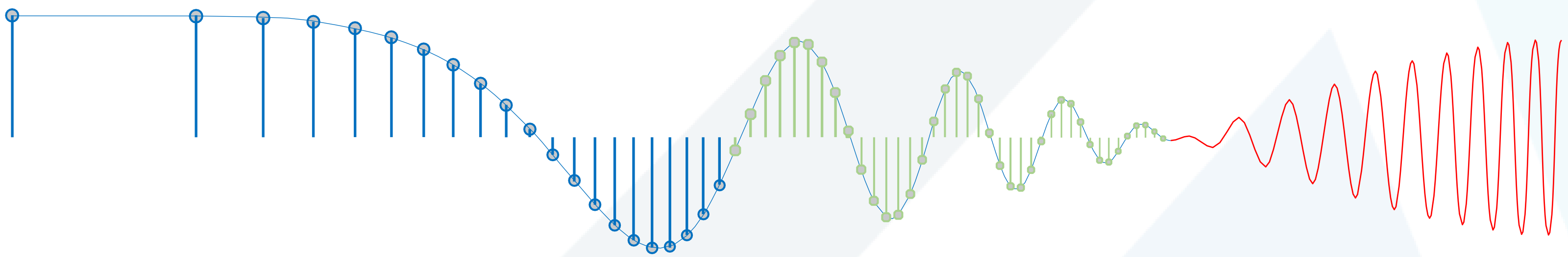
CEDC606: Digital Signal Processing

Lecture Notes 3: The z-transform



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

Chapter 3

The z-transform

1. Introduction

- The z-transform plays the same role in the analysis of DTLTI systems as the Laplace transform does in the analysis of CTLTI systems.
- The z-transform is an extension of the DTFT to address two problems:
  - First, there are many useful signals in practice, such as $nu[n]$, for which the DT Fourier transform does not exist.
  - Second, the transient response of a system due to initial conditions or due to changing inputs cannot be computed using the DTFT approach.
- The decomposition of an arbitrary sequence into a linear combination of scaled and shifted impulses, $x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$ shows that every LTI system can be represented by the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- The impulse response sequence $h[n]$ specifies completely the behavior and the properties of the associated LTI system.
- In general, any sequence that passes through a LTI system changes shape. We now ask: is there any sequence that retains its shape when it passes through an LTI system?

  Let us consider the complex exponential sequence: $x[n] = z^n$, for all n where z is a complex variable defined everywhere on the complex plane

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right) z^n = H(z)z^n, \quad \text{for all } n$$

- Thus, the output sequence is the same complex exponential as the input sequence, multiplied by a constant $H(z)$ that depends on the value of z.

- The quantity $H(z)$, as a function the complex variable z, is known as the system function or transfer function of the system.
- The complex exponential sequences are eigenfunctions of LTI systems.
- The constant $H(z)$, for a specified value of the complex variable z, is the eigenvalue associated with the eigenfunction z^n.

2. The z-transform

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$ two-sided or bilateral z-transform

- Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges.
- The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Let us express the complex variable z in polar form as: $z = re^{j\omega}$

$$X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}$$

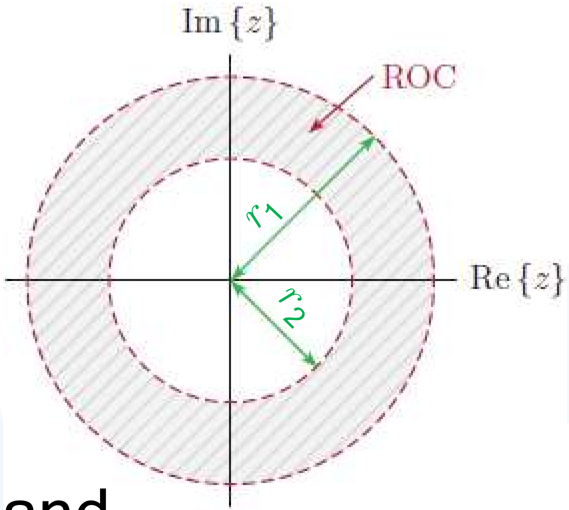
$$\Rightarrow \left|X(z)\right| = \left|\sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]r^{-n}e^{-j\omega n}\right| = \sum_{n=-\infty}^{\infty} \left|x[n]r^{-n}\right|$$

