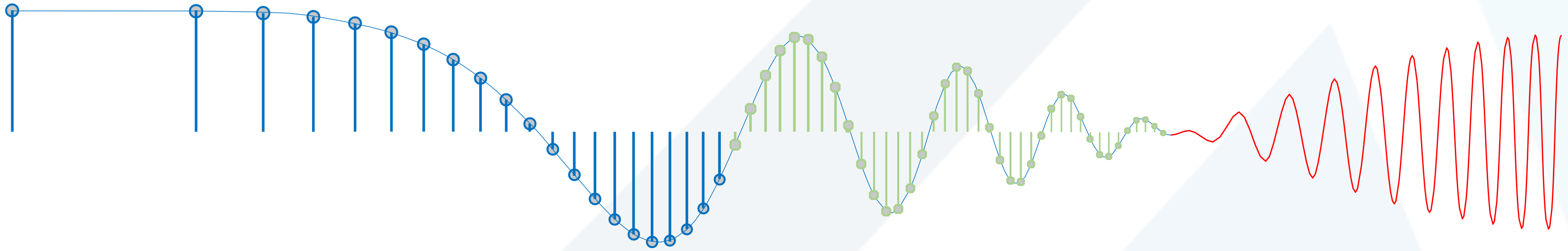
CEDC606: Digital Signal Processing

Lecture Notes 2: Discrete-Time Signals and Systems



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

Chapter 2

Discrete-time signals and systems

1. Discrete-time signals
2. Discrete-time systems
3. Linear time-invariant (LTI) systems
4. Linear constant-coefficient difference equations (LCCDE)

1. Discrete-time signals

- A discrete-time signal $x[n]$ is a sequence of numbers defined for every value of the integer variable n.
- A discrete-time signal is not defined for noninteger values of n. For example, the value of $x[3/2]$ is not zero, just undefined.
- When $x[n]$ is obtained by sampling a continuous-time signal $x(t)$, the interval T_s between two successive samples is known as the sampling period.
- The quantity $F_s = 1/T_s$, called the sampling frequency, equals the number of samples per unit of time.
- The duration or length L_x of a discrete-time signal $x[n]$ is the number of samples from the first nonzero sample $x[n_1]$ to the last nonzero sample $x[n_2]$, that is $L_x = n_2 - n_1 + 1$.

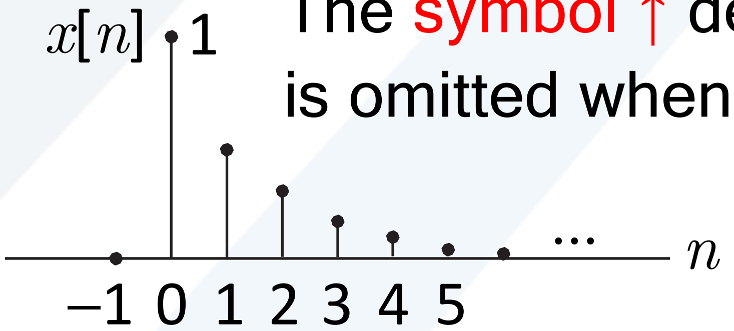
- The range $n_1 \le n \le n_2$, denoted by $[n_1, n_2]$ is called the support of the sequence.
- There are several ways to represent a DT signal. The more widely used are:
  - Functional representation $x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$
  - Tabular representation

n	…	–2	–1	0	1	2	3	…
$x[n]$	…	0	0	1	1/2	1/4	1/8	…

  - Sequence representation $x[n] = \{\cdots, 0, \underset{\uparrow}{1}, 1/2, 1/4, 1/8, \cdots\}$

The symbol ↑ denotes the index $n = 0$; it is omitted when the table starts at $n = 0$.

  - Graphical representation



- The energy of a sequence $x[n]$ is defined by: $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$

- The power of a sequence $x[n]$ is defined by: $\mathcal{P}_x = \lim_{L\to\infty}\left[\frac{1}{2L+1}\sum_{n=-L}^{L}|x[n]|^2\right]$

Elementary discrete-time signals

- Unit impulse sequence: $\delta[n] = \begin{cases} 1 & n = 0 \\ 0, & n \neq 0 \end{cases}$

- Unit step sequence: $u[n] = \begin{cases} 1 & n \geq 0 \\ 0, & n < 0 \end{cases}$

- Real sinusoidal sequence: $x[n] = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty$

  where A (amplitude), ω_0 (frequency) and ϕ (phase) are real constants.

