomials cancel each other, therefore there is neither a zero nor a pole at $z = 1$.

- Example 6: Z-Transform of complex exponential $f(n) = e^{inx}$

$$F(z) = \sum_{n=0}^{\infty} e^{inx} z^{-n} = \sum_{n=0}^{\infty} (e^{ix} z^{-1})^n = \frac{1}{1 - e^{ix} z^{-1}}$$

$$F(z) = \frac{z}{z - e^{ix}} \qquad \left|e^{ix} z^{-1}\right| < 1 \Rightarrow |z| > 1$$

2. Properties of the Z-Transform

Linearity

- Theorem 1 (Linearity of the Z-Transform): The z-transform is a linear operation; that is, for any sequences $f(n)$ and $g(n)$ whose z-transforms exist and any constants a and b, the z-transform of $af + bg$ exists, and

$$Z\{af(n) + bg(n)\} = aZ\{f(n)\} + bZ\{g(n)\}$$

- Example 7: Z-Transform of a cosine and sine

$$\cos(nx) = \tfrac{1}{2} e^{inx} u + \tfrac{1}{2} e^{-inx} \Rightarrow Z\{\cos(nx)\} = \tfrac{1}{2} Z\{e^{inx}\} + \tfrac{1}{2} Z\{e^{-inx}\}$$

$$Z\{\cos(nx)\} = \frac{1/2}{1 - e^{ix} z^{-1}} + \frac{1/2}{1 - e^{-ix} z^{-1}} = \frac{1 - \cos x\, z^{-1}}{1 - 2\cos x\, z^{-1} + z^{-2}} = \frac{z(z - \cos x)}{z^2 - 2\cos x\, z + 1}$$

ROC is $|z| > 1$

$$\sin(nx) = \tfrac{1}{2i} e^{inx} u - \tfrac{1}{2i} e^{-inx} \Rightarrow Z\{\cos(nx)\} = \tfrac{1}{2i} Z\{e^{inx}\} - \tfrac{1}{2i} Z\{e^{-inx}\}$$

$$Z\{\sin(nx)\} = \frac{1/2i}{1-e^{ix}z^{-1}} - \frac{1/2i}{1-e^{-ix}z^{-1}} = \frac{\sin x\, z^{-1}}{1-2\cos x\, z^{-1}+z^{-2}} = \frac{z\sin x}{z^2-2\cos x\, z+1}$$

ROC is $|z| > 1$

- Example 8: Z-Transform of a cosine

Find the Z-transform of $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$Z\left\{\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\} = Z\left\{\cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z(z-\cos\frac{\pi}{2})}{z^2-2\cos\frac{\pi}{2}\,z+1} - \frac{z\sin\frac{\pi}{2}}{z^2-2\cos\frac{\pi}{2}\,z+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

Shifting

- The effect of shifting depends upon whether it is to the right or to the left. For the sequence $f(n-2)$, no values from the sequence $f(n)$ are lost; thus, we anticipate that the z-transform of $f(n-2)$ only involves $F(z)$.
- For the sequence $f(n+2)$, the first two values of $f(n)$ are lost, and we anticipate that the z-transform of $f(n+2)$ cannot be expressed solely in terms of $F(z)$ but must include those two lost pieces of information.

n	$f(n)$	$f(n-2)$	$f(n+2)$
0	1	0	4
1	2	0	8
2	4	1	16
3	8	2	64
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Theorem 2 (Time Shifting): If $\mathcal{Z}\{f(n)\} = F(z)$ and $m \geq 0$, then

$$\mathcal{Z}\{f(n-m)\} = z^{-m}F(z)$$

$$\mathcal{Z}\{f(n+m)\} = z^{m}\left[F(z) - \sum_{r=0}^{m-1} f(r)z^{-r}\right]$$

$$\mathcal{Z}\{f(n+1)\} = z\{F(z) - f(0)\},$$
$$\mathcal{Z}\{f(n+2)\} = z^2\{F(z) - f(0)\} - zf(1),$$
$$\mathcal{Z}\{f(n+3)\} = z^3\{F(z) - f(0)\} - z^2 f(1) - zf(2)$$

- Example 9: Shifting to left

Find $\mathcal{Z}\{4^{n+3}\}$

$$\mathcal{Z}\{4^n\} = \frac{z}{z-4} \Rightarrow \mathcal{Z}\{4^{n+3}\} = z^3\{\frac{z}{z-4} - 4^0\} - z^2 4^1 - z4^2$$

$$\mathcal{Z}\{4^{n+3}\} = z^3\{\frac{z}{z-4} - 4^0\} - z^2 4^1 - z4^2 = \frac{z^4}{z-4} - z^3 - 4z^2 - 16z = \frac{64z}{z-4}$$

ROC is $|z| > 4$

Multiplication by an exponential

- Theorem 3 (Multiplication by an exponential): If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$\mathcal{Z}\{a^n f(n)\} = F(a^{-1}z), \quad |z| > |a|$$

- Example 10: Multiplication by an exponential

Find $\mathcal{Z}\{a^n\}$, $n \geq 0$

$$F(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\mathcal{Z}\{a^n u(n)\} = F(a^{-1}z) = \frac{a^{-1}z}{a^{-1}z-1} = \frac{z}{z-a}, \quad |z| > |a|$$

Derivative of z-transforms

- Theorem 4 (Derivative of z-transforms): If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$\mathcal{Z}\{nf(n)\} = -z\frac{d}{dz}F(z)$$

More generally,

$$\mathcal{Z}\{n^k f(n)\} = -z\frac{d}{dz}\mathcal{Z}\{n^{k-1}f(n)\}, \quad k = 1, 2, \ldots$$

$$\mathcal{Z}\{n^k\} = -z\frac{d}{dz}\mathcal{Z}\{n^{k-1}\}$$

- Example 11: Derivative of z-transforms

Find $\mathcal{Z}\{n\}$, $\mathcal{Z}\{n^2\}$, and $\mathcal{Z}\{n^3\}$

$$Z\{n\} = -z\frac{d}{dz}Z\{n^0\} = -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}$$

$$Z\{n^2\} = -z\frac{d}{dz}Z\{n^1\} = -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}$$

$$Z\{n^3\} = -z\frac{d}{dz}Z\{n^2\} = -z\frac{d}{dz}\frac{z(z+1)}{(z-1)^3} = \frac{z(z^2+4z+1)}{(z-1)^4}$$

Convolution

- **Theorem 5 (Convolution property):** If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$, then the z-transform of the convolution $f(n) * g(n)$ is given by:

$$Z\{f(n) * g(n)\} = Z\{f(n)\}Z\{g(n)\}$$

where $f(n) * g(n) = \sum_{m=0}^{\infty} f(n-m)g(m) = \sum_{m=0}^{\infty} f(m)g(n-m)$

- Example 12: Using the convolution property

$$f(n) = \{4, 3, 2, 1\}, \quad g(n) = \{3, 7, 4\}$$

Determine $h(n) = f(n) * g(n)$ using z-transform techniques.

$$F(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}, \quad G(z) = 3 + 7z^{-1} + 4z^{-2}$$

$$H(z) = F(z)G(z) = 12 + 37z^{-1} + 43z^{-2} + 29z^{-3} + 15z^{-4} + 4z^{-5}$$

$$h(n) = \{12, 37, 43, 29, 15, 4\}$$

Initial Value Theorem

- Theorem 6 (Initial Value Theorem): If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$f(0) = \lim_{z \to \infty} F(z)$$

Also, if $f(0) = 0$, then $f(1) = \lim_{z \to \infty} zF(z)$

- Example 13: Initial Value Theorem

  Verify the initial value theorem for $f(n) = a^n$

$$f(0) = a^0 = 1, \quad \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{z}{z-a} = 1$$

