جامعة المنارة

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قسم الروبوت و الأنظمة الذكية

عميد الكلية

د. إياد حاتم

المحاضرة السادسة

نظم التحكم اللاخطي
Nonlinear Control systems

مدرس المقرر

د.بلال شيحا

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**
- The basic idea in this method is that of isoclines. Consider the dynamics in ($\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$). At a point ($\mathbf{x_1}, \mathbf{x_2}$) in the phase plane, the slope of the tangent to the trajectory can be determined by ($\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$). An isocline is defined to be the locus of the points with a given tangent slope. An isocline with slope $\boldsymbol{\alpha}$ is thus defined to be

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha$$

This is to say that points on the curve

$$f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

all have the same tangent slope $\boldsymbol{\alpha}$.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**
- In the method of isoclines, the phase portrait of a system is generated in two steps. In the first step, a field of directions of tangents to the trajectories is obtained. In the second step, phase plane trajectories are formed from the field of directions .
- Let us explain the isocline method on the mass-spring system in ($\ddot{x} + x = 0$). The slope of the trajectories is easily seen to be

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

- Therefore, the isocline equation for a slope α is

$$x_1 + \alpha x_2 = 0$$

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**
- Therefore, the isocline equation for a slope $\boldsymbol{\alpha}$ is

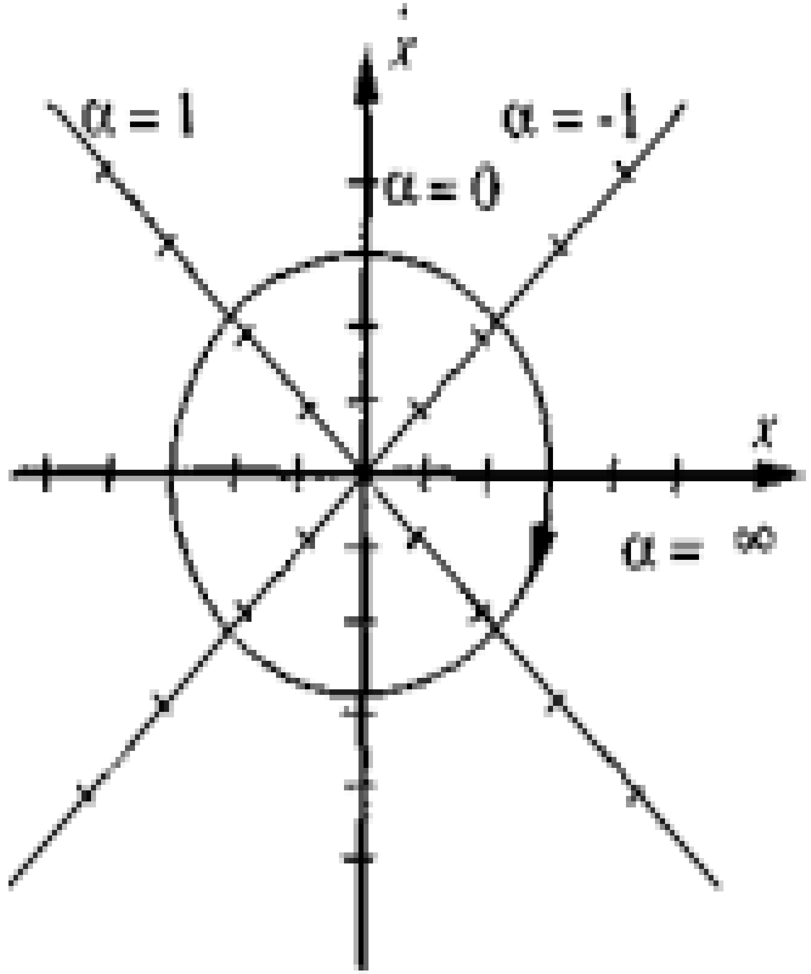
$$\boldsymbol{x_1 + \alpha x_2 = 0}$$

- i.e., a straight line. Along the line, we can draw a lot of short line segments with slope $\boldsymbol{\alpha}$. By taking $\boldsymbol{\alpha}$ to be different values, a set of isoclines can be drawn, and a field of directions of tangents to trajectories are generated, as shown in Figure(7). To obtain trajectories from the field of directions, we assume that the tangent slopes are locally constant. Therefore, a trajectory starting from any point in the plane can be found by connecting a sequence of line segments.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**

- (7)



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**
- Let us use the method of isoclines to study the **Van der Pol** equation, a nonlinear equation.
- For the Van der Pol equation

$$\ddot{x}+0.2(x^2-1)\dot{x}+x=0$$

- an isocline of slope a is defined by

$$\frac{d\dot{x}}{dx}=-\frac{0.2(x^2-1)\dot{x}+x}{\dot{x}}=\alpha$$

- Therefore, the points on the curve

$$0.2(x^2-1)\dot{x}+x+\alpha\dot{x}=0$$

- all have the same slope α .