- Exponential sequence: $x[n] = Aa^n, \quad -\infty < n < \infty$
  where A and a can take real or complex values.
- If both A and a are real then $x[n]$ is termed as a real exponential sequence.
  - If $|a| > 1$, the magnitude of $x[n]$ increases exponentially as n increases.
  - If $|a| < 1$, the magnitude of $x[n]$ decreases exponentially as n increases.
  - If $|a| = 1$, the magnitude of $x[n]$ is a constant, independent of n.
  - The values of $x[n]$ alternate in sign when a is negative.
- If $A = |A|e^{j\phi}$ and $a = e^{j\omega_0}$, then $x[n]$ is termed as a complex sinusoid sequence.

$$x[n] = \underbrace{|A|\cos(\omega_0 n + \phi)}_{\mathrm{Re}\{x[n]\}} + j\underbrace{|A|\sin(\omega_0 n + \phi)}_{\mathrm{Im}\{x[n]\}}$$

Thus Re{$x[n]$} and Im{$x[n]$} are real sinusoids.

- If both $A = |A|e^{j\phi}$ and $a = |a|e^{j\omega_0}$ are complex numbers, then:

$$x[n] = \underbrace{|A||a|^n \cos(\omega_0 n + \phi)}_{\mathrm{Re}\{x[n]\}} + j|a|^n \underbrace{|A|\sin(\omega_0 n + \phi)}_{\mathrm{Im}\{x[n]\}}$$

Thus Re{$x[n]$} and Im{$x[n]$} are each the product of a real exponential and real sinusoid.

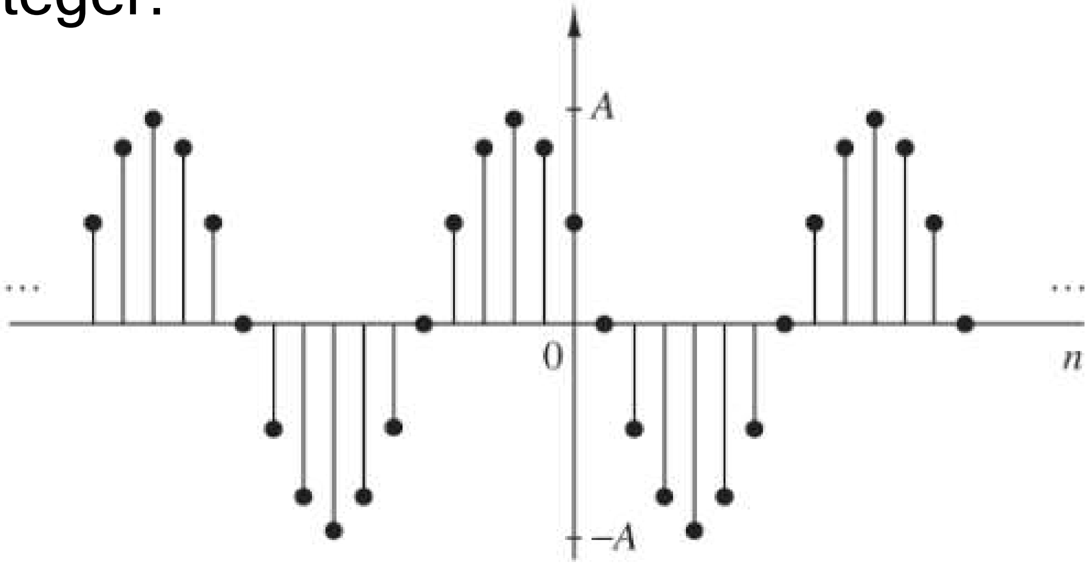
- If $|a| > 1$ Re{$x[n]$} and Im{$x[n]$} are the product of a real sinusoid and a growing real exponential.
- If $|a| < 1$, Re{$x[n]$} and Im{$x[n]$} are the product of a real sinusoid and a decaying real exponential.
- If $|a| = 1$, Re{$x[n]$} and Im{$x[n]$} are real sinusoids.

- A sequence $x[n]$ is called periodic if $x[n] = x[n+N]$, all $n.$ The smallest value of N is known as the fundamental period or simply period of $x[n]$.
- The sinusoidal sequence $\cos(\omega_0 n + \phi)$ is periodic, if $\cos(\omega_0 n + \phi) = \cos(\omega_0 n + \omega_0 N + \phi)$. This is possible if $\omega_0 N = 2\pi k$, where k is an integer ($\omega_0/2\pi$ is a rational number). Therefor the fundamental period is the smallest integer of the form $2\pi k/\omega_0$, where k is a positive integer.

