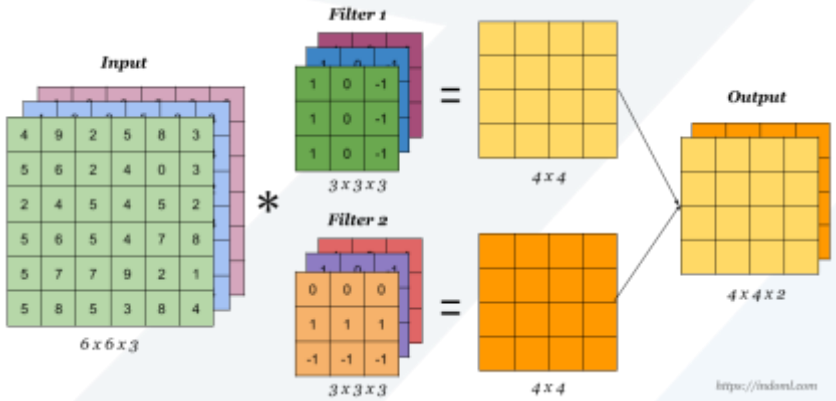
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 3: Matrices: Part B



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Informatics

Manara University

Chapter 2

Matrices

2.1 Operations with Matrices

2.2 Properties of Matrix Operations

2.3 The Inverse of a Matrix

2.4 Elementary Matrices

2.5 Complex Matrices

2.6 Applications of Matrix Operations

2.4 Elementary Matrices

- Row elementary matrix:

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

(1) $R_{ij} = r_{ij}(I)$ — Interchange two rows.

(2) $R_i^{(k)} = r_i^{(k)}(I) \qquad (k \neq 0)$ — Multiply a row by a nonzero constant.

(3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ — Add a multiple of a row to another row.

- Note:

Only do a <u>single</u> elementary row operation.

- Ex 1: (Elementary matrices and non elementary matrices)

$$(a)\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$$

Yes ($r_2^{(3)}(I_3)$)

$$(b)\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$$

No (not square)

$$(c)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

No (Row multiplication must be by a nonzero constant)

$$(d)\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

Yes ($r_{23}(I_3)$)

$$(e)\begin{bmatrix}1&0\\2&1\end{bmatrix}$$

Yes ($r_{12}^{(2)}(I_2)$)

$$(f)\begin{bmatrix}1&0&0\\0&2&0\\0&0&-1\end{bmatrix}$$

No (Use two elementary row operations)

- Theorem 2.6 (Representing elementary row operations):

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- Notes:

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}\ (r_{12}(A)=R_{12}A)$$

$$(b)\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}\ (r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}\ (r_{12}^{(2)}(A)=R_{12}^{(2)}A)$$

- Ex 3: (Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Sol:

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \qquad E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$B = E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- Row-equivalent:

  Matrix B is row-equivalent to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Theorem 2.7: (Elementary matrices are invertible)

  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- Notes:

  (1) $(R_{ij})^{-1} = R_{ij}$

  (2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

  (3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- Ex 4:

Elementary Matrix | Inverse Matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \qquad (R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$

(Elementary Matrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)}$$

(Elementary Matrix)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)}$$

(Elementary Matrix)

- Theorem 2.8 (A property of invertible matrices):

  A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

- Ex 5: Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_1^{(-1)}} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2^{(1/2)}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore $R_{21}^{(-2)} R_2^{(\frac{1}{2})} R_{12}^{(-3)} R_1^{(-1)} A = I$

$$\text{Thus } A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$$

$$= R_1^{(-1)}R_{12}^{(3)}R_2^{(2)}R_{21}^{(2)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Note:

If A is invertible then

$$E_k \cdots E_3 E_2 E_1 A = I$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

- Theorem 2.9 (Equivalent conditions):

If A is an $n\times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\boldsymbol{x}=\boldsymbol{b}$ has a unique solution for every $n\times 1$ column matrix $\boldsymbol{b}$.

(3) $A\boldsymbol{x}=\mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n.

(5) A can be written as the product of elementary matrices.

