CECC122: Linear Algebra and Matrix Theory

Lecture Notes 2: Matrices: Part A

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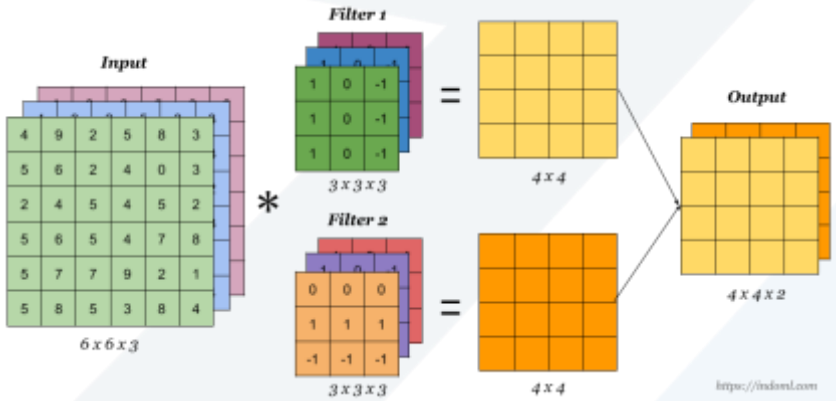

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Chapter 2

Matrices

2.1 Operations with Matrices

2.2 Properties of Matrix Operations

2.3 The Inverse of a Matrix

2.4 Elementary Matrices

2.5 Complex Matrices

2.6 Applications of Matrix Operations

2.1 Operations with Matrices

- Matrix:

$$A = \left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(K)$$

K: a field (R or C) $\quad M_{m\times n}(R)$: Real matrices $\quad M_{m\times n}(C)$: Complex matrices

(i, j)–th entry: a_{ij} $\quad$ row: m $\quad$ column: n $\quad$ size: $m\times n$

- i-th row vector

$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$ $\quad$ row matrix

- j-th column vector

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix} \qquad \text{column matrix}$$

- Square matrix: $m = n$

- Diagonal matrix:

$$A = \operatorname{diag}(d_1,\ d_2,\ \cdots,\ d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}(K)$$

- Trace:

If $A = \left[a_{ij}\right]_{n\times n}$ Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

- Ex 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad \Rightarrow \quad r_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad \Rightarrow \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

- Complex matrices

$$\begin{bmatrix} 1 & -1 & i \\ 3 & 2i & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & i & -i & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix}, \quad \begin{bmatrix} 1+i & 1 \\ i & 1-i \end{bmatrix}$$

- Equal matrix:

If $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$ Then $A = B$ if and only if $a_{ij} = b_{ij}\ \forall\ 1 \le i \le m, 1 \le j \le n$

- Ex 2: (Equal matrix)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$ Then $a = 1,\ b = 2,\ c = 3,\ d = 4$

- Matrix addition:

If $A = \left[a_{ij}\right]_{m\times n}$, $B = \left[b_{ij}\right]_{m\times n}$ Then $A + B = \left[a_{ij}\right]_{m\times n} + \left[b_{ij}\right]_{m\times n} = \left[a_{ij} + b_{ij}\right]_{m\times n}$

- Ex 3: (Matrix addition)

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Scalar multiplication:

If $A = \left[a_{ij}\right]_{m\times n}$, c: scalar $(\in K)$ Then $cA = \left[ca_{ij}\right]_{m\times n}$

- Matrix subtraction:

$$A - B = A + (-1)B$$

- Ex 4: (Scalar multiplication and matrix subtraction)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find $(a)\ 3A$, $(b)\ -B$, $(c)\ 3A - B$

Sol:

(a)

$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)

$$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)

$$3A-B=\begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}-\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- Matrix multiplication:

If $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{ij}\right]_{n\times p}$ Then $AB=\left[a_{ij}\right]_{m\times n}\left[b_{ij}\right]_{n\times p}=\left[c_{ij}\right]_{m\times p}$

Size of AB

where $c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{in}b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

- Notes: (1) $A + B = B + A$, (2) $AB \neq BA$

- Ex 5: (Find AB)