$$x[n] = A\cos(\frac{\pi}{6}n + \frac{\pi}{3})$$

$$N = \frac{2\pi k}{\pi/6} = 12k \Rightarrow N = 12 \text{ (for } k = 1)$$

$$\text{Delay} = 12 \times \frac{\pi/3}{2\pi} = 2 \text{ sampling intervals}$$



2. Discrete-time systems

- A discrete-time system is a computational process or algorithm that transforms or maps a sequence $x[n]$, called the input signal, into another sequence $y[n]$, called the output signal.
  - $y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$ three-point moving average filter
  - $y[n] = \text{median}\{x[n-1], x[n-2], x[n], x[n+1] + x[n+2]\}$
- A system is called causal if the present value of the output does not depend on future values of the input, that is, $y[n_0]$ is determined by the values of $x[n]$ for $n \leq n_0$, only.
- A system is said to be stable, in the Bounded-Input Bounded-Output (BIBO) sense, if every bounded input signal results in a bounded output signal, that is

$$|x[n]| \le M_x < \infty \Rightarrow |y[n]| \le M_y < \infty$$

The three-point moving average filter is stable

$$|x[n]| \le M_x \Rightarrow |y[n]| \le |x[n]| + |x[n-1]| + |x[n-2]| = 3M_x = M_y$$

The accumulator system defined by $y[n] = \sum_{k=0}^{\infty} x[n-k]$

is unstable because the bounded input $x[n] = u[n]$ produces the output $y[n] = (n+1)u[n]$, which becomes unbounded as $n \to \infty$.

- A system $\mathcal{T}$ is linear, if for all functions $x_1[n]$ and $x_2[n]$ and all complex constants α and β, the following condition holds:

$$\mathcal{T}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha \mathcal{T}\{x_1[n]\} + \beta \mathcal{T}\{x_2[n]\}$$

$y[n] = x^2[n]$ is nonlinear system.

- An important consequence of linearity is that a linear system cannot produce an output without being excited. $\mathcal{T}\{x[n]=0\}=y[n]=0$

- A system $\mathcal{T}$ is said to be time invariant (TI) if, for every function $x[n]$ and every integer constant n_0, the following condition holds:

$$\mathcal{T}\{x[n]\}=y[n] \Rightarrow \mathcal{T}\{x[n-n_0]\}=y[n-n_0]$$

  - $y[n]=x[n]\cos\omega_0 n$ is not time invariant system (time-varying).
  - The downsampler system, $y[n]=\mathcal{T}\{x[n]\}=x[nM]$ is linear but time-varying,

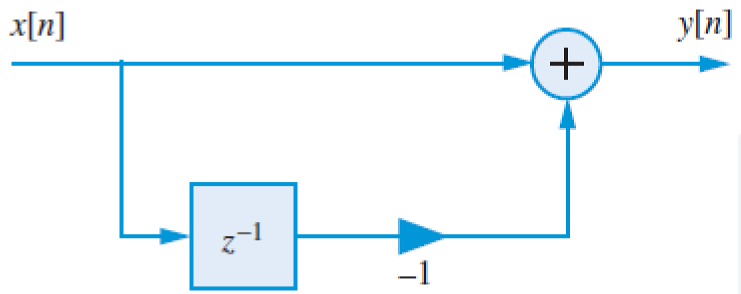
- A system $\mathcal{T}$ is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n. Otherwise it is said to be dynamic.

