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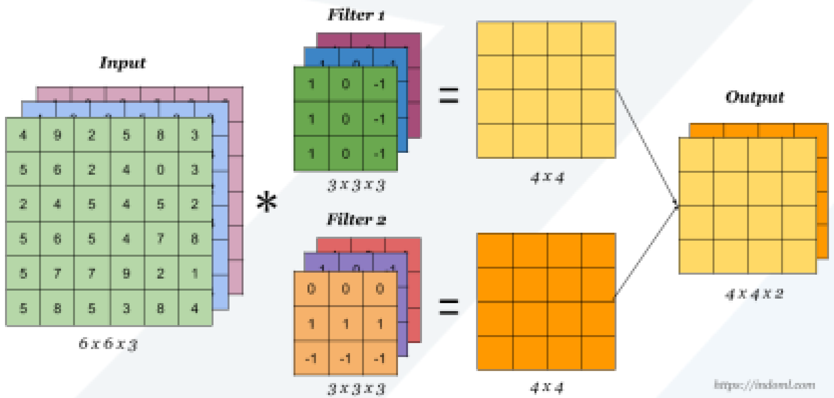
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 12: Eigenvalues and Eigenvectors: Part B



Ramez Koudsieh, Ph.D.

Faculty of Engineering
Department of Informatics
Manara University

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7.3 Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix:

A square matrix A is symmetric if it is equal to its transpose: $A = A^T$

- Ex 1: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

(symmetric) (symmetric) (nonsymmetric)

- Theorem 7.6: (Eigenvalues of symmetric matrices)

If A is an $n\times n$ symmetric matrix, then the following properties are true.

(1) A is diagonalizable.

(2) All eigenvalues of A are real.

(3) If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors.

- Orthogonal matrix:

  A square matrix P is called orthogonal if it is invertible and $P^{-1} = P^T$

- Ex 2: (Orthogonal matrices)

$$\text{(a) } P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is orthogonal because } P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{(b) } P = \begin{bmatrix} \frac{3}{5} & 0 & \frac{-4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \text{ is orthogonal because } P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ \frac{-4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

- Theorem 7.7: (Properties of orthogonal matrices)

  An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set.

- Ex 3: (An orthogonal matrix)

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Sol:

If P is a orthogonal matrix, then $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\text{Let } p_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{-2}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \end{bmatrix}, p_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3\sqrt{5}} \end{bmatrix}, p_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0$$

$$\|p_1\| = \|p_2\| = \|p_3\| = 1$$

$\{p_1, p_2, p_3\}$

is an orthonormal set

- Theorem 7.8: (Properties of symmetric matrices)

  Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are orthogonal.

- Ex 4: (Eigenvectors of a symmetric matrix)

  Show that any two eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

  corresponding to distinct eigenvalues are orthogonal

  Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$$

$\Rightarrow$ Eigenvalues: $\lambda_1 = 2,\ \lambda_2 = 4$

$$(1)\ \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_1 = s\begin{bmatrix} -1 \\ 1 \end{bmatrix},\quad s \neq 0$$

$$(2)\ \lambda_2 = 4 \Rightarrow \lambda_2 I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad t \neq 0$$

$$\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = \begin{bmatrix} -s \\ s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0 \Rightarrow \boldsymbol{x}_1 \text{ and } \boldsymbol{x}_2 \text{ are orthogonal}$$

- Orthogonal Diagonalization

  matrix A is orthogonally diagonalizable when there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal

- Theorem 7.9: (Fundamental theorem of symmetric matrices)

  Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalue if and only if A is symmetric.

- Ex 5: (Determining whether a matrix is orthogonally diagonalizable)

	Symmetric matrix	Orthogonally diagonalizable
$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	○	○
$A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$	×	×
$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	×	×
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$	○	○

Orthogonal diagonalization of a symmetric matrix:

Let A be an $n \times n$ symmetric matrix.

(1) Find all eigenvalues of A and determine the multiplicity of each.

(2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

(3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

(4) The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- Ex 6: (Orthogonal diagonalization)

Find a matrix P that orthogonally diagonalizes $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$

Sol: Characteristic equation:

(1) $|\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$

Eigenvalues: $\lambda_1 = -6$, $\lambda_2 = 3$ (has a multiplicity of 2)

(2) $\lambda_1 = -6, \ \boldsymbol{v}_1 = (1, \ -2, 2) \Rightarrow \boldsymbol{u}_1 = \dfrac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})$

(3) $\lambda_2 = 3, \ \boldsymbol{v}_2 = (2, 1, 0), \ \boldsymbol{v}_3 = (-2, 0, 1)$

Linear Independent

Gram-Schmidt Process:

$$\boldsymbol{w_2} = \boldsymbol{v_2} = (2, 1, 0),\ \ \boldsymbol{w_3} = \boldsymbol{v_3} - \frac{\boldsymbol{v_3} . \boldsymbol{w_2}}{\boldsymbol{w_2} . \boldsymbol{w_2}} \boldsymbol{w_2} = (-\frac{2}{5}, \frac{4}{3}, 1)$$

$$\boldsymbol{u_2} = \frac{\boldsymbol{w_2}}{\|\boldsymbol{w_2}\|} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0),\quad \boldsymbol{u_3} = \frac{\boldsymbol{w_3}}{\|\boldsymbol{w_3}\|} = (-\frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}})$$

$$(4)\ P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \Rightarrow P^{-1}AP = P^{T}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

7.4 Applications of Eigenvalues and Eigenvectors

Systems of Linear Differential Equations (Calculus)

$$\boldsymbol{y}' = A\boldsymbol{y}$$

$$\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad y_i' = \frac{dy_i}{dt}$$

- Ex 1: (Solving a System of Linear Differential Equations)

  Solve the system of linear differential equations
  $$\begin{aligned} y_1' &= 4y_1 \\ y_2' &= -y_2 \\ y_3' &= 2y_3 \end{aligned}$$

  Sol:
  $$\begin{aligned} y_1 &= C_1 e^{4t} \\ y_2 &= C_2 e^{-t} \\ y_3 &= C_3 e^{2t} \end{aligned}$$

- Notes:

(1) The matrix form of the system of linear differential equations is

$$\boldsymbol{y}' = A\boldsymbol{y}$$

$$\boldsymbol{y}' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

So, the coefficients of t in the solutions $y_i = C_i e^{\lambda_i t}$ are the eigenvalues of the matrix A (Diagonal).

(2) If A is not diagonal, find a matrix P that diagonalizes A.

Change of variables $\boldsymbol{y} = P\boldsymbol{w}$ and $\boldsymbol{y}' = P\boldsymbol{w}'$ produces:

$P\boldsymbol{w}' = \boldsymbol{y}' = A\boldsymbol{y} = AP\boldsymbol{w} \Rightarrow \boldsymbol{w}' = P^{-1}AP\boldsymbol{w},$ where $P^{-1}AP$ is a diagonal matrix

- Ex 2: (Solving a System of Linear Differential Equations)

Solve the system of linear differential equations
$$\begin{aligned} y_1' &= 3y_1 + 2y_2 \\ y_2' &= 6y_1 - y_2 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$$ Eigenvalues: –3, 5 Eigenvectors: $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ 3/4 & 1/4 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\boldsymbol{w}' = P^{-1}AP\boldsymbol{w}: \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \begin{aligned} w_1' &= -3w_1 \\ w_2' &= 5w_2 \end{aligned} \Rightarrow \begin{aligned} w_1 &= C_1 e^{-3t} \\ w_2 &= C_2 e^{5t} \end{aligned}$$

$$\boldsymbol{y} = P\boldsymbol{w}: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= w_1 + w_2 = C_1 e^{-3t} + C_2 e^{5t} \\ y_2 &= -3w_1 + w_2 = -3C_1 e^{-3t} + C_2 e^{5t} \end{aligned}$$

Quadratic Forms

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$ Quadratic equation

performing a rotation of axes that eliminates the xy-term

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

a' and c' are eigenvalues of the matrix: $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$

$$ax^2 + bxy + cy^2$$ Quadratic form

A is the matrix of the quadratic form

- Notes:

(1) The matrix A is symmetric

(2) A is diagonal iff its corresponding quadratic form has no xy-term

- Ex 3: (Finding the Matrix of the Quadratic Form)

Find the matrix of the quadratic form associated with each quadratic equation:

$$(a)\ 4x^2 + 9y^2 - 36 = 0 \quad (b)\ 13x^2 - 10xy + 13y^2 - 72 = 0$$

Sol:

(a) $a = 4,\ b = 0$, and $c = 9$, so the matrix is

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

Diagonal matrix (no xy-term)