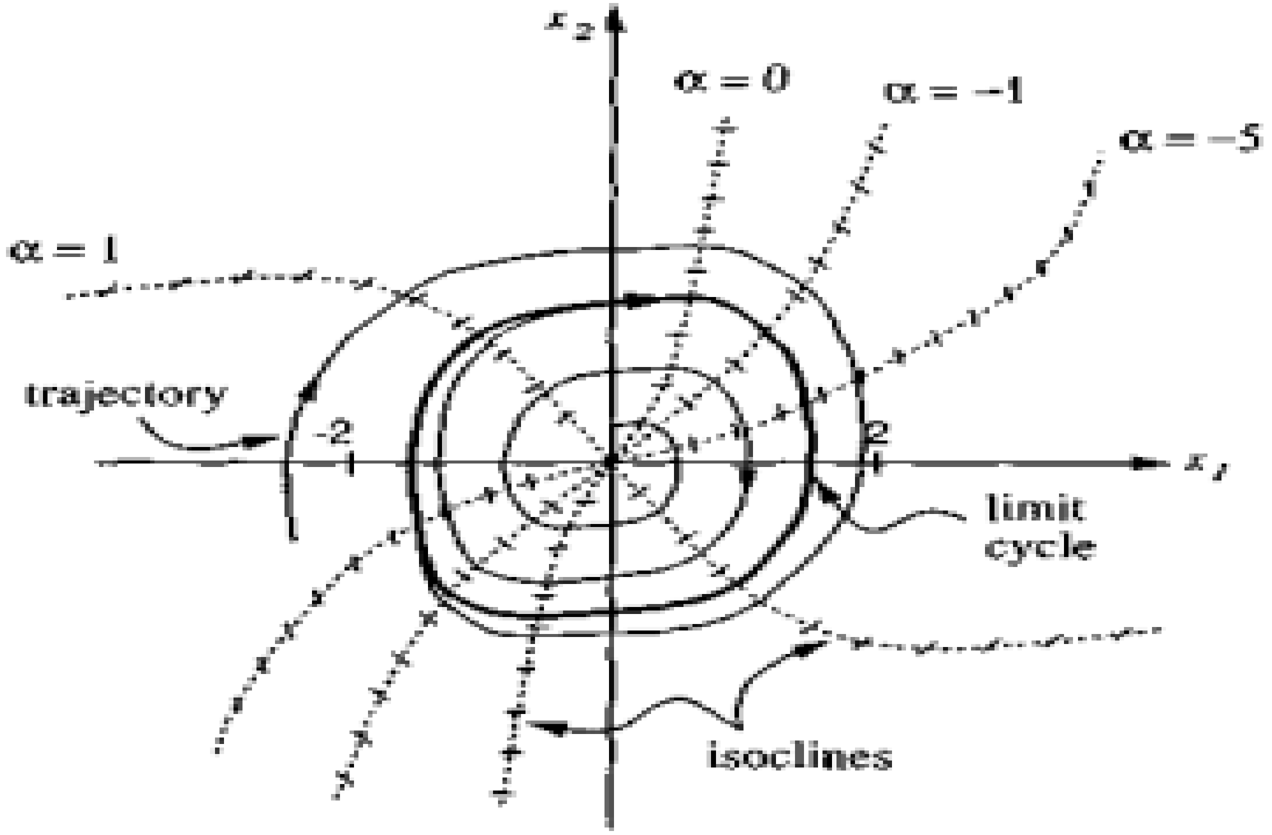
Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**
- By taking α of different values, different isoclines can be obtained, as plotted in Figure(8). Short line segments are drawn on the isoclines to generate a field of tangent directions. The phase portraits can then be obtained, as shown in the plot. It is interesting to note that there exists a closed curve in the portrait, and the trajectories starting from both outside and inside converge to this curve. This closed curve corresponds to a limit cycle, as will be discussed.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**

- (8)



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **THE METHOD OF ISOCLINES**
- Note that the same scales should be used for the $\mathbf{x_1}$ axis and $\mathbf{x_2}$ axis of the phase plane, so that the derivative dx_2/dx_1 equals the geometric slope of the trajectories. Also note that, since in the second step of phase portrait construction we essentially assume that the slope of the phase plane trajectories is locally constant, more isoclines should be plotted in regions where the slope varies quickly, to improve accuracy.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **Determining Time from Phase Portraits**
- Note that time **t** does not explicitly appear in the phase plane having $\mathbf{x_1}$ and $\mathbf{x_2}$ as coordinates. However, in some cases, we might be interested in the time information. For example, one might want to know the time history of the system states starting from a specific initial point. Another relevant situation is when one wants to know how long it takes for the system to move from a point to another point in a phase plane trajectory. We now describe two techniques for computing time history from phase portraits. Both techniques involve a step-by step procedure for recovering time.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **Determining Time from Phase Portraits**
- **Obtaining time from $\Delta \mathbf{t} \cong \Delta \boldsymbol{x}/\dot{\boldsymbol{x}}$**
- In a short time $\Delta \mathbf{t}$, the change of **x** is approximately

$$\Delta \boldsymbol{x} \cong \Delta \mathbf{t} \dot{\boldsymbol{x}}$$

- where $\dot{\boldsymbol{x}}$ is the velocity corresponding to the increment $\Delta \boldsymbol{x}$. Note that for a $\Delta \boldsymbol{x}$ of finite magnitude, the average value of velocity during a time increment should be used to improve accuracy. From (Figure 8), the length of time corresponding to the increment $\Delta \boldsymbol{x}$ is

$$\Delta \mathbf{t} \cong \Delta \boldsymbol{x}/\dot{\boldsymbol{x}}$$

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **Determining Time from Phase Portraits**
- **Obtaining time from $\Delta \mathbf{t} \cong \Delta \boldsymbol{x} / \dot{\boldsymbol{x}}$**
- The above reasoning implies that, in order to obtain the time corresponding to the motion from one point to another point along a trajectory, one should divide the corresponding part of the trajectory into a number of small segments (not necessarily equally spaced), find the time associated with each segment, and then add up the results. To obtain the time history of states corresponding to a certain initial condition, one simply computes the time $\boldsymbol{t}$ for each point on the phase trajectory, and then plots $\boldsymbol{x}$ with respect to $\boldsymbol{t}$ and $\dot{\boldsymbol{x}}$ with respect to $\boldsymbol{t}$,