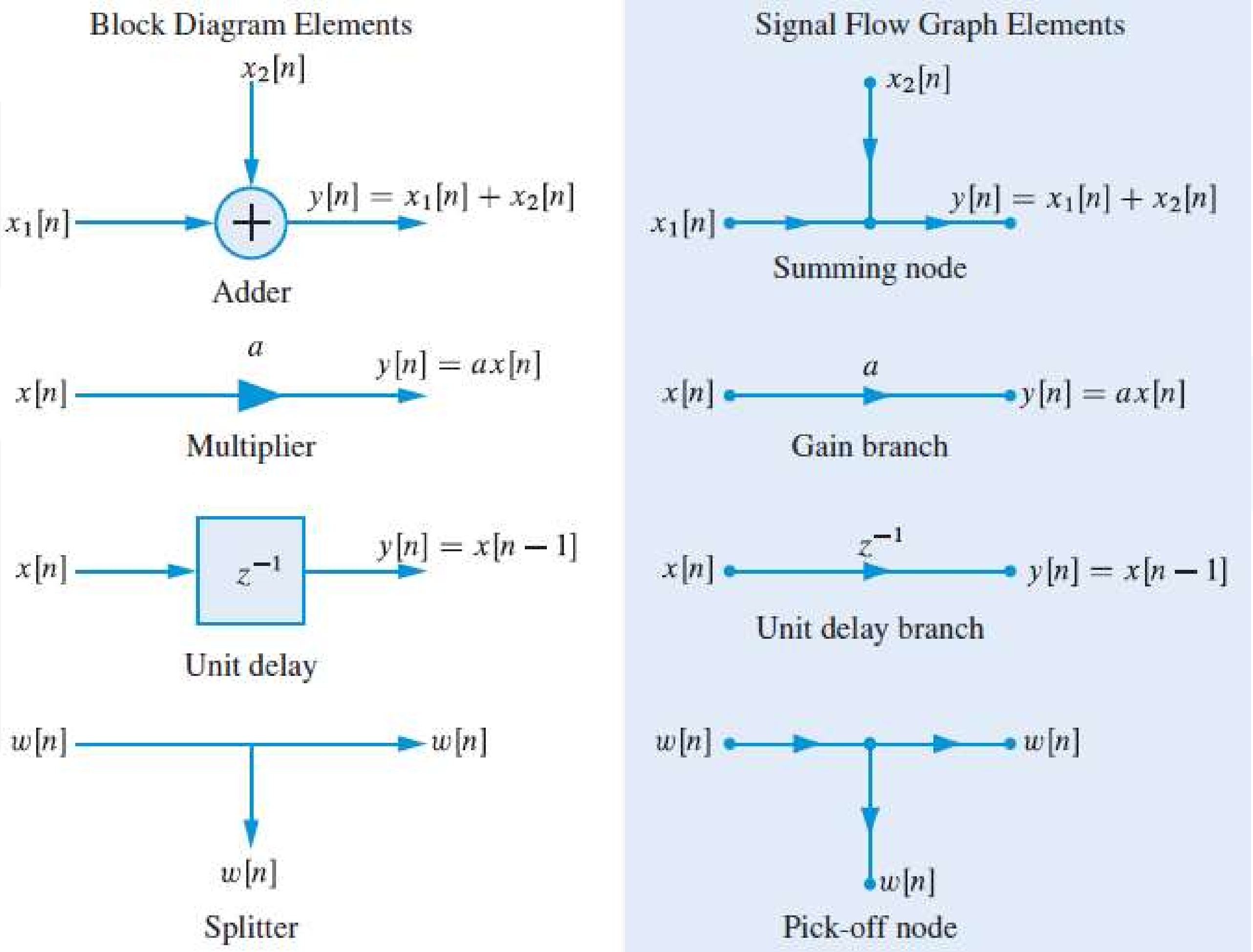
$y[n]=x^2[n]$ is a memoryless system.

Block Diagram Representation of Discrete-Time Systems

- Basic building blocks The most widely used operations for a block diagram representation of discrete-time systems are provided by the four elementary discrete-time systems (or building blocks) shown below.
- The adder, defined by $y[n] = x_1[n] + x_2[n]$, computes the sum of two sequences.
- The constant multiplier, defined by $y[n] = ax[n]$, produces the product of the input sequence by a constant.
- The basic memory element is the unit delay system defined by $y[n] = x[n-1]$ and denoted by the z^{-1}. The unit delay is a memory location which can hold (store) the value of a sample for one sampling interval.
- Finally, the branching element is used to distribute a signal value to different branches.

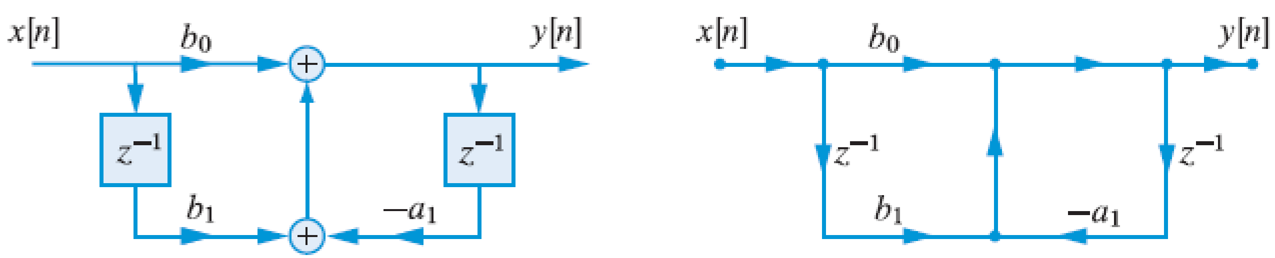
- Figure below shows the block diagram of a system which computes the first difference $y[n] = x[n] - x[n-1]$ of its input.





Basic building blocks and the corresponding signal flow graph elements

- To illustrate these concepts, we consider a first-order IIR system described by: $y[n] = b_0 x[n] + b_1 x[n-1] - a_1 y[n-1]$.

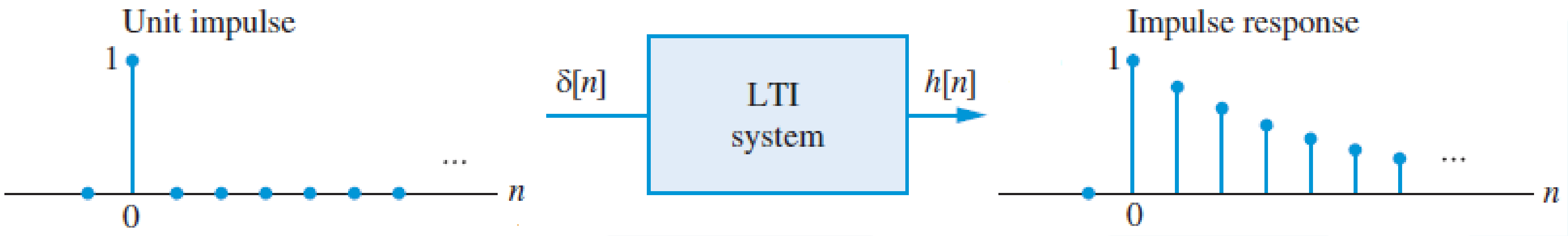


Structure for the first-order IIR system in block diagram, and signal flow graph

- A discrete-time system is called practically realizable if its practical implementation requires:
  - (1) a finite amount of memory for the storage of signal samples and system parameters, and
  - (2) a finite number of arithmetic operations for the computation of each output sample.

3. Linear time-invariant (LTI) systems

- The response of a linear time-invariant (LTI) system to any input can be determined from its response $h[n]$ to the unit sample sequence $\delta[n]$.



$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \qquad -\infty<n<\infty$$

For example, the unit step can be written as: $u[n]=\sum_{k=0}^{\infty}\delta[n-k]=\sum_{k=-\infty}^{n}\delta[k]$

$$y[n]=\mathcal{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}=\sum_{k=-\infty}^{\infty} x[k]\mathcal{T}\{\delta[n-k]\}=\sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

$h_k[n]$ be the response of the system to the input $\delta[n-k]$

The property of time invariance implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad -\infty < n < \infty$$

This equation is referred to as the convolution sum, $y[n] = x[n] * h[n]$

- Example 1: Compute the output $y[n]$ of a LTI system when:

$$x[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}, \qquad h[n] = \{-1, \underset{\uparrow}{2}, 1\}$$

$$y[-1] = x[0]h[-1] = (1)(-1) = -1$$

$$y[0] = x[0]h[0] + x[1]h[-1] = (1)(2) + (2)(-1) = 0, \qquad \cdots$$

$$y[n] = \{-1, \underset{\uparrow}{0}, 2, 4, 6, 14, 5\}$$

k	−3	−2	−1	0	1	2	3	4	5	6	7
$x[k]$				1	2	3	4	5			
$h[k]$			−1	2	1						
$h[-1-k]$		1	2	−1							
$h[-k]$			1	2	−1						
$h[1-k]$				1	2	−1					
$h[2-k]$					1	2	−1				
$h[3-k]$						1	2	−1			
$h[4-k]$							1	2	−1		
$h[5-k]$								1	2	−1	
$y[n]$			−1	0	2	4	6	14	5		
n	−3	−2	−1	0	1	2	3	4	5	6	7

$y[3] = 1\text{x}3 + 2\text{x}4 - 1\text{x}5 = 6$

The computation of convolution in tabular form

- Convolution using direct method

k	–3	–2	–1	0	1	2	3	4	5	6	7
$x[k]$				1	2	3	4	5			
$h[k]$			–1	2	1						x
				2	4	6	8	10			
					1	2	3	4	5		
			–1	–2	–3	–4	–5				+
$y[n]$			–1	0	2	4	6	14	5		
n	–3	–2	–1	0	1	2	3	4	5	6	7

Computation of the convolution sum, the approach is similar to a pencil and paper multiplication calculation, except carries are not performed out of a column.

- Convolution using matrix-vector multiplication

$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} x[0] & 0 & 0 \\ x[1] & x[0] & 0 \\ x[2] & x[1] & x[0] \\ x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ 0 & x[4] & x[3] \\ 0 & 0 & x[4] \end{bmatrix} \begin{bmatrix} h[-1] \\ h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \\ 6 \\ 14 \\ 5 \end{bmatrix}$$

The matrix form of convolution involves a matrix known as Toeplitz.