- LU–factorization:

  If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an LU–factorization of A

- Note:

  If a square matrix A can be row reduced to an upper triangular matrix U using <u>only the row operation of adding a multiple of one row to another row below it</u>, then it is easy to find an LU–factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU \qquad (L = E_1^{-1} E_2^{-1} \cdots E_k^{-1})$$

- Ex 6: $(LU$–factorization)

$$(a)\ A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$$

Sol: (a)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-1)}} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U$$

$$\Rightarrow R_{12}^{(-1)} A = U$$

$$\Rightarrow A = (R_{12}^{(-1)})^{-1} U = LU$$

$$\Rightarrow L = (R_{12}^{(-1)})^{-1} = R_{12}^{(1)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow{r_{23}^{(4)}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = U$$

$$\Rightarrow R_{23}^{(4)} R_{13}^{(-2)} A = U$$

$$\Rightarrow A = (R_{13}^{(-2)})^{-1} (R_{23}^{(4)})^{-1} U = LU$$

$$\Rightarrow L = (R_{13}^{(-2)})^{-1} (R_{23}^{(4)})^{-1} = R_{13}^{(2)} R_{23}^{(-4)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

- Solving $A\boldsymbol{x} = \boldsymbol{b}$ with an LU–factorization of A

  $A\boldsymbol{x} = \boldsymbol{b}$ If $A = LU$, then $LU\boldsymbol{x} = \boldsymbol{b}$

  Let $\boldsymbol{y} = U\boldsymbol{x}$, then $L\boldsymbol{y} = \boldsymbol{b}$

- Two steps:

  (1) Write $\boldsymbol{y} = U\boldsymbol{x}$, and solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$

  (2) Solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$

- Ex 7: (Solving a linear system using LU–factorization)

$$\begin{aligned} x_1 - 3x_2 \qquad\qquad &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $\boldsymbol{y} = U\boldsymbol{x}$, and solve for $L\boldsymbol{y} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 = -14 \end{aligned}$$

(2) Solve the following system $U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

2.5 Complex Matrices

- Conjugate of a matrix:

$$A \in M_{m\times n}(C) = \left[a_{ij}\right]_{m\times n} \quad \Rightarrow \quad \overline{A} \in M_{m\times n}(C) = \left[\overline{a_{ij}}\right]_{m\times n}$$

- Ex 1: $$A = \begin{pmatrix} 1+i & 1 \\ i & 1-i \end{pmatrix} \quad \Rightarrow \quad \overline{A} = \begin{pmatrix} 1-i & 1 \\ -i & 1+i \end{pmatrix}$$

- Properties of the conjugate of a matrix:

(1) $\overline{\overline{A}} = A$ (2) $\overline{A \pm B} = \overline{A} \pm \overline{B}$ (3) $\overline{AB} = \overline{A}\,\overline{B}$

(4) $\overline{cA} = \overline{c}\,\overline{A}, \quad c \in C$ (5) $(\overline{A})^T = \overline{A^T}$

(6) If A is invertible, then $(\overline{A})^{-1} = \overline{A^{-1}}$

- Conjugate transpose of a matrix:

$$A \in M_{m\times n}(C) \quad \Rightarrow \quad A^* = \overline{A^T} \in M_{n\times m}(C)$$

- Ex 2:

$$A = \begin{pmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{pmatrix} \quad \Rightarrow \quad A^* = \overline{A^T} = \begin{pmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{pmatrix}$$

- Properties of the conjugate transpose:

(1) $(A^*)^* = A$ (2) $(A \pm B)^* = A^* \pm B^*$

(3) $(AB)^* = B^* A^*$ (4) $(cA)^* = \bar{c} A^*, \quad c \in C$

- Hermitian matrix:

A square matrix $A \in M_n(C)$ is Hermitian if $A^* = A$

- Ex 3: $$A = \begin{pmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{pmatrix} = A^*$$

- Anti-Hermitian (Skew-Hermitian) matrix:

A square matrix $A \in M_n(C)$ is skew- Hermitian if $A^* = -A$

- Ex 4: $$A = \begin{pmatrix} -i & 2+i \\ -2+i & 0 \end{pmatrix} = -A^*$$

- Notes:

(1) Diagonal entries of an Hermitian matrix are real

(2) Diagonal entries of a Skew-Hermitian matrix are purely imaginary or zero

(3) Every square matrix $A \in M_n(C)$ can be expressed as the sum of a Hermitian matrix B and a skew-Hermitian matrix C

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2}(A - A^*)$$

2.6 More Applications of Matrix Operations

- Cryptography

Encoding a Message = Uncoded Row Matrices $\xrightarrow{A}$ Coded row matrices

0 = _	5 = E	10 = J	15 = O	20 = T	25 = Y
1 = A	6 = F	11 = K	16 = P	21 = U	26 = Z
2 = B	7 = G	12 = L	17 = Q	22 = V	
3 = C	8 = H	13 = M	18 = R	23 = W	
4 = D	9 = I	14 = N	19 = S	24 = X	

- Ex 1:

Use the invertible matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$ to encode the message MEET ME MONDAY

Sol:

[13	5	5]	[20	0	13]	[5	0	13]	[15	14	4]	[1	25	0]
M	*E*	*E*	*T*	_	*M*	*E*	_	*M*	*O*	*N*	*D*	*A*	*Y*	_

Uncoded Row Matrix	Encoding Matrix A	Coded Row Matrix

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$$

$$\vdots$$

The sequence of coded row matrices is

$$13 \;\; -26 \;\; 21 \;\; 33 \;\; -53 \;\; -12 \;\; 18 \;\; -23 \;\; -42 \;\; 5 \;\; -20 \;\; 56 \;\; -24 \;\; 23 \;\; 77$$

Decoding a Message = Coded Row Matrices $\xrightarrow{A^{-1}}$ Decoded row matrices

If $\boldsymbol{x} = [x_1 \ x_2 \ . \ . \ . \ x_n]$ is an uncoded $1\times n$ matrix, then $\boldsymbol{y} = \boldsymbol{x}A$ is the corresponding encoded matrix $\boldsymbol{y}A^{-1} = (\boldsymbol{x}A)A^{-1} = \boldsymbol{x}$.

- Ex 2:

Use the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$ to decode the cryptogram

$$13 \ -26 \ 21 \ 33 \ -53 \ -12 \ 18 \ -23 \ -42 \ 5 \ -20 \ 56 \ -24 \ 23 \ 77$$

Sol:

$$A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

$$[13\ \ -26\ \ 21][33\ \ -53\ \ -12][18\ \ -23\ \ -42][5\ \ -20\ \ 56][-24\ \ 23\ \ 77]$$

Coded Row Matrix | Decoding Matrix A^{-1} | Decoded Row Matrix

$$[13\ \ -26\ \ 21]\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [13\ \ 5\ \ 5]$$

$$[33\ \ -53\ \ -12]\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = [20\ \ 0\ \ 13]$$

$$\vdots$$

The sequence of decoded row matrices is

$$[13\ \ 5\ \ 5]\quad [20\ \ 0\ \ 13]\quad [5\ \ 0\ \ 13]\quad [15\ \ 14\ \ 4]\quad [1\ \ 25\ \ 0]$$

$$M\ \ E\ \ E\ \ T\ \ _\ \ M\ \ E\ \ _\ \ M\ \ O\ \ N\ \ D\ \ A\ \ Y\ \ _$$

- Least Squares Regression Analysis

For a set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ the least squares regression line is the linear function $f(x) = a_0 + a_1 x$ that minimizes the sum of squared error:

$$[y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2$$

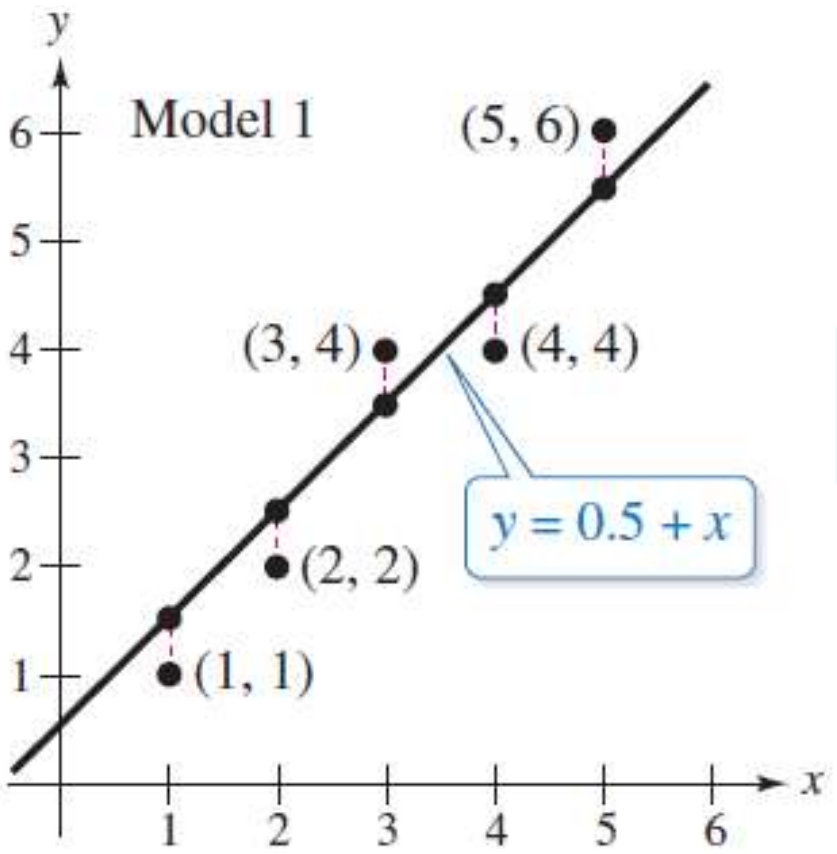
$$e_i = y_i - f(x_i)$$

$$y_1 = (a_0 + a_1 x_1) + e_1$$

$$y_2 = (a_0 + a_1 x_2) + e_2$$

$$\vdots$$

$$y_n = (a_0 + a_1 x_n) + e_n$$



$$\boldsymbol{y} = \boldsymbol{x}A + \boldsymbol{e}; \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Matrix Form for Linear Regression

For the regression model $\boldsymbol{y} = \boldsymbol{x}A + \boldsymbol{e}$, the coefficients of the least squares regression line are given by the matrix equation $A = (\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{y}$

and the sum of squared error is $\boldsymbol{e}^T\boldsymbol{e}$

- Ex 3:

Find the least squares regression line for the points (1, 1), (2, 2), (3, 4), (4, 4), and (5, 6)

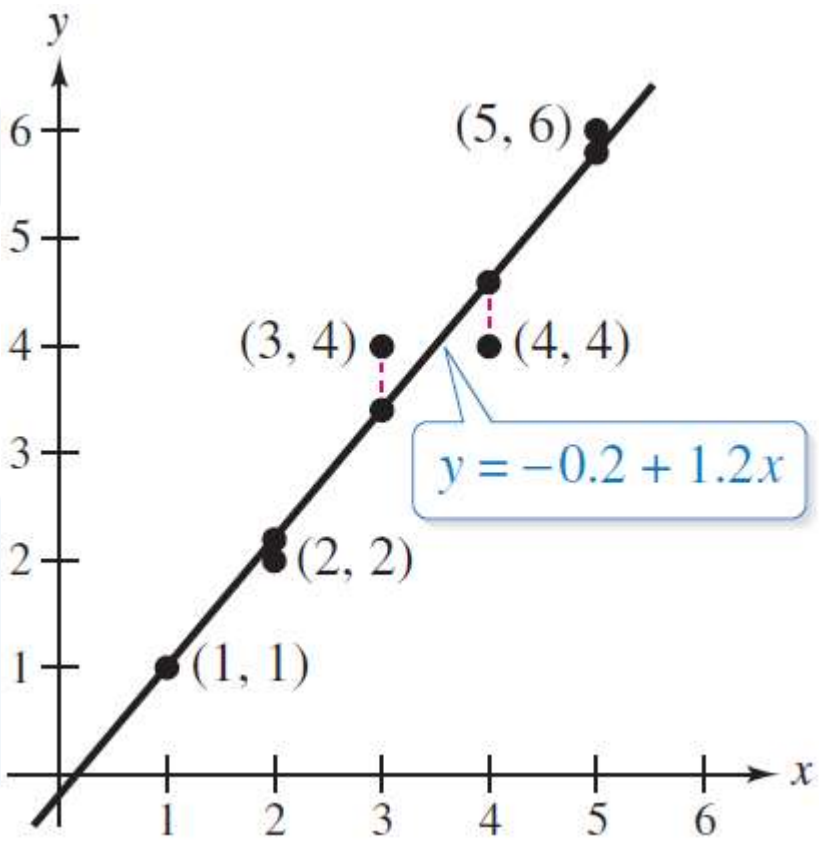
Sol:

$$\boldsymbol{x}^T\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

$$\boldsymbol{x}^T\boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 63 \end{bmatrix}$$

$$(\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{y} = \frac{1}{50}\begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix}\begin{bmatrix} 17 \\ 63 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 1.2 \end{bmatrix}$$

$$y = -0.2 + 1.2x$$



$$\boldsymbol{e}^T\boldsymbol{e} = 0.8$$