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Constructing Phase Portraits**
- **Determining Time from Phase Portraits**
- **Obtaining time from** $\mathbf{t} = \int (\mathbf{1}/\dot{\boldsymbol{x}})\boldsymbol{dx}$
- Since $\dot{\boldsymbol{x}} = (\boldsymbol{dx}/\boldsymbol{dt})$, we can write $\boldsymbol{dt} = (\boldsymbol{dx}/\dot{\boldsymbol{x}})$. Therefore,

$$\mathbf{t} - \mathbf{t}_0 = \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} (\mathbf{1}/\dot{\boldsymbol{x}})\boldsymbol{dx}$$

- where $\boldsymbol{x}$ corresponds to time $\boldsymbol{t}$ and $\boldsymbol{x}_0$ corresponds to time $\boldsymbol{t}_0$. This equation implies that, if we plot a phase plane portrait with new coordinates $\boldsymbol{x}$ and $(\mathbf{1}/\dot{\boldsymbol{x}})$, then the area under the resulting curve is the corresponding time interval.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Linear Systems**
- we describe the phase plane analysis of linear systems. Besides allowing us to visually observe the motion patterns of linear systems, this will also help the development of nonlinear system analysis in the next, because a nonlinear systems behaves similarly to a linear system around each equilibrium point.
- The general form of a linear second-order system is

$$\dot{x}_1 = ax_1 + bx_2 \tag{9a}$$
$$\dot{x}_2 = cx_1 + dx_2 \tag{9b}$$

- To facilitate later discussions, let us transform this equation into a scalar second-order differential equation.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Linear Systems**
- Note from (9a) and (9b) that

$$b\dot{x}_2 = bcx_1 + d(\dot{x}_1 - ax_1)$$

- Consequently, differentiation of (9a) and then substitution of (9b) leads to

$$\ddot{x}_1 = (a+d)\dot{x}_1 + (cb - ad)x_1$$

- Therefore, we will simply consider the second-order linear system described by

$$\ddot{x} + a\dot{x} + bx = 0 \qquad (10)$$

- To obtain the phase portrait of this linear system, we first solve for the time history

$$x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad for\ \lambda_1 \neq \lambda_2 \qquad (11.a)$$

$$x(t) = k_1 e^{\lambda_1 t} + k_2 t e^{\lambda_1 t} \qquad for\ \lambda_1 = \lambda_2 \qquad (11.b)$$

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Linear Systems**
- where the constants $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ are the solutions of the characteristic equation

$$\boldsymbol{s^2 + as + b = (s - \lambda_1)(s - \lambda_2)}$$

- The roots $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ can be explicitly represented as

$$\boldsymbol{\lambda_1 = (-a + \sqrt{a^2 - 4b})/2}$$
$$\boldsymbol{\lambda_2 = (-a - \sqrt{a^2 - 4b})/2}$$

- For linear systems described by ($\boldsymbol{\ddot{x} + a\dot{x} + bx = 0}$), there is only one singular point (assuming $\boldsymbol{b} \neq 0$), namely the origin. However, the trajectories in the vicinity of this singularity point can display quite different characteristics, depending on the values of ***a*** and ***b***.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Linear Systems**
- The following cases can occur

1. $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ are both real and have the same sign (positive or negative)
2. $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$are both real and have opposite signs
3. $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$are complex conjugate with non-zero real parts
4. $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$are complex conjugates with real parts equal to zero

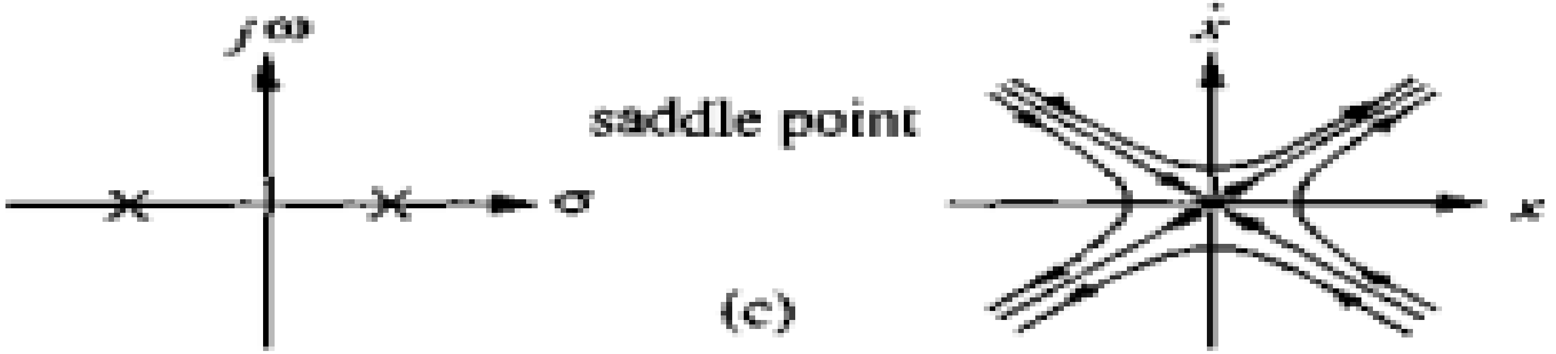
- We now briefly discuss each of the above four cases.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Linear Systems**
- **STABLE OR UNSTABLE NODE**
- The first case corresponds to ***a*** *node*. A node can be stable or unstable. If the eigenvalues are negative, the singularity point is called a stable node because both x and $\dot{x}$ converge to zero exponentially, as shown in Figure (a). If both eigenvalues are positive, the point is called an unstable node, because both x and $\dot{x}$ diverge from zero exponentially, as shown in Figure (b). Since the eigenvalues are real, there is no oscillation in the trajectories.