$$\left|X(z)\right| \le \sum_{n=-\infty}^{-1} \left|x[n]r^{-n}\right| + \sum_{n=0}^{\infty} \left|\frac{x[n]}{r^n}\right| = \sum_{n=1}^{\infty} \left|x[-n]r^{n}\right| + \sum_{n=0}^{\infty} \left|\frac{x[n]}{r^n}\right|$$

$\sum_{n=1}^{\infty} \left|x[-n]r^{n}\right|$ converges for all points inside a circle of radius r_1

$\sum_{n=0}^{\infty} \left|\frac{x[n]}{r^n}\right|$ converges for all points outside a circle of radius r_2

$X(z)$ converges for all points within an annular region of the form $r_2 < r < r_1$



- Note: The ROC depends only on r and not on ω.

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \mathcal{F}\{x[n]\}$$

- Note: The discrete-time Fourier transform $X(e^{j\omega})$ may be viewed as a special case of the z-transform $X(z)$.
- The values of z for which $X(z) = 0$ are called zeros of $X(z)$, and the values of z for which $X(z)$ is infinite are known as poles.
- Note: The ROC cannot include any poles.
- For finite duration sequences the ROC is the entire z-plane, with the possible exception of $z = 0$ or $z = \infty$.

- For infinite duration sequences the ROC can have one of the following shapes:
  - Right-sided ($x[n]=0,\ n<n_0$) $\Rightarrow$ ROC: $|z|>r$.
  - Left-sided ($x[n]=0,\ n>n_0$) $\Rightarrow$ ROC: $|z|<r$.
  - Two-sided $\Rightarrow$ ROC: $r_2<|z|<r_1$.
- Example 1: z-Transform of the unit-impulse

$$X(z)=\mathcal{Z}\{\delta[n]\}=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=x[0]z^{0}=1$$

It converges at every point in the z-plane

- Example 2: z-Transform of a causal exponential signal $x[n]=a^n u[n]$

$$X(z)=\sum_{n=-\infty}^{\infty}a^n u[n]z^{-n}=\sum_{n=0}^{\infty}a^n z^{-n}=\sum_{n=0}^{\infty}(az^{-1})^n=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$$

converge if: $|az^{-1}| < 1 \Rightarrow |z| > |a|$

The inverse z-transform

- The recovery of a sequence $x[n]$ from its z-transform ($X(z)$ and ROC) can be formally done using the formula (inverse z-transform):

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_{\Gamma} X(z) z^{n-1} dz$$

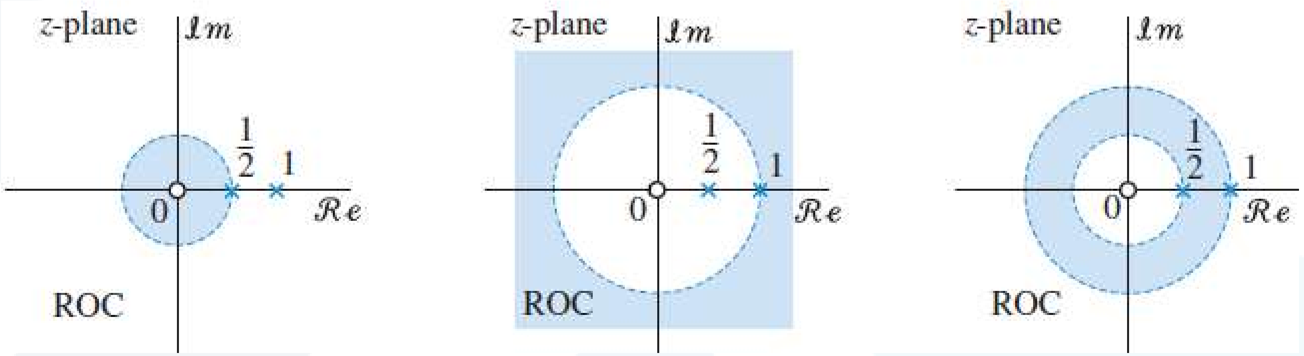
where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.