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

Sol: $$AB = \begin{bmatrix} (-1)(-3)+(3)(-4) & (-1)(2)+(3)(1) \\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1) \\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1) \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

- Matrix form of a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\boldsymbol{b}}$$

m linear equation

$\downarrow$

single matrix equation

$$\underset{m \times n}{A}\,\underset{n \times 1}{\boldsymbol{x}} = \underset{m \times 1}{\boldsymbol{b}}$$

2.2 Properties of Matrix Operations

- Three basic matrix operators:
  (1) matrix addition
  (2) scalar multiplication
  (3) matrix multiplication

- Zero matrix: $O_{m\times n}$
- Identity matrix of order n: I_n
- Properties of matrix addition and scalar multiplication:

If $A,\ B,\ C \in M_{m\times n}(K)$ then

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd)A = c(dA)$

(4) $1A = A$

(5) $c(A + B) = cA + cB$

(6) $(c + d)A = cA + dA$

- Properties of zero matrices:

If $A \in M_{m\times n}(K)$, c: scalar then

(1) $A + O_{m\times n} = A$

(2) $A + (-A) = O_{m\times n}$

(3) $cA = O_{m\times n} \Rightarrow c = 0$ or $A = O_{m\times n}$

- Notes:

(1) $O_{m\times n}$: the additive identity for the set of all $m\times n$ matrices

(2) $-A$: the additive inverse of A

- Properties of matrix multiplication:

$$(1)\ A(BC) = (AB)C$$

$$(2)\ A(B + C) = AB + AC$$

$$(3)\ (A + B)C = AC + BC$$

$$(4)\ c(AB) = (cA)B = A(cB)$$

- Properties of identity matrix:

If $A \in M_{m\times n}(K)$, then

$$(1)\ AI_n = A$$

$$(2)\ I_m A = A$$

- Transpose of a matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(K)$$

$$\text{Then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}(K)$$

- Ex 1: (Find the transpose of the following matrix)

$$(a)\ A=\begin{bmatrix}2\\8\end{bmatrix} \qquad (b)\ A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \qquad (c)\ A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}$$

Sol: $(a)\ A=\begin{bmatrix}2\\8\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$

$$(b)\ A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

$$(c)\ A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

- Properties of transposes:

(1) $(A^T)^T = A$

(2) $(A+B)^T = A^T + B^T$

(3) $(cA)^T = c(A^T)$

(4) $(AB)^T = B^T A^T$

- Symmetric matrix:

A square matrix A is symmetric if $A^T = A$

- Skew-symmetric matrix:

A square matrix A is skew-symmetric if $A^T = -A$

- Ex 2:

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix} \text{ is symmetric, find } a,\ b,\ c?$$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix},\ A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix} \qquad A = A^T \Rightarrow a = 2,\ \ b = 3,\ \ c = 5$$

- Ex 3:

$$\text{If } A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix} \text{ is a skew-symmetric, find } a,\ b,\ c?$$

Sol:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}, \quad -A^T = \begin{bmatrix} 0 & -a & -b \\ -1 & 0 & -c \\ -2 & -3 & 0 \end{bmatrix}$$

$$A = -A^T \Rightarrow a = -1, \ b = -2, \ c = -3$$

- Notes:

(1) AA^T is symmetric

(2) Every square matrix $A \in M_n(R)$ can be expressed as the sum of a symmetric matrix B and a skew-symmetric matrix C

$$B = \tfrac{1}{2}(A + A^T), \quad C = \tfrac{1}{2}(A - A^T)$$

- Real (Complex) number:

$ab = ba$ (Commutative law for multiplication)

- Matrix:

$$\underset{m \times n}{A}\,\underset{n \times p}{B} \neq \underset{n \times p}{B}\,\underset{m \times n}{A}$$

Three situations:

(1) If $m \neq p$, then AB is defined, BA is undefined

(2) If $m = p$, $m \neq n$, then $AB \in M_{m\times m}(K)$, $BA \in M_{n\times n}(K)$

(Sizes are not the same)

(3) If $m = p = n$, then $AB \in M_{m\times m}(K)$, $BA \in M_{m\times m}(K)$

(Sizes are the same, but matrices are not equal)

