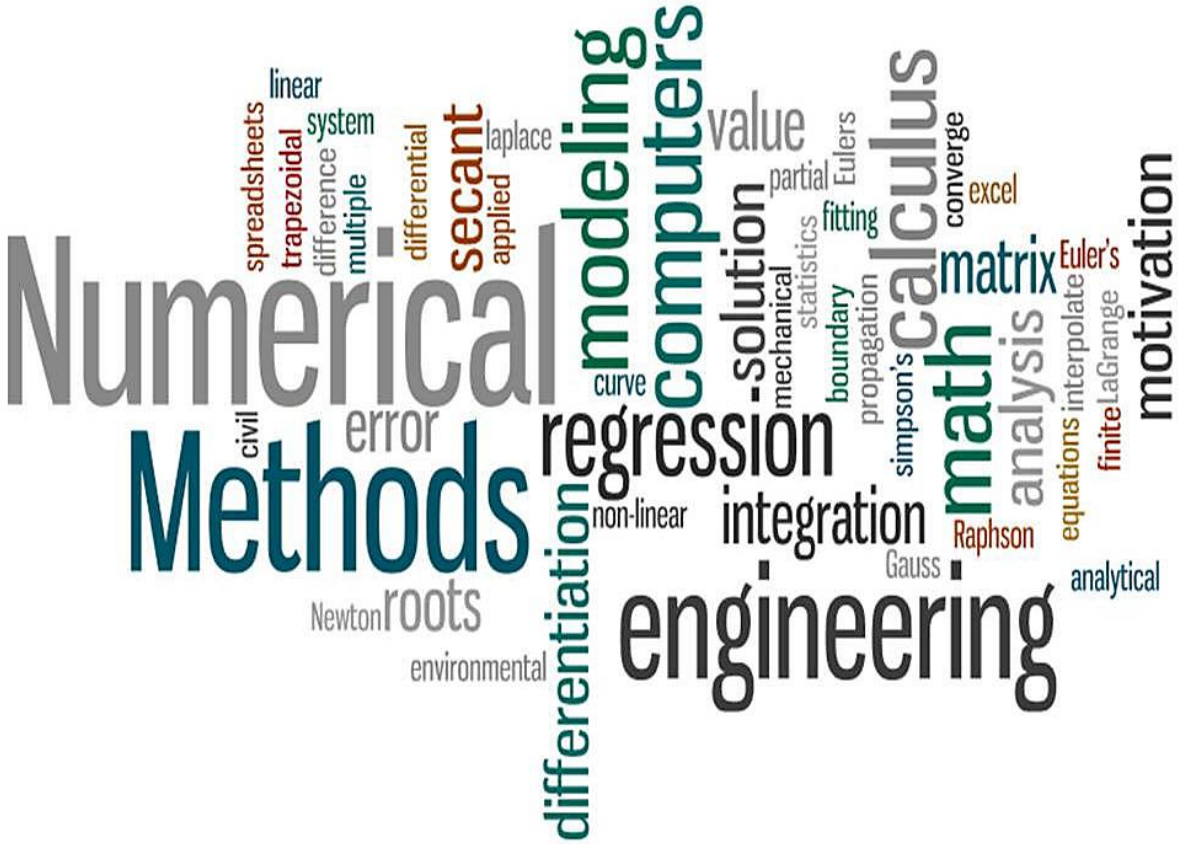
Programming and Numerical Analysis



Approximation and Round-Off Errors

التحليل العددي هو أحد فروع الرياضيات الهامة وهو الذي يربط بين الرياضيات التحليلية والحاسب الآلي ويستخدم عادة في إيجاد حلول بعض المسائل والمشاكل التي لا يمكن حلها بالرياضيات التحليلية حيث تكون النتيجة التي نحصل عليها نتيجة تقريبية. بما أننا نحصل على نتيجة تقريبية أو حل تقريبي هذا يعني أنه يوجد خطأ وعلينا حساب الخطأ إلا أنه لو استطعنا إيجاد الخطأ لاستطعنا إيجاد الحل الفعلي (الحقيقي) الأمر الذي يعني أن إيجاد الخطأ غير ممكن ونسعى بالتالي إلى إيجاد تقريب للخطأ أو حجم الخطأ أي تلك القيمة التي لا يتجاوزها الخطأ. وتتلخص مهمة التحليل العددي في إيجاد الحل التقريبي لمسألة ما وتقويم الخطأ.

إن معظم الأعداد التي نتعامل معها هي أعداد تقريبية، لأنها غالباً ما تمثل أطوال . وقياسات أو قيم لمقادير فيزيائية بنتيجة القياس وهي بحد ذاتها تقريبية. كذلك فإن الكثير من الأعداد الحقيقية لا يمكن التعبير عنها بعدد منته من الأرقام فمثلاً العدد π يساوي تقريباً 3.14159.