- We do not usually compute the inverse z-transform using the above equation.
- For rational functions, the inverse z-transform can be more easily computed using partial fraction expansions (PFE).

- Example 3: Finding the inverse z-transform using partial fractions

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

$$X(z) = \frac{4}{1 - z^{-1}} - \frac{3}{1 - 0.5z^{-1}}$$



If ROC: $|z| > 1$, both fractions are the z-transform of causal sequences. Hence

$$x[n] = 4u[n] - 3\left(\tfrac{1}{2}\right)^n u[n]$$ causal

If ROC: $|z| < ½$, $x[n] = -4u[-n-1] + 3\left(\tfrac{1}{2}\right)^n u[-n-1]$ anticausal

If ROC: $½ < |z| < 1$, $x[n] = -4u[-n-1] - 3\left(\tfrac{1}{2}\right)^n u[n]$ noncausal

Properties of z-Transform

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Time shifting	$x[n-k]$	$X(z)z^{-k}$	$R \pm \{0 \text{ or } \infty\}$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1}
Multiply by exp.	$a^n x[n]$	$X(z/a)$	$\|a\|R$
Differentiate in z	$nx[n]$	$-z\, dX(z)/dz$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)\, X_2(z)$	$\supset (R_1 \cap R_2)$
Summation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{z}{z-1} X(z)$	$\supset (R \cap (z > 1))$

- Example 4: z-Transform of a cosine signal

$$x[n] = \cos(\omega_0 n)u[n]$$

$$\cos(\omega_0 n)u[n] = \tfrac{1}{2}e^{j\omega_0 n}u[n] + \tfrac{1}{2}e^{-j\omega_0 n}u[n]$$

$$\mathcal{Z}\{\cos(\omega_0 n)u[n]\} = \tfrac{1}{2}\mathcal{Z}\{e^{j\omega_0 n}u[n]\} + \tfrac{1}{2}\mathcal{Z}\{e^{-j\omega_0 n}u[n]\}$$

$$= \frac{1/2}{1 - e^{j\omega_0}z^{-1}} + \frac{1/2}{1 - e^{-j\omega_0}z^{-1}} = \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

ROC is $|z| > 1$

- Example 5: z-Transform of a sine signal

$$x[n] = \sin(\omega_0 n)u[n]$$

$$\sin(\omega_0 n)u[n] = \tfrac{1}{2j}e^{j\omega_0 n}u[n] - \tfrac{1}{2j}e^{-j\omega_0 n}u[n]$$

$$\mathcal{Z}\{\sin(\omega_0 n)u[n]\} = \frac{1}{2j}\mathcal{Z}\{e^{j\omega_0 n}u[n]\} - \frac{1}{2j}\mathcal{Z}\{e^{-j\omega_0 n}u[n]\}$$

$$= \frac{1/2j}{1-e^{j\omega_0}z^{-1}} - \frac{1/2j}{1-e^{-j\omega_0}z^{-1}} = \frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

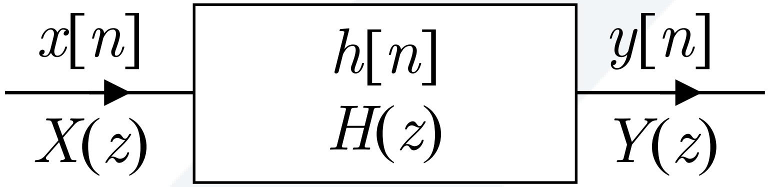
ROC is $|z| > 1$

- Initial and final value properties of the z-transform applies to causal signals only.