- A simpler approach, from a programming viewpoint, is to express the above equations as a linear combination of column vectors:

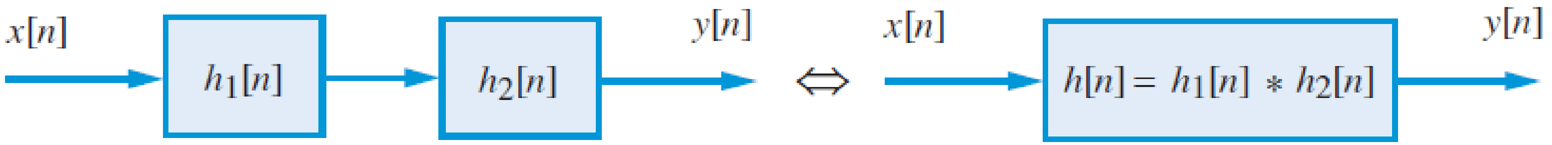
$$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = h[-1]\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ 0 \\ 0 \end{bmatrix} + h[0]\begin{bmatrix} 0 \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ 0 \end{bmatrix} + h[1]\begin{bmatrix} 0 \\ 0 \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

Properties of linear time-invariant systems

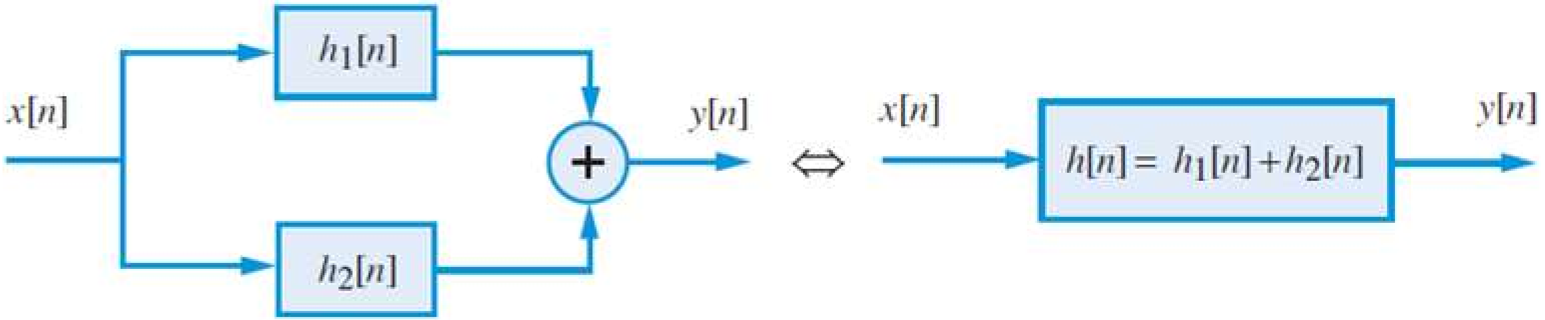
- Properties of Convolution
  - Convolutional identity: $x[n] * \delta[n] = x[n]$
  - Commutative: $x[n] * h[n] = h[n] * x[n]$
  - Associative: $(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$

- Distributive: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$

- Note: The convolution of two non-periodic sequences: $x[n]$, $0 \le n \le M-1$ and $h[n]$, $0 \le n \le N-1$ has length $M+N-1$.
- Cascade interconnection of two LTI systems



- Parallel interconnection of two LTI systems

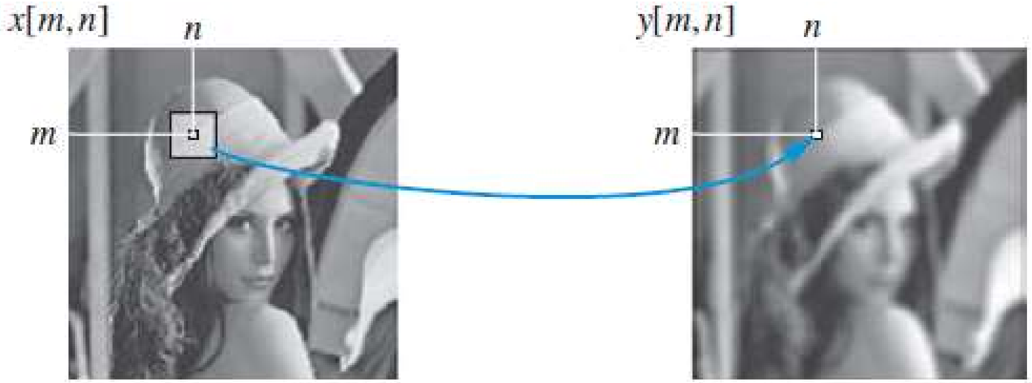


- Causality and stability
  - A LTI system is causal if its impulse response $h[n] = 0$ for $n < 0$.
  - A LTI system is stable if and only if its impulse response is absolutely summable, $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Convolution in two dimensions

- Spatial filters are very popular and useful in the processing of digital images to implement visual effects like noise filtering, edge detection, etc.
- Smoothing images consists of replacing each pixel by its average over a local region.
- Consider a 3×3 region around the pixel $x[m, n]$. Then the smoothed pixel value $y[m, n]$ can be computed as:

$$y[m,\, n] = \sum_{k=-1}^{1} \sum_{k=-1}^{1} \left(\frac{1}{9}\right) x[m-k,\, n-l]$$



We next define a 2D sequence $h[m,\, n]$

$$h[m,\, n] = \begin{cases} \frac{1}{9}, & -1 \leq m, n \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

which can be seen as a spatial filter impulse response:

$$y[m,\, n] = \sum_{k=-1}^{1} \sum_{k=-1}^{1} h[k,\, l] x[m-k,\, n-k]$$

which is a 2D convolution of image $x[m,\, n]$ with a spatial filter $h[m,\, n]$. A general expression for 2D convolution, when the spatial filter has finite symmetric support $(2K+1) \times (2L+1)$, is given by:

$$y[m,\, n] = \sum_{k=-K}^{K} \sum_{l=-L}^{L} h[k,\, l] x[m-k,\, n-l]$$

4. Linear constant-coefficient difference equations (LCCDE)

- An important class of LTI systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Example 2: The accumulator system defined by: $y[n] = \sum_{k=-\infty}^{n} x[k]$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k] = x[n] + y[n-1] \Rightarrow y[n] - y[n-1] = x[n]$$

Solution of Linear Constant-Coefficient Difference Equations

- The goal is to determine the output $y[n]$, $n \geq 0$, of the system given a specific input $x[n]$, $n \geq 0$, and a set of initial conditions.

- A solution to a LCCDE can be obtained in the form: $y[n] = y_h[n] + y_p[n]$.

  where $y_h[n]$ is is the solution of the homogeneous linear difference equation found by setting $x[n] = 0$:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

  and $y_p[n]$ is due to the input signal $x[n]$ being applied to the system. It is referred to as the particular solution of the difference equation.

- A solution to LCCDE can also be obtained in the form: $y[n] = y_{zi}[n] + y_{zs}[n]$.

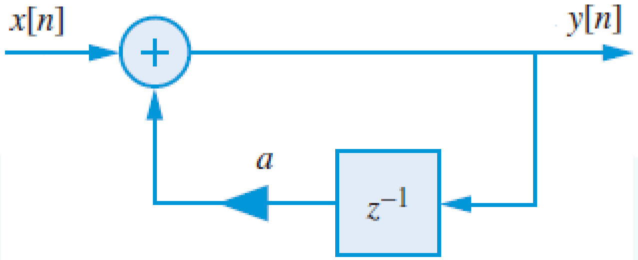
  where $y_{zi}[n]$ is is called the zero-input solution, due to the initial conditions alone (assuming they exist), and $y_{zs}[n] = h[n] * x[n]$ is called the zero-state solution, due to the input $x[n]$ alone (initial conditions assumed to be zero).

- A solution to LCCDE can also be obtained in the form: $y[n] = y_{tr}[n] + y_{ss}[n]$.

where $y_{tr}[n]$ is the transient response due to the initial state of the system; It disappears over time, and $y_{ss}[n]$ is the steady-state response; It remains.

- Example 3: A causal and stable LTI system

$$y[n] = ay[n-1] + x[n], \quad |a| < 1$$



We apply an input signal $x[n]$ to the system for $n \geq 0$.

We make no assumptions about the input signal for $n < 0$, but we do assume the existence of the initial condition $y[-1]$.

Computing successive values of $y[n]$: $y[n] = a^{n+1}y[-1] + \sum_{k=0}^{n} a^k x[n-k], \quad n \geq 0$

If the system is initially relaxed at time $n = 0$, then its memory (i.e., the output of the delay) should be zero. Hence $y[-1] = 0$. We say that the system is at zero state and its corresponding output is called the zero-state response,

$$y_{zs}[n] = h[n] * x[n] = \sum_{k=0}^{n} a^k x[n-k],\ n \ge 0 \quad \Rightarrow \quad h[n] = a^n u[n]$$

Now, suppose that the system is initially nonrelaxed, $y[-1] \neq 0$, and the input $x[n] = 0$ for all n. Then the output of the system with zero input is called the zero-input response: $y_{zi}[n] = y[-1]a^{n+1}, \quad n \ge 0$

$$y[n] = y_{zi}[n] + y_{zs}[n] = \underbrace{y[-1]a^{n+1}}_{zero-input} + \underbrace{\sum_{k=0}^{n} a^k x[n-k]}_{zero\text{-}state}, \quad n \ge 0$$

Then, if $y[-1] = 0$, the system is LTI. If $y[-1] \neq 0$, the system is linear in a more general sense that involves linearity with respect to both input and ICs.