Introduction

- For many engineering problems, we cannot obtain analytical solutions.
- Numerical methods yield approximate results, results that are close to the exact analytical solution. We cannot exactly compute the errors associated with numerical methods.
  - ❖Only rarely given data are exact, since they originate from measurements. Therefore there is probably error in the input information.
  - ❖Algorithm itself usually introduces errors as well, e.g., unavoidable round-offs, etc.
  - ❖The output information will then contain error from both of these sources.
- How confident we are in our approximate result?
- The question is "*how much error is present in our calculation and is it tolerable?*"

Error Definition

- Since numerical solutions are approximated results, we have to specify how different the approximated results are from the true values, i.e. how large the error is.

True Error: The difference between the true solution value and the approximated (numerical) solution value,

True Value = Approximation + True Error (Et)

Et = True value – Approximation (+/-)

True Error

True fractional relative error :obtained by dividing the absolute error in the quantity by the quantity itself.

$$\text{True fractional relative error} = \left|\frac{\text{true error}}{\text{true value}}\right|$$

True percent relative error :

$$\text{True percent relative error, } \varepsilon_t = \left|\frac{\text{true error}}{\text{true value}}\right| \times 100\%$$

Error Definition

Example

Problem Statement: Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (a) the true error and (b) the true percent relative error for each case.

Solution:

(a) The error for measuring the bridge is (True Value = Approximation + Error) : Et = 10,000 – 9999 = 1 cm

and for the rivet it is: Et = 10 – 9 = 1 cm

(b) The percent relative error for the bridge is $\varepsilon_t = \frac{\text{true error}}{\text{true value}} \times 100\%$: $\varepsilon_t = \frac{1}{10{,}000} \times 100\% = 0.01\%$

and for the rivet it is $\varepsilon_t = \frac{1}{10} \times 100\% = 10\%$

Thus, although both measurements have an error of 1 cm, the relative error for the rivet is much greater. We would conclude that we have done an adequate job of measuring the bridge, whereas our estimate for the rivet leaves something to be desired

Error Definition con.

Estimated Relative Error:

- For numerical methods, the true value will be known only when we deal with functions that can be solved analytically (simple systems). In real world applications, we usually not know the answer a priori. Then:

$$\varepsilon_a = \frac{\text{approximate error}}{\text{approximation}} \times 100\%$$

$$\underset{(+\,/\,-)}{\varepsilon_a} = \frac{\text{current approximation} - \text{previous approximation}}{\text{current approximation}} \times 100\%$$

Error Definition con.

- Computations are repeated until stopping criterion is satisfied.

$$|\varepsilon_a| = \varepsilon_s$$

Pre-specified % tolerance based on the knowledge of your solution

- If the following criterion is met

$$\varepsilon_s = \left(0.5 \times 10^{(2-n)}\right) \times 100\%$$

- you can be sure that the result is correct to at least ***n significant*** figures.

Determine the true relative error and estimated relative error from approximating of $e^{0.5}$ by using the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ up to 6th term. And

1st term estimate:

2nd term estimate:

True relative error:

Estimated relative error:

Repeat for approximation to 3rd, 4th…term, we can get

Terms	Results	ε_t	ε_a
1	1	39.3	
2	1.5	9.02	33.3
3	1.625	1.44	7.69
4	1.645833333	0.175	1.27
5	1.648437500	0.0172	0.158
6	1.648697917	0.00142	0.0158

Truncation Errors and the Taylor Series

Source of Error

- **Round-off error:**
  - **Caused by the limited number of digits that represent numbers in a computer and**
  - **The ways numbers are stored and additions and substructions are performed in a computer**

- **Truncation Error:**
  - **Caused by approximation used in the mathematical formula of the scheme**

Background of Truncation Error

- Numerical solutions are mostly approximations for exact solution
- Most numerical methods are based on approximating function by polynomials
- How accurately the polynomial is approximating the true function ?
- Comparing the polynomial to the exact solution it becomes possible to evaluate the error, called truncation error

Taylor Series

- The most important polynomials used to derive numerical schemes and analyze truncation errors
- With an infinite power series, it is exactly represents a function within a certain radius about a given point

Taylor Series - Taylor's Theorem

- If the function f and its first n + 1 derivatives are continuous on an interval containing a and x, then the value of the function at x is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

- With the remainder Rn is defined as:

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

Taylor Series

- Taylor series is of great value in the study of numerical methods. In essence, the Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point. In particular, the theorem states that any smooth function can be approximated as a polynomial.
- A useful way to gain insight into the Taylor series is to build it term by term. For example, the first term in the series is:

$$f(x_{i+1}) \cong f(x_i)$$

Taylor Series

$$f(x_{i+1}) \cong f(x_i)$$

Zero-order approximation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

First-order approximation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

Second-order approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x_{i+1} - x_i)^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

Taylor Series

n^{th} order approximation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \ldots$$

$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

$(x_{i+1}-x_i)= h$ *step size* (define first)

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)}$$

- Reminder term, R_n, accounts for all terms from (n+1) to infinity.

Taylor Series Approximation of a Polynomial

Example

Problem Statement: Use zero- through fourth-order Taylor series expansions to approximate the function:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

From $x_i = 0$ with h=1, that is, predict the function's value at $x_{i+1} = 1$

Solution:

Because we are dealing with a known function, we can compute values for *f(x)* between 0 and 1.

Taylor Series Approximation of a Polynomial