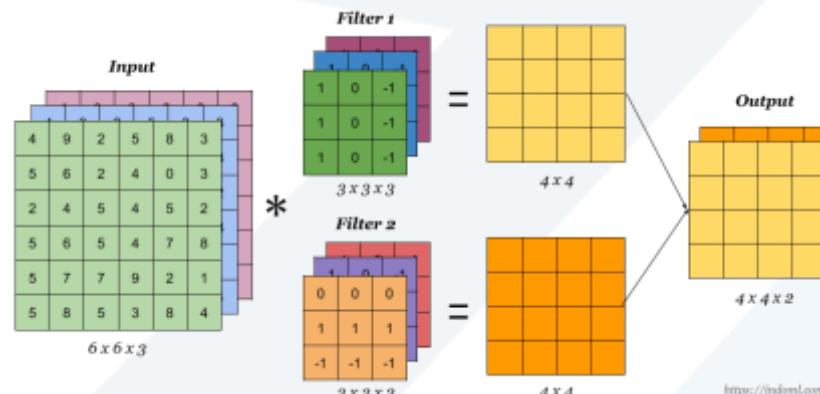


CECC122: Linear Algebra and Matrix Theory

Lecture Notes 11: Eigenvalues and Eigenvectors: Part A



Ramez Koudsieh, Ph.D.

Faculty of Engineering
Department of Informatics

Manara University

- 7.1 Eigenvalues and Eigenvectors
- 7.2 Diagonalization
- 7.3 Symmetric Matrices and Orthogonal Diagonalization
- 7.4 Applications of Eigenvalues and Eigenvectors

7.1 Introduction Eigenvalues and Eigenvectors

- **Eigenvalue problem:**

If A is an $n \times n$ matrix, do there exist nonzero vectors x in R^n such that Ax is a scalar multiple of x ?