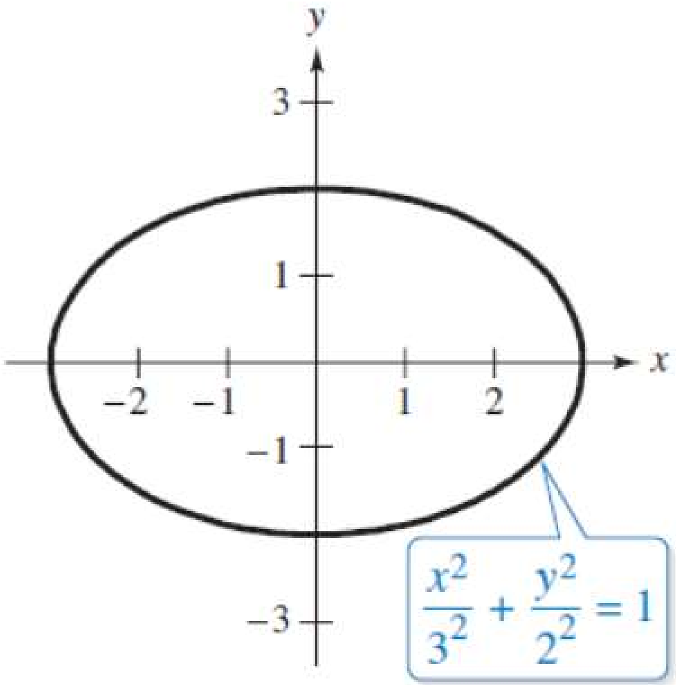
(b) $a = 13,\ b = -10$, and $c = 13$, so the matrix is

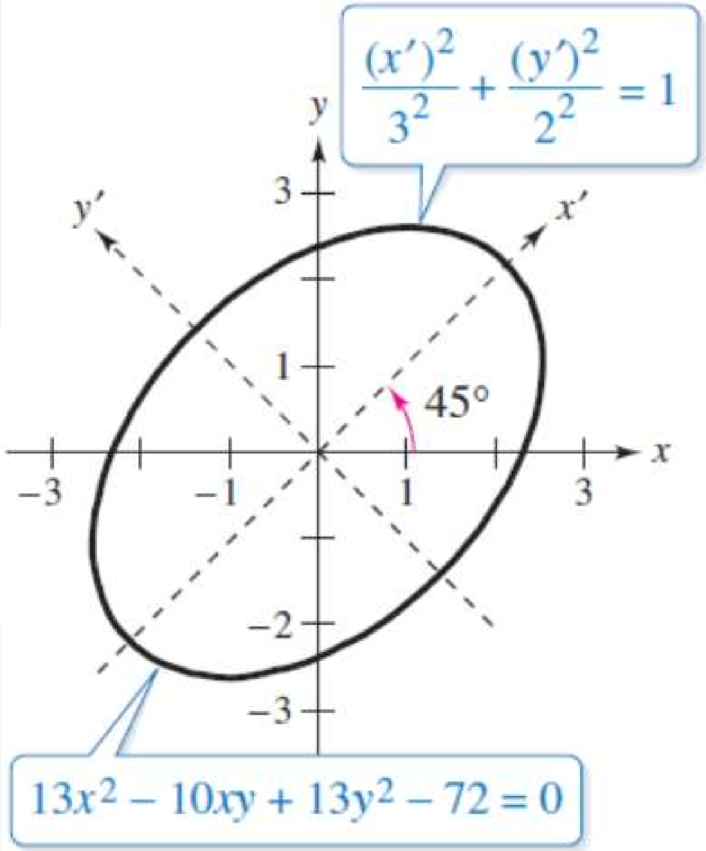
$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}$$

Nondiagonal matrix (xy-term)

- **Note:**

$13x^2 - 10xy + 13y^2 - 72 = 0$ is a 45° counterclockwise rotation of $4x^2 + 9y^2 - 36 = 0$





- **Theorem 7.10: (Principal Axes Theorem)**

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation $X = PX'$ eliminates the xy-term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes the matrix of the quadratic form A.

That is, $P^T AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1 and λ_2 are eigenvalues of A.

The equation of the rotated conic is $\lambda_1(x')^2 + \lambda_2(y')^2 + [d\ e]PX' + f = 0$

- **Ex 4: (Rotation of a Conic)**

Eliminate the xy-term in $13x^2 - 10xy + 13y^2 - 72 = 0$

Sol:

The characteristic polynomial of A is $(\lambda - 8)(\lambda - 18)$

Eigenvalues: $\lambda_1 = 8, \lambda_2 = 18$

Eigenvectors: $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Normalization $\Rightarrow \quad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$

Then, the equation of the rotated conic is: $8(x')^2 + 18(y')^2 - 72 = 0$

which, when written in the standard form: $\dfrac{(x')^2}{3^2} + \dfrac{(y')^2}{2^2} = 1$