Phase Plane Analysis
Concepts of Phase Plane Analysis

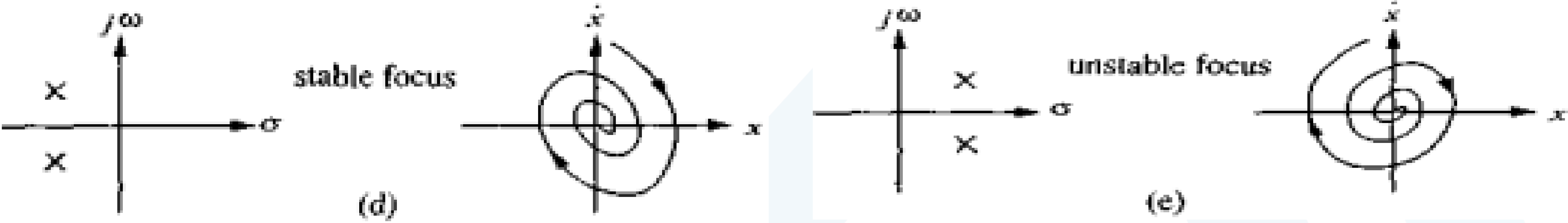
- **Phase Plane Analysis of Linear Systems**
- **SADDLE POINT**
- The second case (say $\boldsymbol{\lambda_1 < 0}$ and $\boldsymbol{\lambda_2 > 0}$) corresponds to a saddle point (Figure (c)). The phase portrait of the system has the interesting "saddle" shape shown in Figure (c). Because of the unstable pole $\boldsymbol{\lambda_2}$, almost all of the system trajectories diverge to infinity. In this figure, one also observes two straight lines passing through the origin. The diverging line (with arrows pointing to infinity) corresponds to initial conditions which make $\boldsymbol{k_2}$ (i.e., the unstable component) equal zero. The converging straight line corresponds to initial conditions which make $\boldsymbol{k_1}$ equal zero.



(c)

Phase Plane Analysis
Concepts of Phase Plane Analysis

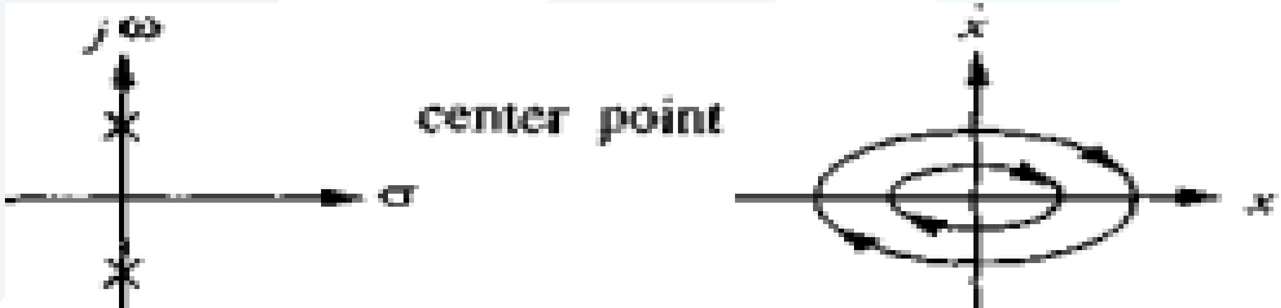
- **Phase Plane Analysis of Linear Systems**
- **STABLE OR UNSTABLE FOCUS**
- The third case corresponds to a focus. A stable focus occurs when the real part of the eigenvalues is negative, which implies that $x(t)$ and $\dot{x}$ *(t)* both converge to zero. The system trajectories in the vicinity of a stable focus are depicted in Figure (d). Note that the trajectories encircle the origin one or more times before converging to it, unlike the situation for a stable node. If the real part of the eigenvalues is positive, then $x(t)$ and $\dot{x}$ *(t)* both diverge to infinity, and the singularity point is called an unstable focus. The trajectories corresponding to an unstable focus are sketched in Figure (e).



(d) (e)

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Linear Systems**
- **CENTER POINT**
- The last case corresponds to a center point, as shown in Figure (f). The name comes from the fact that all trajectories are ellipses and the singularity point is the center of these ellipses. The phase portrait of the undamped mass-spring system belongs to this category.



(f)