- **Eigenvalue and eigenvector:**

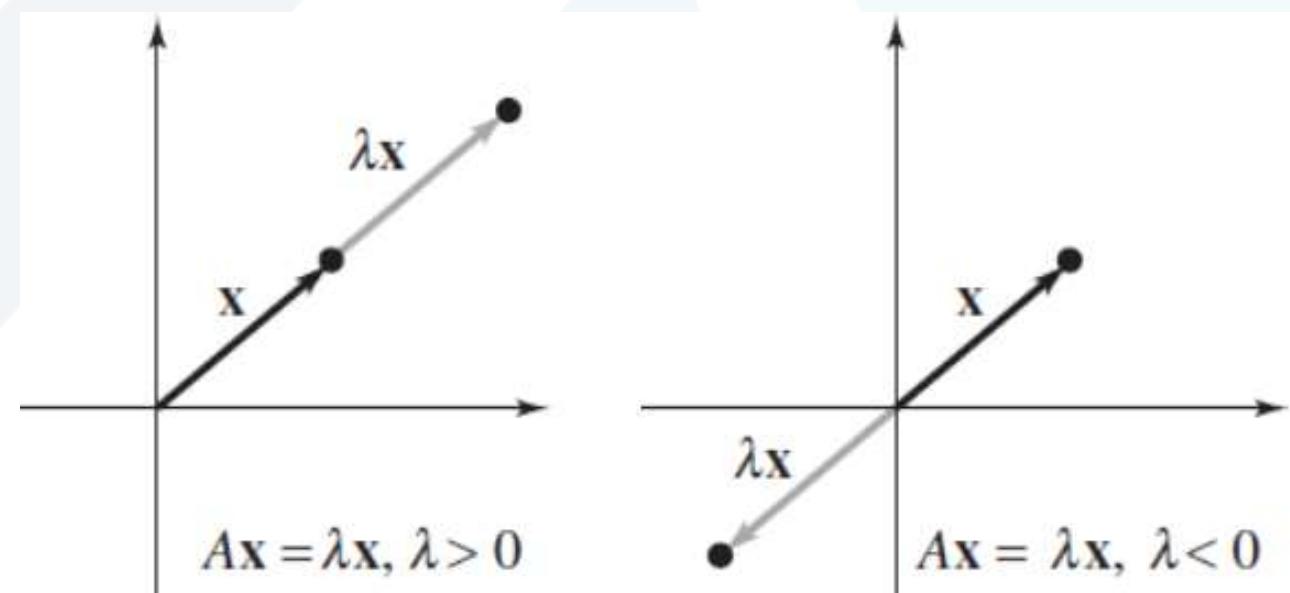
A : an $n \times n$ matrix

λ : a scalar

x : a nonzero vector in R^n

$$Ax = \lambda x$$

Eigenvalue \downarrow
 Eigenvector \uparrow \uparrow



- **Geometrical Interpretation:**

- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue

$$Ax_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2x_1$$

↑
Eigenvector

Eigenvalue

$$Ax_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)x_2$$

↑
Eigenvector

- Theorem 7.1: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n \times n}$)
Let A is an $n \times n$ matrix.
 - (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$
 - (2) The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)x = \mathbf{0}$
- Note:
 $Ax = \lambda x \Rightarrow (\lambda I - A)x = \mathbf{0}$ (homogeneous system)
If $(\lambda I - A)x = \mathbf{0}$ has nonzero solutions iff $\det(\lambda I - A) = 0$
- Characteristic polynomial of $A \in M_{n \times n}$:
$$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic equation of A : $\det(\lambda I - A) = 0$
- Ex 2: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol:

Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$
$$\Rightarrow \lambda = -1, -2$$

Eigenvalues: $\lambda_1 = -1, \lambda_2 = -2$

$$(1) \lambda_1 = -1 \Rightarrow (\lambda_1 I - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2) \lambda_2 = -2 \Rightarrow (\lambda_2 I - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Check: $Ax = \lambda x$

- Ex 3: (Finding eigenvalues)

Find the eigenvalues and corresponding eigenvectors for the matrix A .

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol:

Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0 \quad \text{Eigenvalue: } \lambda = 2$$

$\lambda = 2 \Rightarrow$ Eigenvectors:

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

- Note:

If an eigenvalue λ_1 occurs as a multiple root (k times) for the characteristic polynomial, then λ_1 has multiplicity k .

- Theorem 7.2: (Eigenvalues of triangular matrices)

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 4: (Finding eigenvalues)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

Sol:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$
$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

7.2 Diagonalization

- **Diagonalization problem:**

For a square matrix A , does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- **Diagonalizable matrix:**

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**. (P **diagonalizes** A)

- **Notes:**

(1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

(2) The eigenvalue problem is related closely to the diagonalization prob.

- Theorem 7.3: (Similar matrices have the same eigenvalues)

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues

- Ex 1: (A diagonalizable matrix)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

Eigenvalues: $\lambda_1 = 4, \lambda_2 = -2, \lambda_3 = -2$

$$(1) \lambda_1 = 4 \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$(2) \lambda_2 = -2 \Rightarrow \text{Eigenvectors: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [p_1 \ p_2 \ p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1) P = [p_2 \ p_1 \ p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2) P = [p_2 \ p_3 \ p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- **Theorem 7.4: (Condition for diagonalization)**

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

- **Note:**

If n linearly independent vectors do not exist, then an $n \times n$ matrix A is not diagonalizable.

- **Ex 2: (A matrix that is not diagonalizable)**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0 \quad \text{Eigenvalue: } \lambda_1 = 1$$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A does not have two ($n = 2$) L. I. eigenvectors, so A is not diagonalizable

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors p_1, p_2, \dots, p_n for A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

Step 2: Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$

Step 3: Let $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, where $Ap_i = \lambda_i p_i, i = 1, 2, \dots, n$

- Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D .

- Ex 3: (Diagonalizing a matrix)

Find a matrix P such that is $P^{-1}AP$ diagonal $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$

$$(1) \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(2) \lambda_2 = -2 \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/4)t \\ -(1/4)t \\ t \end{bmatrix} = \frac{1}{4} t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$(3) \lambda_2 = 3 \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \ p_2 \ p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: k is a positive integer

$$(1) D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2) D = P^{-1}AP \Rightarrow D^k = (P^{-1}AP)^k = P^{-1}A^kP \\ \Rightarrow A^k = PD^kP^{-1}$$

- Theorem 7.5: (Sufficient conditions for diagonalization)

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Ex 4: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol:

Because A is a triangular matrix, its eigenvalues are the main diagonal entries.

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable.