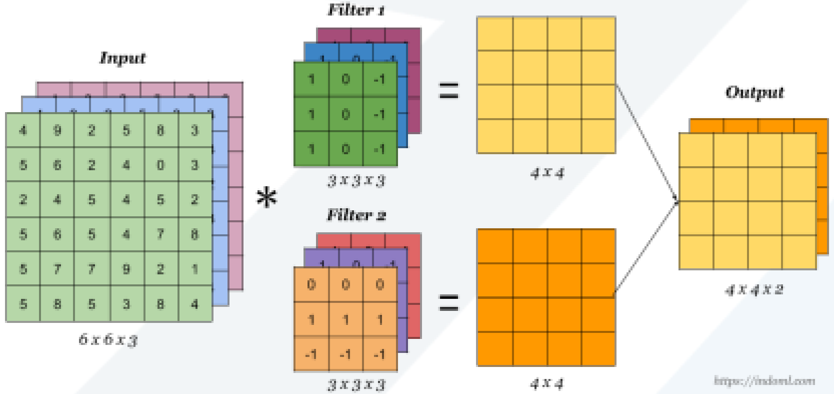
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 7: Inner Product Spaces: Part A



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5.1 Length and Dot Product in R^n

- **Length:**

The length of a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in R^n is given by

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- **Note:** The length of a vector is also called its norm.

- **Notes: Properties of length**

$(1)\ \|\boldsymbol{v}\| \geq 0$

$(2)\ \|\boldsymbol{v}\| = 1 \Rightarrow \boldsymbol{v}$ is called a unit vector

$(3)\ \|\boldsymbol{v}\| = 0$ iff $\boldsymbol{v} = 0$

- Ex 1:

(*a*) In R^5, the length of $\boldsymbol{v} = (0, -2, 1, 4, -2)$ is given by

$$\|\boldsymbol{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5$$

(*b*) In R^3 the length of $\boldsymbol{v} = \left(\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ is given by

$$\|\boldsymbol{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1$$

($\boldsymbol{v}$ is a unit vector)

- A standard unit vector in R^n:

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\} = \{(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1)\}$$

- Ex 2:

the standard unit vector in R^2: $\{\boldsymbol{i}, \boldsymbol{j}\} = \{(1, 0), (0, 1)\}$

the standard unit vector in R^3: $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

- Notes: (Two nonzero vectors are parallel)

$$\boldsymbol{u} = c\boldsymbol{v}$$

(1) $c > 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ have the same direction

(2) $c < 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ have the opposite direction

- Theorem 5.1: (Length of a scalar multiple)

  Let $\boldsymbol{v}$ be a vector in R^n and c be a scalar, then $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$

- Theorem 5.2: (Unit vector in the direction of $\boldsymbol{v}$)

  If $\boldsymbol{v}$ is a nonzero vector in R^n, then the vector $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ has length 1 and has the same direction as $\boldsymbol{v}$.

  This vector $\boldsymbol{u}$ is called the unit vector in the direction of $\boldsymbol{v}$.

- Note:

  The process of finding the unit vector in the direction of $\boldsymbol{v}$ is called normalizing the vector $\boldsymbol{v}$.

- Ex 3: (Finding a unit vector)

  Find the unit vector in the direction of $\boldsymbol{v}=(3, -1, 2)$, and verify that this vector has length 1.

- Sol:

$$\|\boldsymbol{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}=\frac{(3\,,-1\,,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3\,,-1\,,2)=\left(\frac{3}{\sqrt{14}}\,,\frac{-1}{\sqrt{14}}\,,\frac{2}{\sqrt{14}}\right)$$

$$\sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1\Rightarrow \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \text{ is a unit vector}$$

- Distance between two vectors:

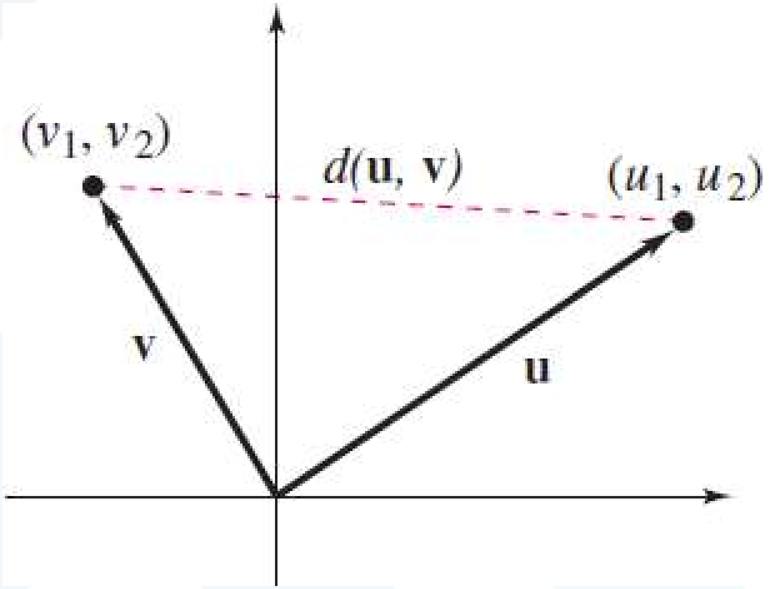
  The distance between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in R^n is: $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$

- Notes: (Properties of distance)

  (1) $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$

  (2) $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{u} = \boldsymbol{v}$

  (3) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$



- Ex 4: (Distance between 2 vectors)

The distance between $\boldsymbol{u} = (0, 2, 2)$ and $\boldsymbol{v} = (2, 0, 1)$ is

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \|(0-2), 2-0, 2-1)\| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

- Dot product in R^n:

The dot product of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is the scalar quantity

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Ex 5: (Finding the dot product of two vectors)

The dot product of $\boldsymbol{u} = (1, 2, 0, -3)$ and $\boldsymbol{v} = (3, -2, 4, 2)$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

- Theorem 5.3: (Properties of the dot product)

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in R^n and c is a scalar, then the following properties are true.

(1) $\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{v}.\boldsymbol{u}$

(2) $\boldsymbol{u}.(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u}.\boldsymbol{v} + \boldsymbol{u}.\boldsymbol{w}$

(3) $c(\boldsymbol{u}.\boldsymbol{v}) = (c\boldsymbol{u}).\boldsymbol{v} = \boldsymbol{u}.(c\boldsymbol{v})$

(5) $\boldsymbol{v}.\boldsymbol{v} \geq 0$, and $\boldsymbol{v}.\boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$

(4) $\boldsymbol{v}.\boldsymbol{v} = \|\boldsymbol{v}\|^2$

- Euclidean n-space:

R^n was defined to be the set of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called Euclidean n-space.

- Ex 6: (Finding dot products)

$$\boldsymbol{u} = (2, -2),\ \boldsymbol{v} = (5, 8),\ \boldsymbol{w} = (-4, 3)$$

$(a)\ \boldsymbol{u}.\boldsymbol{v}$ $(b)\ (\boldsymbol{u}.\boldsymbol{v})\boldsymbol{w}$ $(c)\ \boldsymbol{u}.(2\boldsymbol{v})$ $(d)\ \|\boldsymbol{w}\|^2$ $(e)\ \boldsymbol{u}.(\boldsymbol{v} - 2\boldsymbol{w})$

Sol:

$$(a)\ \boldsymbol{u}.\boldsymbol{v} = (2)(5) + (-2)(8) = -6$$

$$(b)\ (\boldsymbol{u}.\boldsymbol{v})\boldsymbol{w} = -\boldsymbol{w} = -6(-4, 3) = (24, -18)$$

$$(c)\ \boldsymbol{u}.(2\boldsymbol{v}) = 2(\boldsymbol{u}.\boldsymbol{v}) = 2(-6) = -12$$

$$(d)\ \|\boldsymbol{w}\|^2 = \boldsymbol{w}.\boldsymbol{w} = (-4)(-4) + (3)(3) = 25$$

$$(e)\ (\boldsymbol{v} - 2\boldsymbol{w}) = (5 - (-8), 8 - 6) = (13, 2)$$

$$\boldsymbol{u}.(\boldsymbol{v} - 2\boldsymbol{w}) = (2)(13) + (-2)(2) = 22$$

- Ex 7: (Using the properties of the dot product)

Given $\boldsymbol{u}.\boldsymbol{u} = 39,\ \boldsymbol{u}.\boldsymbol{v} = -3,\ \boldsymbol{v}.\boldsymbol{v} = 79$

Find $(\boldsymbol{u}+2\boldsymbol{v}).(3\boldsymbol{u}+\boldsymbol{v})$

Sol:

$$
\begin{aligned}
(\boldsymbol{u}+2\boldsymbol{v}).(3\boldsymbol{u}+\boldsymbol{v}) &= \boldsymbol{u}.(3\boldsymbol{u}+\boldsymbol{v}) + 2\boldsymbol{v}.(3\boldsymbol{u}+\boldsymbol{v}) \\
&= \boldsymbol{u}.(3\boldsymbol{u}) + \boldsymbol{u}.\boldsymbol{v} + (2\boldsymbol{v}).(3\boldsymbol{u}) + (2\boldsymbol{v}).\boldsymbol{v} \\
&= 3(\boldsymbol{u}.\boldsymbol{u}) + \boldsymbol{u}.\boldsymbol{v} + 6(\boldsymbol{v}.\boldsymbol{u}) + 2(\boldsymbol{v}.\boldsymbol{v}) \\
&= 3(\boldsymbol{u}.\boldsymbol{u}) + 7(\boldsymbol{u}.\boldsymbol{v}) + 2(\boldsymbol{v}.\boldsymbol{v}) \\
&= 3(39) + 7(-3) + 2(79) = 254
\end{aligned}
$$

- Theorem 5.4: (The Cauchy - Schwarz inequality)

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n, then $|\boldsymbol{u}.\boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$

- Ex 8: (An example of the Cauchy - Schwarz inequality)

  Verify the Cauchy - Schwarz inequality for $\boldsymbol{u} = (1, -1, 3)$ and $\boldsymbol{v} = (2, 0, -1)$

Sol:

$$\boldsymbol{u}.\boldsymbol{u} = 11, \ \boldsymbol{u}.\boldsymbol{v} = -1, \ \boldsymbol{v}.\boldsymbol{v} = 5$$

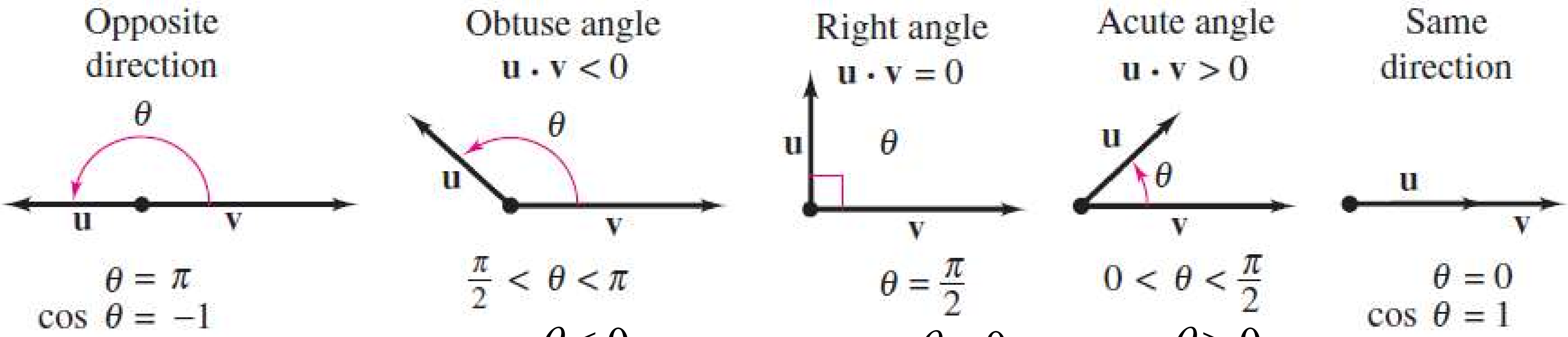
$$|\boldsymbol{u}.\boldsymbol{v}| = |-1| = 1$$

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| = \sqrt{\boldsymbol{u}.\boldsymbol{u}}\sqrt{\boldsymbol{v}.\boldsymbol{v}} = \sqrt{11}\sqrt{5} = \sqrt{55}$$

$$\Rightarrow |\boldsymbol{u}.\boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- The angle between two vectors in R^n:

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|},\ 0 \le \theta \le \pi$$



$\cos\theta < 0$ $\qquad$ $\cos\theta = 0$ $\qquad$ $\cos\theta > 0$

- Note:

The angle between the zero vector and another vector is not defined.

- Ex 9: (Finding the angle between two vectors)

$$\boldsymbol{u} = (-4, 0, 2, -2),\ \boldsymbol{v} = (2, 0, -1, 1)$$

Sol:

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}.\boldsymbol{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}.\boldsymbol{v}} = \sqrt{(2)^2 + 0^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\boldsymbol{u}.\boldsymbol{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = \frac{-12}{\sqrt{144}} = -1$$

$\Rightarrow \theta = \pi$ $\boldsymbol{u}$ and $\boldsymbol{v}$ have opposite directions ($\boldsymbol{u} = -2\boldsymbol{v}$)

- Orthogonal vectors:

  Tow vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in R^n are orthogonal if $\boldsymbol{u.v} = \mathbf{0}$

- Note: The vector $\mathbf{0}$ is said to be orthogonal to every vector

- Ex 10: (Finding orthogonal vectors)

  Determine all vectors in R^2 that are orthogonal to $\boldsymbol{u} = (4, 2)$

Sol:

$$\text{Let } \boldsymbol{v} = (v_1, v_2) \Rightarrow \boldsymbol{u.v} = (4, 2)\mathbf{.}(v_1, v_2) = 4v_1 + 2v_2 = 0 \Rightarrow 2v_1 + v_2 = 0$$

letting $v_2 = t$ (free variable), then $\{\boldsymbol{v} = (-t/2,\ t) | t \in R\}$

- Theorem 5.5: (The Triangle inequality)

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n, then $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

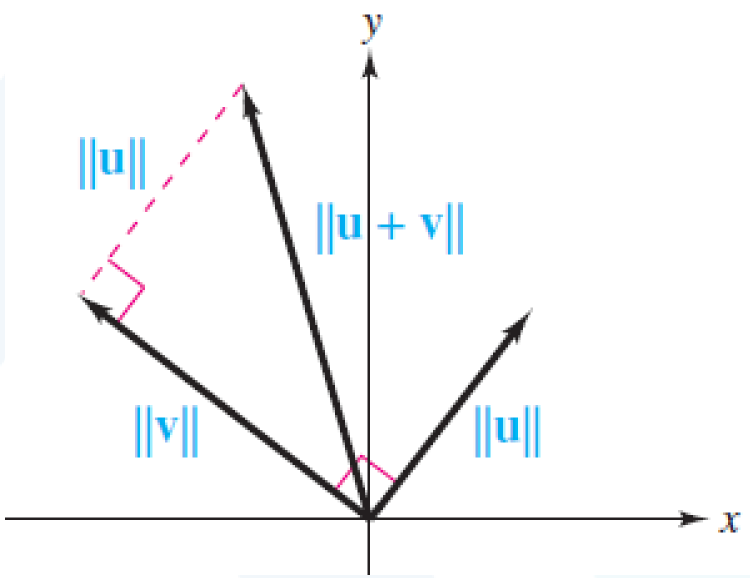
- Note:

  Equality occurs in the triangle inequality if and only if the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same direction.

- Theorem 5.6: (The Pythagorean theorem)

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$



- Dot product and matrix multiplication:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n is represented as an $n \times 1$ column matrix)

$$\mathbf{u}.\mathbf{v} = \mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 + u_2v_2 + \cdots + u_nv_n \end{bmatrix}$$

5.2 Inner Product Spaces

- **Inner Product:**

Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $<\boldsymbol{u}, \boldsymbol{v}>$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and satisfies the following axioms.