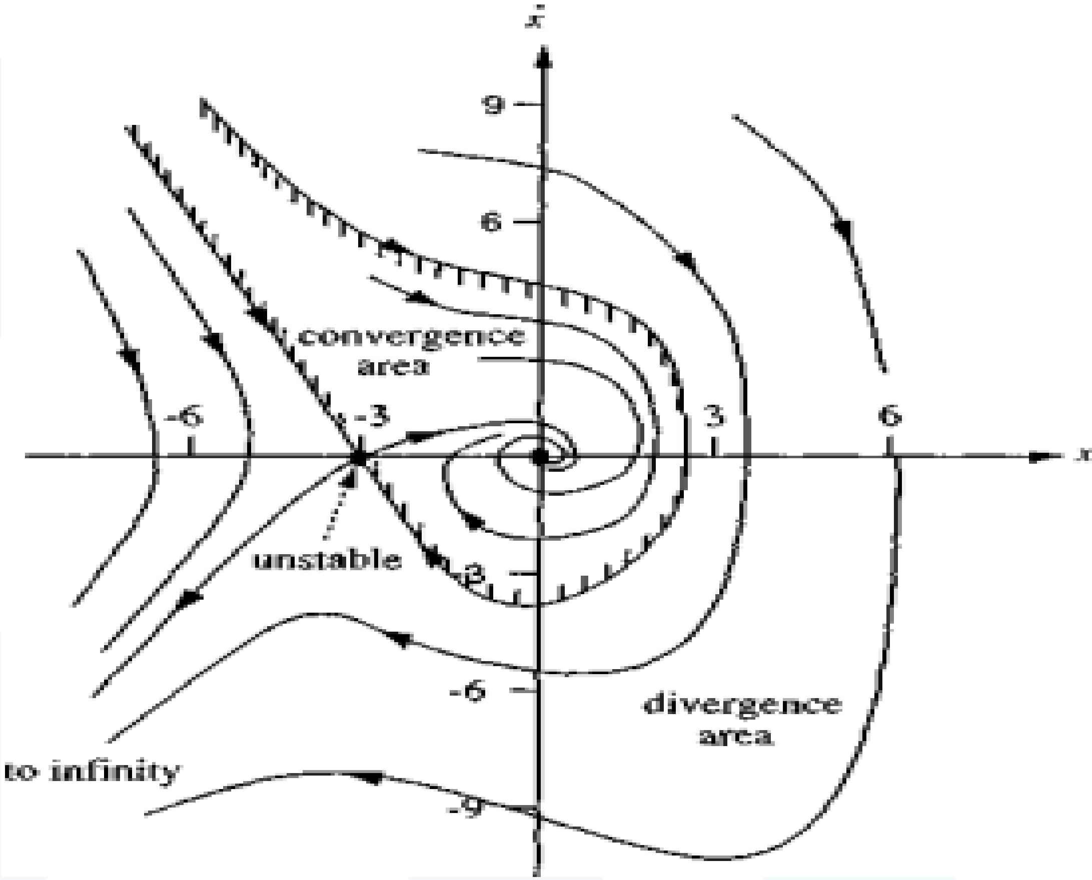
- Note that the stability characteristics of linear systems are uniquely determined by the nature of their singularity points. This, however, is not true for nonlinear systems.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- In discussing the phase plane analysis of nonlinear systems, two points should be kept in mind. Phase plane analysis of nonlinear systems is related to that of linear systems, because the local behavior of a nonlinear system can be approximated by the behavior of a linear system. Yet, nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles. We now discuss these points in more detail.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LOCAL BEHAVIOR OF NONLINEAR SYSTEMS**
- In the phase portrait of Figure 2, one notes that, in contrast to linear systems, there are two singular points, **(0,0)** and **(-3,0)**. However, we also note that the features of the phase trajectories in the neighborhood of the two singular points look very much like those of linear systems, with the first point corresponding to a stable focus and the second to a saddle point. This similarity to a linear system in the local region of each singular point can be formalized by linearizing the nonlinear system, as we now discuss.



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LOCAL BEHAVIOR OF NONLINEAR SYSTEMS**
- If the singular point of interest is not at the origin, by defining the difference between the original state and the singular point as a new set of state variables, one can always shift the singular point to the origin. Therefore, without loss of generality, we may simply consider Equation ($\dot{x}_1=f_1(x_1, x_2)$, $\dot{x}_2=f_2(x_1, x_2)$ with a singular point at **0**. Using Taylor expansion, Equations ($\dot{x}_1=f_1(x_1, x_2)$) and ($\dot{x}_2=f_2(x_1, x_2)$ can be rewritten as

$$\dot{x}_1 = \mathrm{a}x_1 + bx_2 + g_1(x_1, x_2)$$
$$\dot{x}_2 = \mathrm{c}x_1 + dx_2 + g_2(x_1, x_2)$$

- where g_1 and g_2 contain higher order terms.

Phase Plane Analysis
Concepts of Phase Plane Analysis

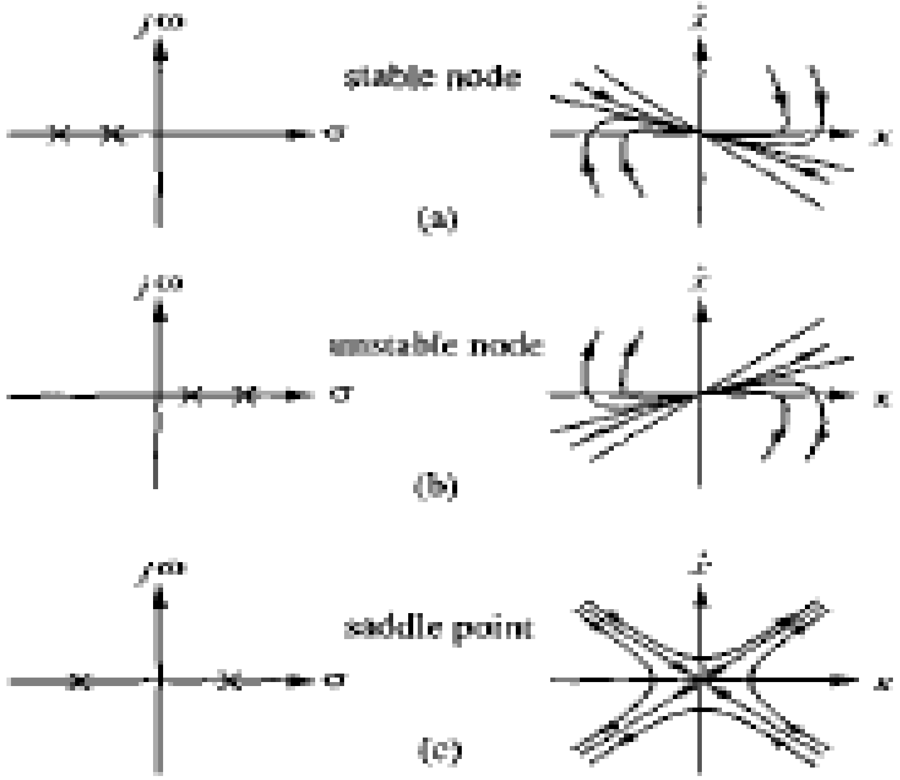
- **Phase Plane Analysis of Nonlinear System**
- **LOCAL BEHAVIOR OF NONLINEAR SYSTEMS**
- In the vicinity of the origin, the higher order terms can be neglected, and therefore, the nonlinear system trajectories essentially satisfy the linearized equation