To obtain the step response of the system we set $x[n] = u[n]$:

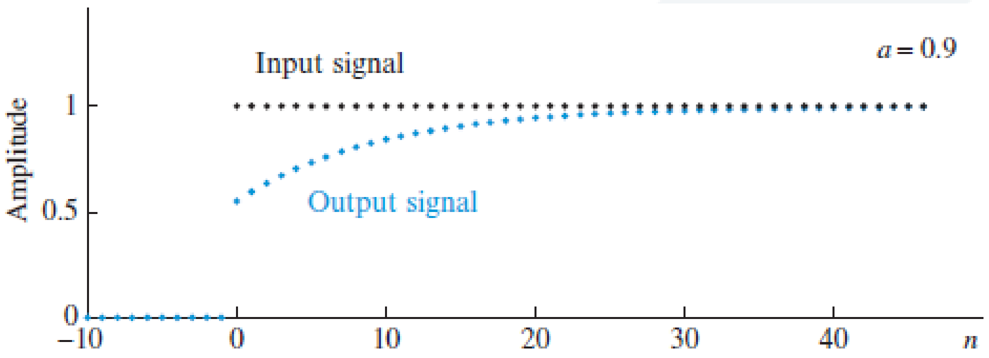
$$y[n] = y_{ss}[n] + y_{tr}[n] = \underbrace{\frac{1}{1-a}}_{steady\text{-}state} + \underbrace{y[-1]a^{n+1} - \frac{a^{n+1}}{1-a}}_{transient}, \quad n \ge 0$$

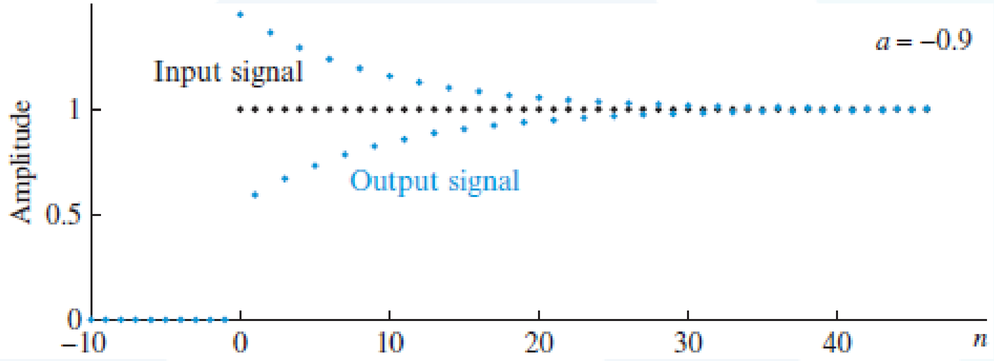
For a stable system, that is, when $|a| < 1$, we have:

$$y_{ss}[n] = \lim_{n\to\infty} y[n] = \frac{1}{1-a},\ n \geq 0 \qquad y_{tr}[n] = y[-1]a^{n+1} - \frac{a^{n+1}}{1-a} \underset{n\to\infty}{\to} 0$$

$$y[n] = \underbrace{\frac{1}{1-a}}_{steady\text{-}state} + \underbrace{y[-1]a^{n+1} - \frac{a^{n+1}}{1-a}}_{transient} = \underbrace{\frac{1-a^{n+1}}{1-a}}_{zero\text{-}state} + \underbrace{y[-1]a^{n+1}}_{zero-input}$$

- Note: In general, we have $y_{zi}[n] \neq y_{tr}[n]$, and $y_{ss}[n] \neq y_{zs}[n]$.
- Note: If the system is stable $y_{ss}[n] = \lim_{n\to\infty} y_{zs}[n]$.





FIR versus IIR systems

- If the unit impulse response of an LTI system is of finite duration, then the system is called a finite-duration impulse response (or FIR) filter.

  The following difference equation describes a causal FIR filter:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- If the impulse response of an LTI system is of infinite duration, then the system is called an infinite-duration impulse response (or IIR) filter. The following difference equation describes a recursive IIR filter:

$$\sum_{k=0}^{N} a_k y[n-k] = x[n]$$

System	Equation	Linear	Time-invariant	LTI	Causal	Stable
Multiplier	$y[n] = 2x[n]$	√	√	√	√	√
Offset	$y[n] = x[n] + 1$	✗	√	✗	√	√
Squarer	$y[n] = x^2[n]$	✗	√	✗	√	√
Delay	$y[n] = x[n - n_0]$	√	√	√	$n_0 \geq 0$	√
Average	$y[n] = (x[n-1]+x[n]+x[n+1])/3$	√	√	√	✗	√
Summer	$y[n] = \sum_{k=-\infty}^{n} x[k]$	√	√	√	√	✗
LCCDE	$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$	√	√	√	√	√ or ✗
Switch	$y[n] = x[n]u[n]$	√	✗	✗	√	√

Summary of system properties of example systems