Initial value: $x[0] = \lim_{z\to\infty} X(z)$

Final value: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

3. Transfer function of LTI systems



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$Y(z) = X(z)H(z)$$

- Example 6: Determine the response of a system with impulse response $h[n] = a^n u[n]$, $|a| < 1$ to the input $x[n] = u[n]$ using the convolution theorem.

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, |z| > |a| \quad \text{and} \quad X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, |z| > 1$$

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}, \quad |z| > \max\{|a|, 1\} = 1$$

$$Y(z) = \frac{1}{1 - a}\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - az^{-1}}\right), \quad |z| > 1$$

$$y[n] = \frac{1}{1 - a}(u[n] - a^{n+1}u[n]) = \frac{1 - a^{n+1}}{1 - a}u[n]$$

which is exactly the steady-state response

Causality and stability

- A TF $H(z)$ with the ROC that is the exterior of a circle, including ∞, is a necessary and sufficient condition for DTLTI system to be causal.
- An LTI system with transfer function $H(z) = N(z)/D(z)$ is causal if and only if: 1. the ROC is $|z| > |p|$, where p is the outermost pole and 2. deg $N \leq$ deg D.
- An LTI system is stable if and only if the ROC of $H(z)$ includes the unit circle $|z| = 1$.
- A causal LTI system with rational transfer function $H(z)$ is stable if and only if all poles of $H(z)$ are inside the unit circle.

▪ The conditions for causality and stability are different and that one does not imply the other.

- For example, a causal system may be stable or unstable, just as a noncausal system may be stable or unstable.
- Similarly, an unstable system may be either causal or noncausal, just as a stable system may be causal or noncausal.
- Example 7: A linear time-invariant system is characterized by the transfer function:

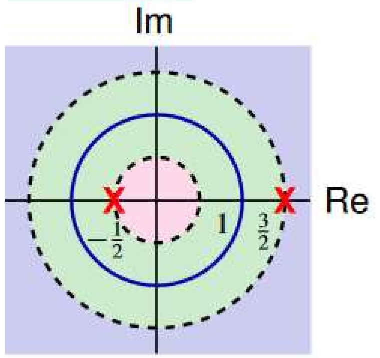
$$H(z) = \frac{1}{2}\left[\frac{z}{z-\frac{3}{2}} - \frac{z}{z+\frac{1}{2}}\right]$$

If ROC: $|z| > 3/2$, the system is causal and unstable

$$h[n] = \tfrac{1}{2}\left(\tfrac{3}{2}\right)^n u[n] - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n u[n]$$

If ROC: $1/2 < |z| < 3/2$, the system is noncausal and stable

$$h[n] = -\tfrac{1}{2}\left(\tfrac{3}{2}\right)^n u[-n-1] - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n u[n]$$

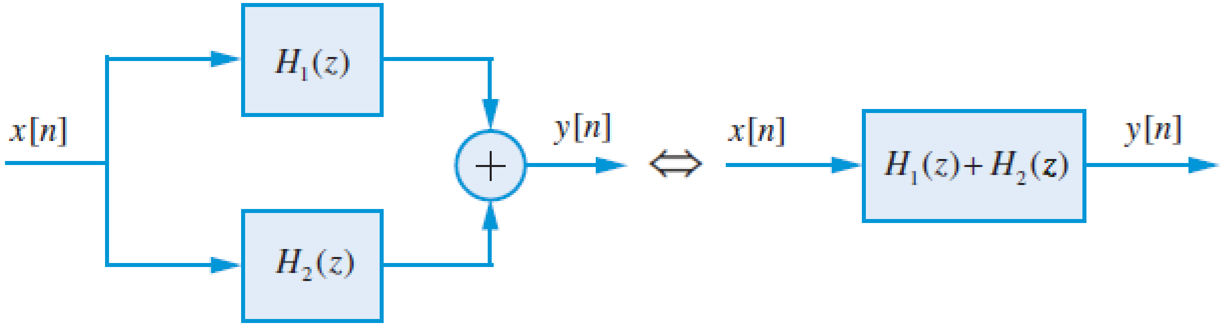


If ROC: $|z| < 1/2$, the system is anticausal and unstable

$$h[n] = -\frac{1}{2}\left(\frac{3}{2}\right)^n u[-n-1] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[-n-1]$$

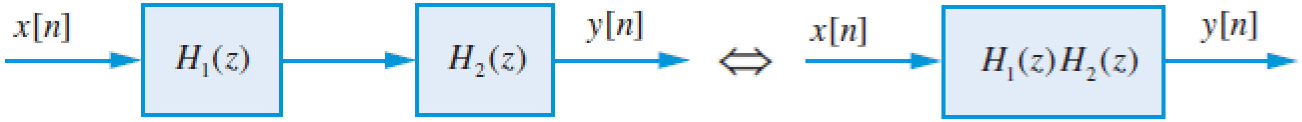
Interconnection of two LTI systems

parallel interconnection



$$h[n] = h_1[n] + h_2[n]$$
$$H(z) = H_1(z) + H_2(z)$$

cascade interconnection



$$h[n] = h_1[n] * h_2[n]$$
$$H(z) = H_1(z)H_2(z)$$

4. Linear constant-coefficient difference equations

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)} = \frac{b_0 z^{-M}\left(z^M + \cdots + \frac{b_M}{b_0}\right)}{z^{-N}\left(z^N + \cdots + a_N\right)}$$

$$H(z) = b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(p - p_k)} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

where z_l's are the system zeros and p_k's are the system poles, and b_0 is a constant gain term.

Impulse response

- The transfer function $H(z)$ with distinct poles can be expressed in the form:

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

where $A_k = (1 - p_k z^{-1})\, X(z)\big|_{z=p_k}$

and $C_k = 0$ when $M < N$, that is, when the rational function $H(z)$ is proper.

If we assume that the system is causal, then the ROC is the exterior of a circle starting at the outermost pole, and the impulse response is:

$$h[n] = \sum_{k=0}^{M-N} C_k \delta[n-k] + \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

5. Connections between pole-zero locations and time-domain behavior

The TF $H(z)$ with distinct poles: $H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$

where the first summation is included only if $M \geq N$

- The roots of a polynomial with real coefficients either must be real or must occur in complex conjugate pairs.

$$H(z) = \sum_{k=0}^{M-N} C_k z^{-k} + \sum_{k=1}^{K_1} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