Final Value Theorem

- Theorem 7 (Final Value Theorem): If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}\{(z-1)F(z)\}$$

  provided the limits exist

- Example 14: Final Value Theorem

  Verify the initial value theorem for the sequence with the z-transform:

$$F(z)=\frac{10z^2+2z}{(z-1)(5z-1)^2}$$

$$\lim_{z\to 1}\left\{(z-1)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\}=\lim_{z\to 1}\left\{\frac{10z^2+2z}{(5z-1)^2}\right\}=\frac{3}{4}$$

Verification:

$$F(z)=\frac{10z^2+2z}{(z-1)(5z-1)^2}=\frac{3}{4}\frac{z}{z-1}-\frac{3}{4}\frac{z}{z-1/5}-\frac{1}{5}\frac{z}{(z-1/5)^2}$$

$$f(n)=\frac{3}{4}-\frac{3}{4}\left(\frac{1}{5}\right)^n-n\left(\frac{1}{5}\right)^n$$

$$\lim_{n\to\infty}f(n)=\frac{3}{4}$$

- **Note:**

$$F(z) = \mathcal{Z}\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \cdots$$

$$f(0) = \lim_{z\to\infty} F(z)$$

$$f(1) = \lim_{z\to\infty} z\left[F(z) - f(0)\right]$$

$$f(2) = \lim_{z\to\infty} z^2\left[F(z) - f(0) - \frac{f(1)}{z}\right]$$

$$f(3) = \lim_{z\to\infty} z^3\left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2}\right]$$

$$\vdots$$

- Example 15: Initial Value Theorem

If $F(z)=\dfrac{2z^2+5z+14}{(z-1)^4}$, evaluate *f*(2), *f*(3)

$$f(0)=\lim_{z\to\infty}F(z)=\lim_{z\to\infty}\frac{2z^2+5z+14}{(z-1)^4}=0$$

$$f(1)=\lim_{z\to\infty}z\left[F(z)-f(0)\right]=\lim_{z\to\infty}z\,\frac{2z^2+5z+14}{(z-1)^4}=0$$

$$f(2)=\lim_{z\to\infty}z^2\left[F(z)-f(0)-\frac{f(1)}{z}\right]=\lim_{z\to\infty}z^2\,\frac{2z^2+5z+14}{(z-1)^4}=2$$

$$f(3)=\lim_{z\to\infty}z^3\left[F(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}\right]=\lim_{z\to\infty}\left\{z^3\,\frac{2z^2+5z+14}{(z-1)^4}-2z\right\}=13$$

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Z-Transforms of Some Commonly Used Sequences

	$f(n),\ n \geq 0$	$F(z)$	ROC
1	$\delta(n)$	1	$\lvert z\rvert > 0$
2	$u(n)$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
3	a^n	$\dfrac{z}{z-a}$	$\lvert z\rvert > a$
4	n	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$\dfrac{a^n}{n!}$	$e^{a/z}$	$\lvert z\rvert > 0$

	$f(n),\ n \geq 0$	$F(z)$	ROC
6	$\cos nx$	$\dfrac{z(z-\cos x)}{z^2-2z\cos x+1}$	$\lvert z\rvert > 1$
7	$\sin nx$	$\dfrac{z\sin x}{z^2-2z\cos x+1}$	$\lvert z\rvert > 1$
8	$\cosh nx$	$\dfrac{z(z-\cosh x)}{z^2-2z\cosh x+1}$	$\lvert z\rvert > e^{\lvert x\rvert}$
9	$\sinh nx$	$\dfrac{z\sinh x}{z^2-2z\cosh x+1}$	$\lvert z\rvert > e^{\lvert x\rvert}$
10	$(\ln a)^n / n!$	$a^{1/z}$	$\lvert z\rvert > 0$