$$\dot{x}_1 = \mathrm{a}x_1 + bx_2$$
$$\dot{x}_2 = \mathrm{c}x_1 + dx_2$$

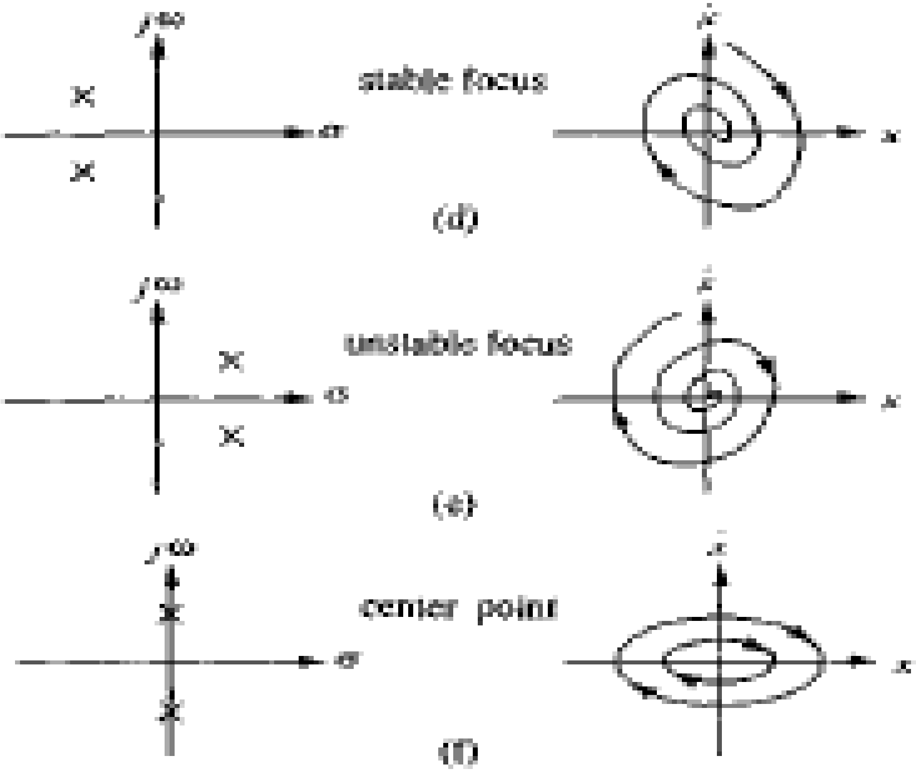
- As a result, the local behavior of the nonlinear system can be approximated by the patterns shown in Figure 9.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LOCAL BEHAVIOR OF NONLINEAR SYSTEMS**
-