First-order systems

$$H(z) = \frac{b}{1 - az^{-1}}, \quad a, b \text{ real}$$

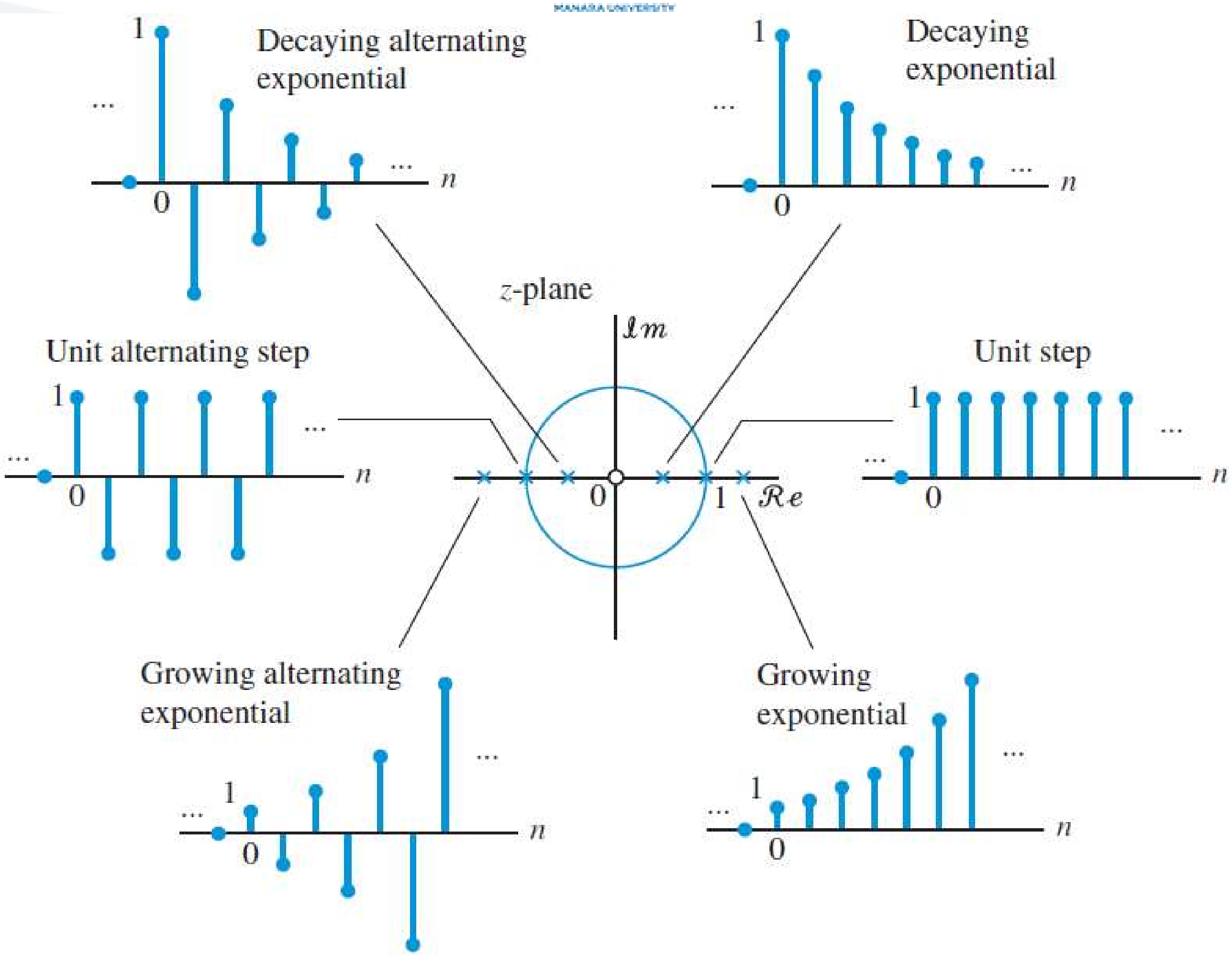
Assuming a causal system, the impulse response is given by the following real exponential sequence: $h[n] = ba^n u[n]$