- Ex 4:

Show that AB and BA are not equal for the matrices.

$$A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}2 & 5\\ 4 & -4\end{bmatrix}$$

$$BA=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 4 & -2\end{bmatrix}$$

- Note: $AB \neq BA$

- Real (Complex) number:

$ac = bc,\ c \neq 0 \Rightarrow a = b$ (Cancellation law)

- Matrix:

$AC = BC, \quad C \neq O$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Ex 5: (An example in which cancellation is not valid)

Show that $AC = BC$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Sol:

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \qquad BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

So $AC = BC$ but $A \neq B$

2.3 The Inverse of a Matrix

- Inverse matrix:

Consider $A \in M_{n\times n}(K)$

If there exists a matrix $B \in M_{n\times n}(K)$ such that $AB = BA = I_n$,

Then (1) A is invertible (or nonsingular)

(2) B is the inverse of A

- Note:

  A matrix that does not have an inverse is called noninvertible (or singular).

- Theorem 2.1 (The inverse of a matrix is unique):

  If B and C are both inverses of the matrix A, then $B = C$.

  Consequently, the inverse of a matrix is unique.

- Notes:

  (1) The inverse of A is denoted by A^{-1}

  (2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$\left[A \mid I\right] \longrightarrow \left[I \mid A^{-1}\right] \quad \text{Gauss-Jordan Elimination}$$

- Ex 1: (Find the inverse of the matrix)

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Sol:

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_{11} &+ 4x_{21} = 1 \\ -x_{11} &- 3x_{21} = 0 \end{aligned} \quad (1)$$

$$\begin{aligned} x_{12} &+ 4x_{22} = 0 \\ -x_{12} &- 3x_{22} = 1 \end{aligned} \quad (2)$$

$$(1) \Rightarrow \left[\begin{array}{cc:c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3,\ x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{cc:c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4,\ x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \quad (AX = I = AA^{-1})$$

- Note:

$$\begin{array}{c}\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \\ A \qquad\quad I\end{array} \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \begin{array}{c}\left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \\ I \qquad\quad A^{-1}\end{array}$$

If A can't be row reduced to I, then A is singular.

- Ex 2: (Find the inverse of the following matrix)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Sol: $$\left[A \ \vdots \ I\right] = \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_{13}^{(6)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{r_{3}^{(-1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] = \left[\begin{array}{c:c} I & A^{-1} \end{array}\right]$$

So the matrix A is invertible, and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

- Check: $AA^{-1} = A^{-1}A = I$

- Power of a square matrix:

$$(1)\ A^0 = I$$

$$(2)\ A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3)\ A^r \cdot A^s = A^{r+s} \qquad r,\ s\text{: integers}$$

$$(4)\ (A^r)^s = A^{rs}$$

$$(5)\ D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- Theorem 2.2 (Properties of inverse matrices):

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) A^k is invertible and $(A^k)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k = A^{-k}$

(3) cA is invertible and $(cA)^{-1} = \dfrac{1}{c}A^{-1}$, $c \neq 0$

(4) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

- Theorem 2.3 (The inverse of a product):

  If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

- Note:

$$\left(A_1A_2A_3\cdots A_n\right)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- Theorem 2.4 (Cancellation properties)

  If C is an invertible matrix, then the following properties hold:

  (1) If $AC = BC$, then $A = B$ (Right cancellation property)

  (2) If $CA = CB$, then $A = B$ (Left cancellation property)

- Note: If C is not invertible, then cancellation is not valid.

- Theorem 2.5 (Systems of equations with unique solutions):

  If A is an invertible matrix, then the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

- Note:

  For square systems (those having the same number of equations as variables), Theorem 2.5 can be used to determine whether the system has a unique solution.

- Note: $A\boldsymbol{x} = \boldsymbol{b}$ (A is an invertible matrix)

$$\left[A \mid \boldsymbol{b}\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}\boldsymbol{b}\right] = \left[I \mid A^{-1}\boldsymbol{b}\right]$$

- Ex 3: Use an inverse matrix to solve each system

$$(a)\quad \begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$(a)\quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$(b)\quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$