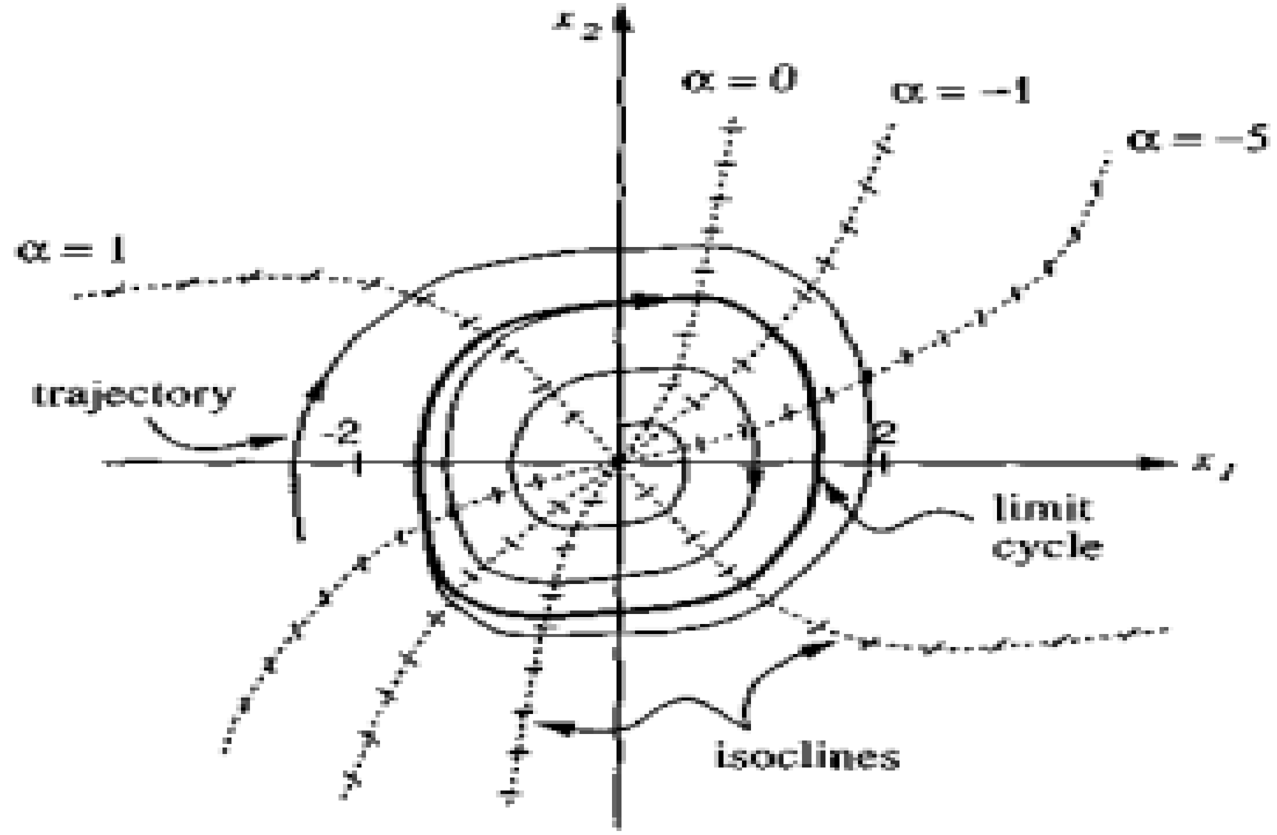


(9)



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- In the phase portrait of the nonlinear Van der Pol equation, shown in Figure.8, one observes that the system has an unstable node at the origin. Furthermore, there is a closed curve in the phase portrait. Trajectories inside the curve and those outside the curve all tend to this curve, while a motion started on this curve will stay on it forever, circling periodically around the origin. This curve is an instance of the so-called "limit cycle" phenomenon. Limit cycles are unique features of nonlinear systems.



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- In the phase plane, a limit cycle is defined as an isolated closed curve. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with nearby trajectories converging or diverging from it). Thus, while there are many closed curves in the phase portraits of the mass-spring-damper system in Example 1 or the satellite system in Example 5, these are not considered limit cycles in this definition, because they are not isolated.

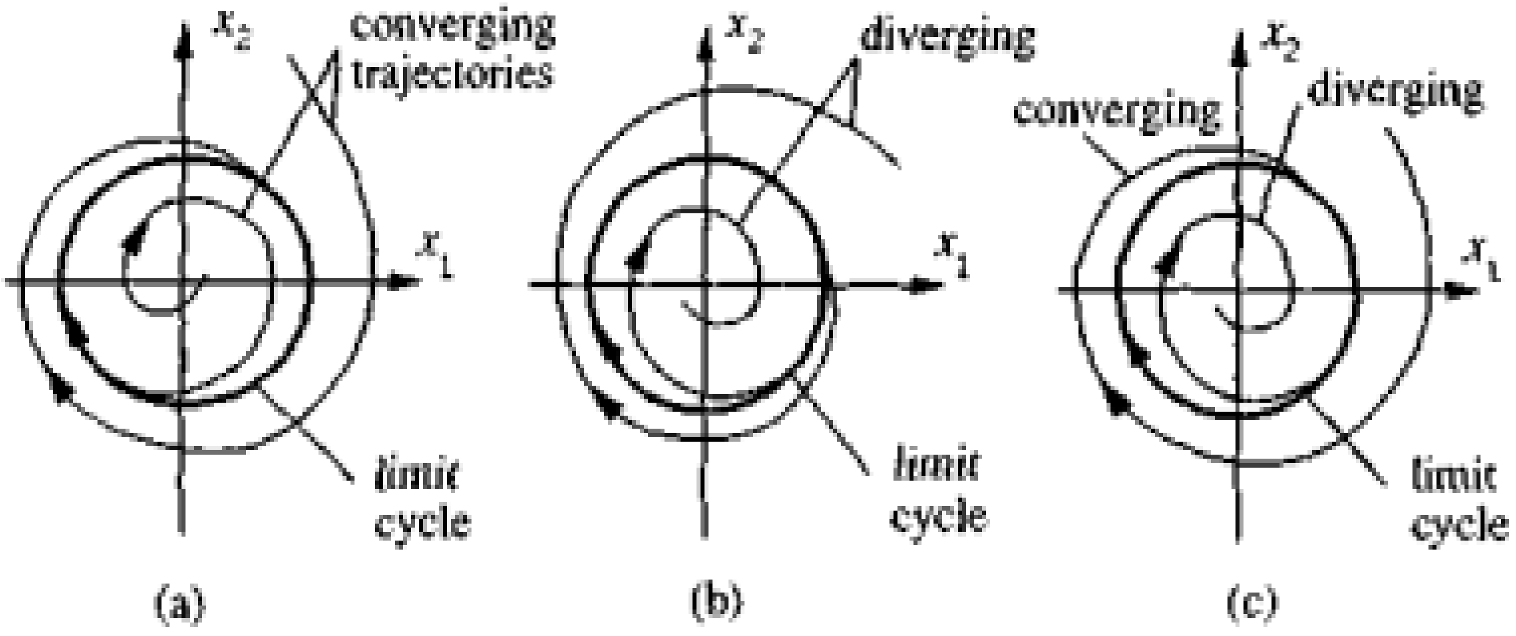
Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, one can distinguish three kinds of limit cycles

1. **Stable Limit Cycles:** all trajectories in the vicinity of the limit cycle converge to it as $\boldsymbol{t} \rightarrow \infty$ (Figure 10(a));
2. **Unstable Limit Cycles:** all trajectories in the vicinity of the limit cycle diverge from it as $\boldsymbol{t} \rightarrow \infty$ (Figure 10(b));
3. **Semi-Stable Limit Cycles:** some of the trajectories in the vicinity converge to it, while the others diverge from it as $\boldsymbol{t} \rightarrow \infty$ (Figure 10(c));

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- As seen from the phase portrait of Figure 8, the limit cycle of the Van der Pol equation is clearly stable. Let us consider some additional examples of stable, unstable, and semi-stable limit cycles.



Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- **Example 2.7: stable, unstable, and semi-stable limit cycles**
- Consider the following nonlinear systems
- $(a)\quad \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \qquad \dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2 - 1) \qquad (2.12)$
- $(b)\quad \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \qquad \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \qquad (2.13)$
- $(c)\quad \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \qquad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \qquad (2.14)$
- Let us study system (a) first. By introducing polar coordinates

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- Let us study system (a) first. By introducing polar coordinates

$$r = \left(x_1^2 + x_2^2\right)^{1/2}$$
$$\theta = \tan^{-1}\left(x_2/x_1\right)$$

- the dynamic equations (12) are transformed as

$$\frac{dr}{dt} = -r\left(r^2 - 1\right)$$
$$\frac{d\theta}{dt} = -1$$

- When the state starts on the unit circle, the above equation shows that ***ŕ(t) = 0***. Therefore, the state will circle around the origin with a period **1/2π**. When ***r* <** 1, then ***ŕ* > 0**. This implies that the state tends to the circle from inside. When ***r* > 1**, then ***ŕ* < 0**. This implies that the state tends toward the unit circle from outside. Therefore, the unit circle is a stable limit cycle.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **LIMIT CYCLES**
- When the state starts on the unit circle, the above equation shows that ***ŕ(t) = 0***. Therefore, the state will circle around the origin with a period **1/2π**. When ***r*** **<** 1, then ***ŕ*** **> 0**. This implies that the state tends to the circle from inside. When ***r*** **> 1**, then ***ŕ*** **< 0**. This implies that the state tends toward the unit circle from outside. Therefore, the unit circle is a stable limit cycle.
- This can also be concluded by examining the analytical solution of (12)
- $\boldsymbol{r(t) = \frac{1}{(1+c_0 e^{-2t})^{1/2}}} \qquad \boldsymbol{\theta(t) = \theta_0 - t}$
- Where
- $\boldsymbol{c_0 = \frac{1}{r_0^2} - 1}$
- Similarly, one can find that the system (b) has an unstable limit cycle and system (c) has a semistable limit cycle.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **Existence of Limit Cycles**
- It is of great importance for control engineers to predict the existence of limit cycles in control systems. In this section, we state three simple classical theorems to that effect. These theorems are easy to understand and apply.
- The first theorem to be presented reveals a simple relationship between the existence of a limit cycle and the number of singular points it encloses. In the statement of the theorem, we use N to represent the number of nodes, centers, and foci enclosed by a limit cycle, and S to represent the number of enclosed saddle points.
- **Theorem 1 (Poincare)** if a limit cycle exists in the second-order autonomous system (1), then $N = S + 1$.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **Existence of Limit Cycles**
- This theorem is sometimes called the index theorem. Its proof is mathematically involved (actually, a family of such proofs led to the development of algebraic topology) and shall be omitted here. One simple inference from this theorem is that a limit cycle must enclose at least one equilibrium point. The theorem's result can be verified easily on Figures 8 and 10.
- The **second theorem** is concerned with the asymptotic properties of the trajectories of second-order systems.
- **Theorem 2** (Poincare-Bendixson) If a trajectory of the second-order autonomous system remains in a finite region **Ω**, then one of the following is true:
- (a) the trajectory goes to an equilibrium point
- (b) the trajectory tends to an asymptotically stable limit cycle
- (c) the trajectory is itself a limit cycle
- While the proof of this theorem is also omitted here, its intuitive basis is easy to see, and can be verified on the previous phase portraits.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **Existence of Limit Cycles**
- The third theorem provides a sufficient condition for the non-existence of limit cycles.
- **Theorem 3 (Bendixson)** For the nonlinear system (1), no limit cycle can exist in a region **Ω**. of the phase plane in which $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not vanish and does not change sign.
- **Proof**: Let us prove this theorem by contradiction. First note that, from (5), the equation
  - $$f_2 dx_1 - f_1 dx_2 = 0 \qquad (15)$$
- is satisfied for any system trajectories, including a limit cycle. Thus, along the closed curve L of a limit cycle, we have
  - $$\int_L (f_2 dx_1 - f_1 dx_2) = 0 \qquad (16)$$
- Using Stokes' Theorem in calculus, we have
-
- $\int_L (f_2 dx_1 - f_1 dx_2) = \iint (\partial f_1/\partial x_1 + \partial f_2/\partial x_2) dx_1\, dx_2$
- where the integration on the right-hand side is carried out on the area enclosed by the limit cycle.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **Existence of Limit Cycles**
- By Equation (16), the left-hand side must equal zero. This, however, contradicts the fact that the right-hand side cannot equal zero because by hypothesis $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not vanish and does not change sign.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Phase Plane Analysis of Nonlinear System**
- **Let us illustrate the result on an example**
- Example 8: Consider the nonlinear system
- $\dot{x}_1 = g(x_2) + 4x_1x_2^2$
- $\dot{x}_2 = h(x_1) + 4x_1^2x_2$
- Since
- $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2)$
- which is always strictly positive (except at the origin), the system does not have any limit cycles anywhere in the phase plane.
- The above three theorems represent very powerful results. It is important to notice, however, that they have no equivalent in higher-order systems, where exotic asymptotic behaviors other than equilibrium points and limit cycles can occur.

Phase Plane Analysis
Concepts of Phase Plane Analysis

- **Summary**
- Phase plane analysis is a graphical method used to study second-order dynamic

systems. The major advantage of the method is that it allows visual examination of the global behavior of systems. The major disadvantage is that it is mainly limited to second-order systems (although extensions to third-order systems are often achieved with the aid of computer graphics). The phenomena of multiple equilibrium points and of limit cycles are clearly seen in phase plane analysis. A number of useful classical theorems for the prediction of limit cycles in second-order systems are also presented.