Second-order systems

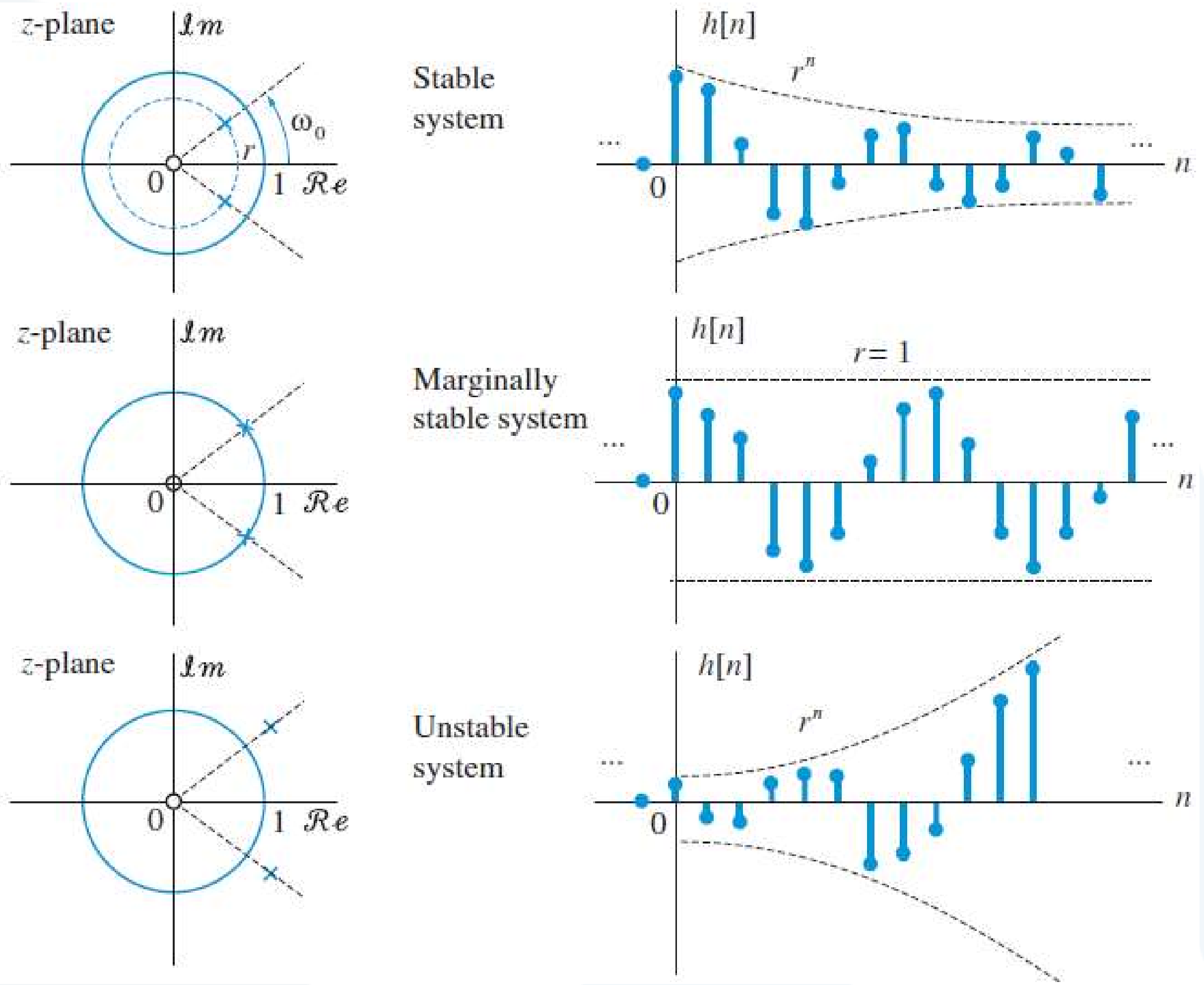
$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z(b_0 z + b_1)}{z^2 + a_1 z + a_2}$$

There are three possible cases for poles: 1. Real and distinct, 2. Real and equal, 3. Complex conjugate.

