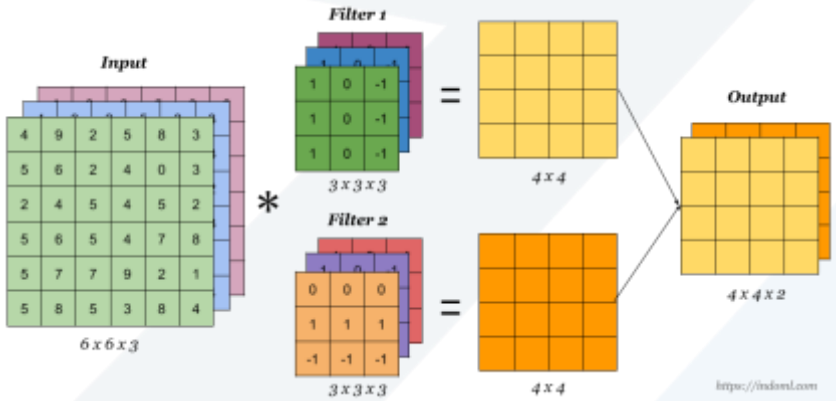
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 4: Determinants



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3.1 The Determinant of a Matrix

3.2 Determinant and Elementary Operations

3.3 Properties of Determinants

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3.1 The Determinant of a Matrix

The determinant is a function $\det: M_n(K) \to K;\ A \mapsto \det(A) = |A|$

K: a field (R or C)

- The determinant of a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Ex. 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7 \qquad \begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

- Note: The determinant of a matrix can be positive, zero, or negative

- Minor of the entry a_{ij}:

The determinant of the matrix determined by deleting the i-th row and j-th column of A

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

- Cofactor of a_{ij}:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- Ex. 2: Find all the minors and cofactors of A

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of A

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \qquad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5, \qquad M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2, \qquad M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \qquad M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \qquad M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3, \qquad M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

(2) All the cofactors of A

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \qquad C_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5, \qquad C_{13} = +\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \qquad C_{22} = +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \qquad C_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \qquad C_{32} = -\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3, \qquad C_{33} = +\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- Theorem 3.1: (Expansion by cofactors)

Let A is a square matrix of order n. Then the determinant of A is given by

$$(a)\ \det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(Cofactor expansion along the i-th row, $i = 1, 2, \ldots, n$)

or

$$(b)\ \det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(Cofactor expansion along the j-th column, $j = 1, 2, \ldots, n$)

- Ex. 3: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

From Ex. 2:

$$C_{11} = -1,\ C_{12} = 5,\quad C_{13} = 4$$
$$C_{21} = -2,\ C_{22} = -4,\ C_{23} = 8$$
$$C_{31} = 5,\quad C_{32} = 3,\quad C_{33} = -6$$

Sol:

$$\begin{aligned}
\Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\
&= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\
&= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\
&= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\
&= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14
\end{aligned}$$

- Ex. 4: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix} = 7 \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} = -8$$

$$\Rightarrow \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= (0)(7) + (2)(5) + (1)(-8) = 2$$

- Note:

  The row (or column) containing the most zeros is the best choice for expansion by cofactors.

- Ex. 5: The determinant of a matrix of order 4

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3\left[(0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 3\left[0 + (2)(1)(-4) + (3)(-1)(-7)\right]$$

$$= (3)(13)$$

$$= 39$$

- The determinant of a matrix of order 3 (Sarrus Rule)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Subtract these three products.

Add these three products.

$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- Ex. 6:

$$\begin{matrix} & & & -4 & 0 & 6 \\ A = & & & 0 & 2 & \\ & & & 3 & -1 & \\ & & & 4 & -4 & \\ & & & 0 & 16 & -12 \end{matrix} \qquad \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4 + 0 + 6) = 2$$

- Upper triangular matrix:

  All the entries below the main diagonal are zeros.

- Lower triangular matrix:

  All the entries above the main diagonal are zeros.

- Diagonal matrix:

  All the entries above and below the main diagonal are zeros.

- Note:

  A matrix that is both upper and lower triangular is called diagonal.

- Ex. 7:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular lower triangular diagonal

- Theorem 3.2: (Determinant of a Triangular Matrix)

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

- Ex. 8:

$$(a)\quad A=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} \qquad (b)\quad B=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a)\quad |A| = (2)(-2)(1)(3) = -12$$

$$(b)\quad |B| = (-1)(3)(2)(4)(-2) = 48$$

3.2 Evaluation of a determinant using elementary operations

- Theorem 3.3: (Elementary row operations and determinants)

  Let A and B be square matrices.

$$(a)\ B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad \left(\text{i.e. } \left|r_{ij}(A)\right| = -\left|A\right|\right)$$

$$(b)\ B = r_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad \left(\text{i.e. } \left|r_{i}^{(k)}(A)\right| = k\left|A\right|\right)$$

$$(c)\ B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad \left(\text{i.e. } \left|r_{ij}^{(k)}(A)\right| = \left|A\right|\right)$$

- Ex. 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

- Note:

A row-echelon form of a square matrix is always <u>upper triangular</u>.