(1) $<\boldsymbol{u}, \boldsymbol{v}> = <\boldsymbol{v}, \boldsymbol{u}>$

(2) $<\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}> = <\boldsymbol{u}, \boldsymbol{v}> + <\boldsymbol{u}, \boldsymbol{w}>$

(3) $c<\boldsymbol{u}, \boldsymbol{v}> = <c\boldsymbol{u}, \boldsymbol{v}>$

(4) $<\boldsymbol{v}, \boldsymbol{v}> \geq 0$ and $<\boldsymbol{v}, \boldsymbol{v}> = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$

- Notes:

$\boldsymbol{u}.\boldsymbol{v}$ = dot product (Euclidean inner product for R^n)

$<\boldsymbol{u}, \boldsymbol{v}>$ = general inner product for vector space V

- Notes:

A vector space V with an inner product is called an inner product space.

Vector space: $(V, +, .)$

Inner product space: $(V, +, ., <,>)$

- Ex 1: (Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\boldsymbol{u} = (u_1, u_2, \ldots, u_n), \ \boldsymbol{v} = (v_1, v_2, \ldots, v_n)$$

$$< \boldsymbol{u}, \boldsymbol{v} > = \boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

- Ex 2: (A different inner product for R^n)

  Show that the function defines an inner product on R^2, where $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$

$$<\boldsymbol{u}, \boldsymbol{v}> = u_1 v_1 + 2 u_2 v_2$$

Sol:

(1) $<\boldsymbol{u}, \boldsymbol{v}> = u_1 v_1 + 2 u_2 v_2 = v_1 u_1 + 2 v_2 u_2 = <\boldsymbol{v}, \boldsymbol{u}>$

(2) $\boldsymbol{w} = (w_1, w_2)$

$$\begin{aligned} < \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} > &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2) \\ &= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2 \\ &= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2) \\ &= < \boldsymbol{u}, \boldsymbol{v} > + < \boldsymbol{u}, \boldsymbol{w} > \end{aligned}$$

(3) $c < \boldsymbol{u}, \boldsymbol{v} > = c\,(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = < c\boldsymbol{u}, \boldsymbol{v} >$

(4) $< \boldsymbol{v}, \boldsymbol{v} > = v_1^{\,2} + 2v_2^{\,2} \geq 0$

$$< \boldsymbol{v}, \boldsymbol{v} > = 0 \Rightarrow v_1^{\,2} + 2v_2^{\,2} = 0 \Rightarrow v_1 = v_2 = 0 \quad (\boldsymbol{v} = \boldsymbol{0})$$

- Note: (An inner product on R^n)

$$< \boldsymbol{u}, \boldsymbol{v} > = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n, \quad c_i > 0 \qquad \text{(weights)}$$

- Ex 3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3

$$< \boldsymbol{u}, \boldsymbol{v} > = u_1v_1 - 2u_2v_2 + u_3v_3$$

Sol:

Let $\boldsymbol{v} = (1, 2, 1)$, then $<\boldsymbol{v}, \boldsymbol{v}> = (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied. Thus this function is not an inner product on R^3

- Theorem 5.7: (Properties of inner products)

Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in an inner product space V, and let c be any real number.

(1) $<\mathbf{0}, \boldsymbol{v}> = <\boldsymbol{v}, \mathbf{0}> = \mathbf{0}$

(2) $<\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}> = <\boldsymbol{u}, \boldsymbol{w}> + <\boldsymbol{v}, \boldsymbol{w}>$

(3) $<\boldsymbol{u}, c\boldsymbol{v}> = c <\boldsymbol{u}, \boldsymbol{v}>$

- Norm (length) of $\boldsymbol{u}$: $\|\boldsymbol{u}\| = \sqrt{< \boldsymbol{u}, \boldsymbol{u} >}$

- Note: $\|\boldsymbol{u}\|^2 = < \boldsymbol{u}, \boldsymbol{u} >$

- Distance between $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle}$$

- Angle between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}, \; 0 \le \theta \le \pi$$

- Orthogonal: $(\boldsymbol{u} \perp \boldsymbol{v})$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $<\boldsymbol{u}, \boldsymbol{v}> = 0$

- Notes:

(1) If $\|\boldsymbol{v}\| = 1$, then $\boldsymbol{v}$ is called a unit vector

(2) $\|\boldsymbol{v}\| \neq 1$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ (the unit vector in the direction of $\boldsymbol{v}$)

$\boldsymbol{v} \neq \mathbf{0}$

not a unit vector

- Properties of norm:

(1) $\|\boldsymbol{u}\| \geq 0$

(2) $\|\boldsymbol{u}\| = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$

(3) $\|c\boldsymbol{u}\| = |c|\|\boldsymbol{u}\|$

- (Properties of distance)

  (1) $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$

  (2) $d(\boldsymbol{u}, \boldsymbol{v}) = 0$ if and only if $\boldsymbol{u} = \boldsymbol{v}$

  (3) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$

- Theorem 5.8:

  Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product space V.

  (1) Cauchy-Schwarz inequality: $|< \boldsymbol{u}.\boldsymbol{v} >| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|$ Theorem 5.4

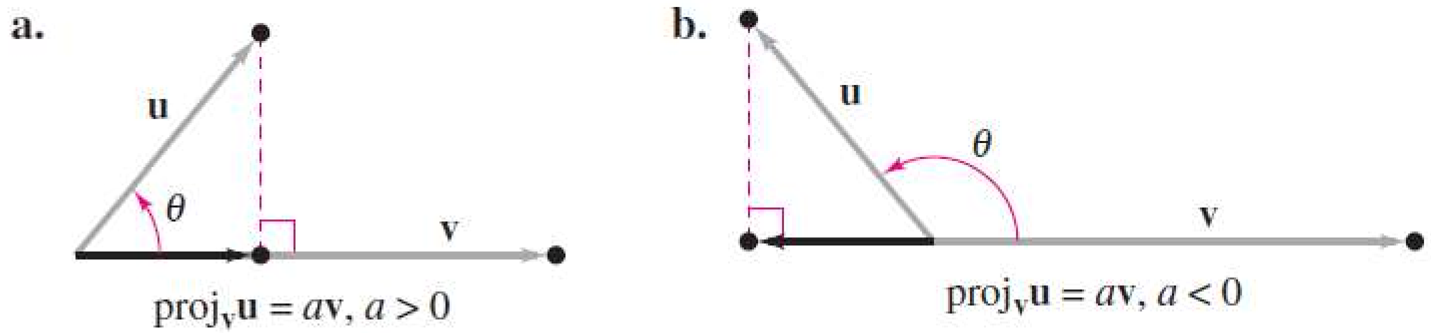
  (2) Triangle inequality: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ Theorem 5.5

  (3) Pythagorean theorem: Theorem 5.6

  $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if $\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$

- Orthogonal projections in inner product spaces:

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in an inner product space V, such that $\boldsymbol{v} \neq 0$. Then the orthogonal projection of u onto $\boldsymbol{v}$ is given by $\operatorname{proj}_{\boldsymbol{v}}\boldsymbol{u} = \dfrac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\langle \boldsymbol{v}, \boldsymbol{v}\rangle}\boldsymbol{v}$



- Note:

If $\boldsymbol{v}$ is a unit vector, then $< \boldsymbol{v}, \boldsymbol{v} > = \|\boldsymbol{v}\|^2 = 1.$

The formula for the orthogonal projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ takes the following simpler form:

$$\operatorname{proj}_{\boldsymbol{v}}\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v}\rangle\boldsymbol{v}$$

- Ex 4: (Finding an orthogonal projection in R^3)

Use the Euclidean inner product in R^3 to find the orthogonal projection of $\boldsymbol{u} = (6, 2, 4)$ onto $\boldsymbol{v} = (1, 2, 0)$.

Sol:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (6)(1) + (2)(2) + (4)(0) = 10$$

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1^2 + 2^2 + 0^2 = 5$$

$$\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}\boldsymbol{v} = \frac{10}{5}(1, 2, 0) = (2, 4, 0)$$

- Note:

$$\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = (6,2,4) - (2,4,0) = (4, -2, 4) \text{ is orthogonal to } \boldsymbol{v} = (1,2,0)$$