The impulse response of a causal system with a pair of complex conjugate poles: $h[n] = 2|A| r^n \cos(\omega_0 n + \phi) u[n]$, where A is the PFE coefficient



Impulse responses associated with real poles in the z-plane



Impulse responses associated with a pair of complex conjugate poles in the z-plane

- **Example 8**: Given that causal system: $H(z) = \dfrac{z+1}{z^2 - 0.9z + 0.81}$, find

  a. its difference equation representation, and

  b. its impulse response representation.

a. $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{z+1}{z^2 - 0.9z + 0.81} = \dfrac{z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$

$$Y(z) - 0.9z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) + z^{-2}X(z)$$

$$y[n] - 0.9y[n-1] + 0.81y[n-2] = x[n-1] + x[n-2]$$

$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n-1] + x[n-2]$$

b. $H(z) = 1.2346 + \dfrac{-0.6173 + j0.9979}{1 - 0.9e^{-j\pi/3}z^{-1}} + \dfrac{-0.6173 - j0.9979}{1 - 0.9e^{j\pi/3}z^{-1}}, \quad |z| > 0.9$

From z-transform table:

$$h[n] = 1.2346\delta[n] + [(-0.6173 + j0.9979)0.9^n e^{-j\pi n/3} + (-0.6173 - j0.9979)0.9^n e^{j\pi n/3}]u[n]$$

$$h[n] = 1.2346\delta[n] + 0.9^n[-1.2346\cos(\pi n/3) + 1.9958\sin(\pi n/3)]u[n]$$

$$h[0] = 0 \Rightarrow h[n] = 0.9^n[-1.2346\cos(\pi n/3) + 1.9958\sin(\pi n/3)]u[n-1]$$

6. The one-sided z-transform

$$X^+(z) = \mathcal{Z}^+\{x[n]\} = \mathcal{Z}\{x[n]u[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

one-sided or unilateral z-transform

- Almost all properties we have studied for the two-sided z-transform carry over to the one-sided z-transform with the exception of the time shifting property.

$$\mathcal{Z}^+\{x[n-1]\} = x[-1] + x[0]z^{-1} + x[1]z^{-2} + \cdots = x[-1] + z^{-1}(x[0] + x[1]z^{-1} + \cdots)$$

$$= x[-1] + z^{-1}X^+(z)$$

$$\mathcal{Z}^{+}\{x[n-2]\} = x[-2] + x[-1]z^{-1} + z^{-2}X^{+}(z)$$

In general, for any $k > 0$, we can show that

$$\mathcal{Z}^{+}\{x[n-k]\} = z^{-k}X^{+}(z) + \sum_{m=1}^{k} x[-m]z^{m-k}$$

- This property makes possible the solution of linear constant-coefficient difference equations with nonzero initial conditions.

- Example 9: A linear time-invariant system

$$y[n] = ay[n-1] + x[n], \quad n \geq 0 \quad \text{with } y[-1] \neq 0$$

$$Y^{+}(z) = ay[-1] + az^{-1}Y^{+}(z) + X^{+}(z) \Rightarrow Y^{+}(z) = \underbrace{\frac{ay[-1]}{1-az^{-1}}}_{initial\ condition} + \underbrace{\frac{1}{1-az^{-1}}X^{+}(z)}_{zero-state}$$

If the input $x[n] = 0$ for all $n \geq 0$, then: $y_{zi}[n] = ay[-1]a^{n} = y[-1]a^{n+1}$, $n \geq 0$

If the initial condition is zero then the system is at rest or at zero-state:

$$Y^+(z) = H(z)X^+(z), \quad H(z) = \frac{1}{1-az^{-1}} \text{ or } h[n] = a^n u[n]$$

and hence the second term can be identified as the zero-state response $y_{zs}[n]$.

The complete response is given by:

$$y[n] = y[-1]a^{n+1} + \sum_{k=0}^{n} h[k]x[n-k], \quad n \geq 0$$

If we set $x[n] = u[n]$

$$Y^+(z) = \frac{ay[-1]}{1-az^{-1}} + \frac{1}{1-az^{-1}}\frac{1}{1-z^{-1}} = \frac{ay[-1]}{1-az^{-1}} + \frac{1/(1-a)}{1-z^{-1}} - \frac{a/(1-a)}{1-az^{-1}}$$

$$y[n] = y[-1]a^{n+1} + \frac{1}{1-a}(1-a^{n+1}), \quad n \geq 0$$