- Ex. 2:

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \det(A) = ?$$

Sol:

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{12}^{(-2)}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{23}^{(-1)}}{=} (7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} \overset{r_3^{(-1)}}{=} (7)(-1)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (7)(-1)(1)(1)(1) = -7$$

Determinants and elementary column operations

- Theorem 3.4: (Elementary columns operations and determinants)

Let A and B be square matrices.

$$(a)\ B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad \text{(i.e. } \left|c_{ij}(A)\right| = -\left|A\right|\text{)}$$

$$(b)\ B = c_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad \text{(i.e. } \left|c_{i}^{(k)}(A)\right| = k\left|A\right|\text{)}$$

$$(c)\ B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad \text{(i.e. } \left|c_{ij}^{(k)}(A)\right| = \left|A\right|\text{)}$$

- Ex. 3:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \quad \Rightarrow \quad \det(A_1) = \det(c_1^{(\frac{1}{2})}(A)) = \frac{1}{2}\det(A) = (\frac{1}{2})(-8) = -4$$

$$A_2 = c_{12}(A) \quad \Rightarrow \quad \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \quad \Rightarrow \quad \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- Theorem 3.5: (Conditions that yield a zero determinant)

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$

(a) An entire row (or an entire column) consists of zeros.

(b) One row (or column) is a multiple of another row (or column).

- Ex. 4:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Note:

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

The maximum number of additions alone for the cofactor expansion of an $n \times n$ matrix is $n! - 1$. 30! is approximately equal to 2.65×10^{32}.

For a 30×30 matrix, If a computer could do one trillion operations per second, more than 22 trillion years to compute the determinant.

- Ex. 5: (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \overset{c_{13}^{(2)}}{=} \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = (-3)(-1)^{3+1} \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = (-3)(-1) = 3$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \overset{r_{12}^{(\frac{4}{5})}}{=} \begin{vmatrix} -3 & 5 & 2 \\ -2/5 & 0 & 3/5 \\ -3 & 0 & 6 \end{vmatrix}$$

$$= (5)(-1)^{1+2} \begin{vmatrix} -2/5 & 3/5 \\ -3 & 6 \end{vmatrix} = (-5)(-\frac{3}{5}) = 3$$

- Ex. 6: (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{r_{25}^{(-1)}}{\overset{r_{24}^{(1)}}{=}} \begin{vmatrix} 2 & \boxed{0} & 1 & 3 & -2 \\ -2 & \boxed{1} & 3 & 2 & -1 \\ 1 & \boxed{0} & -1 & 2 & 3 \\ 1 & \boxed{0} & 5 & 6 & -4 \\ 3 & \boxed{0} & 0 & 0 & 1 \end{vmatrix}$$

$$= (1)(-1)^{2+2} \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{c_{41}^{(-3)}}{=}\begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ \boxed{0 \quad 0 \quad 0 \quad 1} \end{vmatrix} = (1)(-1)^{4+4}\begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \overset{r_{21}^{(1)}}{=} \begin{vmatrix} \boxed{0 \quad 0 \quad 5} \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3}\begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

3.3 Properties of Determinants

- Theorem 3.6: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

- Note:

$$(1)\ \det(A + B) \neq \det(A) + \det(B)$$

- Ex. 1: (Determinant of a matrix product)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7, \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

Check: $|AB| = |A|\,|B|$ $\qquad -77 = -7 \text{ x } 11$

- Theorem 3.7: (Determinant of a scalar multiple of a matrix)

If A is an $n x n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

- Ex. 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5 \qquad \text{Find } |A|$$

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- Theorem 3.8: (Determinant of an invertible a matrix)

  A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$

- Ex. 3: (Classifying square matrices as singular or nonsingular)

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \Rightarrow A$ has no inverse (it is singular).

$|B| = -12 \neq 0 \Rightarrow B$ has an inverse (it is nonsingular).

- Theorem 3.9: (Determinant of an inverse matrix)

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

- Theorem 3.10: (Determinant of a transpose)

If A is a square matrix, then $\det(A^{\mathrm{T}}) = \det(A)$.

- Ex. 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad (a)\ \left|A^{-1}\right| = ? \qquad (b)\ \left|A^{T}\right| = ?$$

Sol:

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 \quad \Rightarrow \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}, \quad \left|A^{T}\right| = |A| = 4$$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\boldsymbol{x} = b$ has a unique solution for every $n \times 1$ matrix $\boldsymbol{b}$.

(3) $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex. 5: Which of the following system has a unique solution?

(a)
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

(b)
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

(a) $A\boldsymbol{x} = \boldsymbol{b} \Rightarrow |A| = 0$. This system does not have a unique solution.

(b) $B\boldsymbol{x} = \boldsymbol{b} \Rightarrow |B| = -12 \neq 0$. This system has a unique solution.

3.4 Applications of Determinants

- The Adjoint of a Matrix $\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$

- The Inverse of a Matrix Using its Adjoint

  If A is an $n \times n$ invertible matrix, then $A^{-1} = \dfrac{1}{\det A} \operatorname{adj}(A)$

- Ex. 1: Find the inverse of a matrix using its adjoint

  Use the adjoint of A to find A^{-1}, where $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

Sol:

$$\operatorname{adj}(A)=\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}=\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$|A|=\begin{vmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{vmatrix}=3$$

$$A^{-1}=\frac{1}{\det A}\operatorname{adj}(A)=\frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}=\begin{bmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{bmatrix}$$

- Cramer's Rule

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
A\boldsymbol{x} = \boldsymbol{b}, \quad A = \left[a_{ij}\right]_{n\times n} = \left[\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \cdots, \boldsymbol{A}^{(n)}\right], \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0 \quad \text{(this system has a unique solution)}
$$

$$A_j = \left[\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \cdots, \boldsymbol{A}^{(j-1)}, \boldsymbol{b}, \boldsymbol{A}^{(j+1)}, \cdots, \boldsymbol{A}^{(n)} \right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$(\text{i.e. } \det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \quad j = 1, 2, \cdots, n$$

- Ex. 2: Use Cramer's rule to solve the system of linear equations

$$\begin{array}{rcrcrcl} -x & + & 2y & - & 3z & = & \boxed{1} \\ 2x & & & + & z & = & \boxed{0} \\ 3x & - & 4y & + & 4z & = & \boxed{2} \end{array}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 2 \end{matrix}} & \begin{matrix} 2 \\ 0 \\ -4 \end{matrix} & \begin{matrix} -3 \\ 1 \\ 4 \end{matrix} \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & \boxed{\begin{matrix} 1 \\ 0 \\ 2 \end{matrix}} & -3 \\ 2 & & 1 \\ 3 & & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & \boxed{\begin{matrix} 1 \\ 0 \\ 2 \end{matrix}} \\ 2 & 0 & \\ 3 & -4 & \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5}, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2}, \quad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

- Determinants as Area or Volume

If A is a 2x2 matrix, the area of the parallelogram determined by the columns of A is

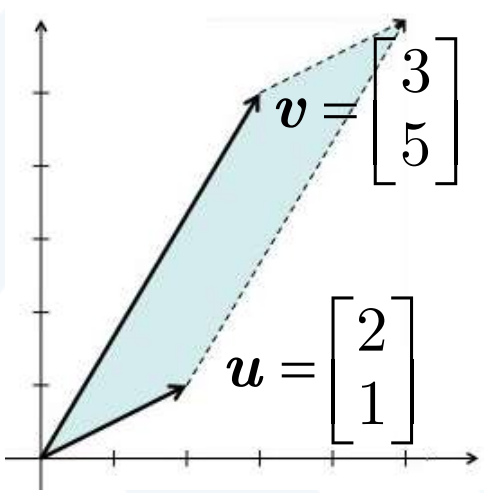
$$\text{Area} = |\det(A)|$$

If A is a 3x3 matrix, the volume of the parallelepiped determined by the columns of A is

$$\text{Volume} = |\det(A)|$$

- Ex. 3:

Find area of parallelogram formed by $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$



Sol:

$$\text{Area} = |\det(A)| = \left|\det\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}\right| = |10 - 3| = 7$$

- Ex. 4:

Find volume of parallelepiped formed by $\boldsymbol{u}=\begin{bmatrix}2\\1\\1\end{bmatrix}, \boldsymbol{v}=\begin{bmatrix}1\\1\\4\end{bmatrix}, \boldsymbol{w}=\begin{bmatrix}1\\3\\1\end{bmatrix}$

Sol:

$$\text{Volume} = |\det(A)| = \left|\det\begin{bmatrix}2 & 1 & 1\\1 & 1 & 3\\1 & 4 & 1\end{bmatrix}\right| = |2+3+4-(1+1+24)| = 17$$

- The Vandermonde matrix

Let $x_1, x_2, \ldots, x_n$ be real numbers. The Vandermonde matrix, which arises in signal processing and coding theory, is given by

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}, \qquad \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i \le j \le n} (x_j - x_i)$$

For any $n > 1$

For $n = 3$, $\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$