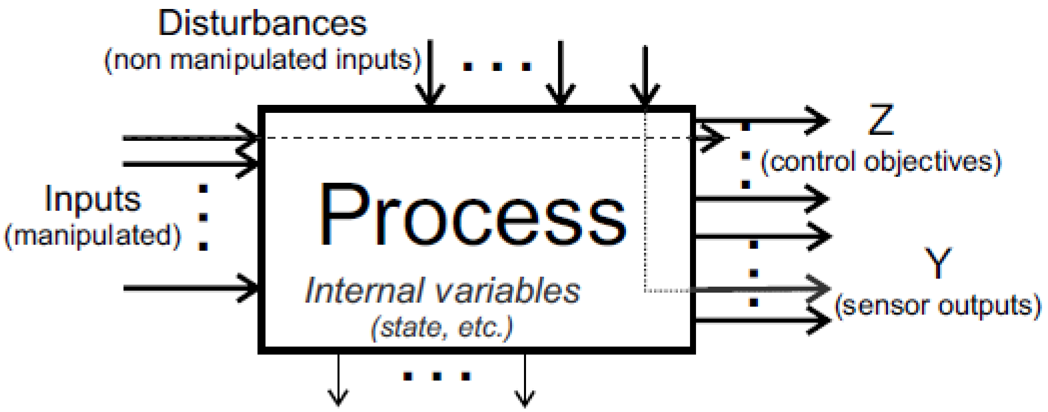
كلية الهندسة
جامعة المنارة
MANARA UNIVERSITY

نظم متعددة المتغيرات
Multivariable Systems



الروبوت والأنظمة الذكية

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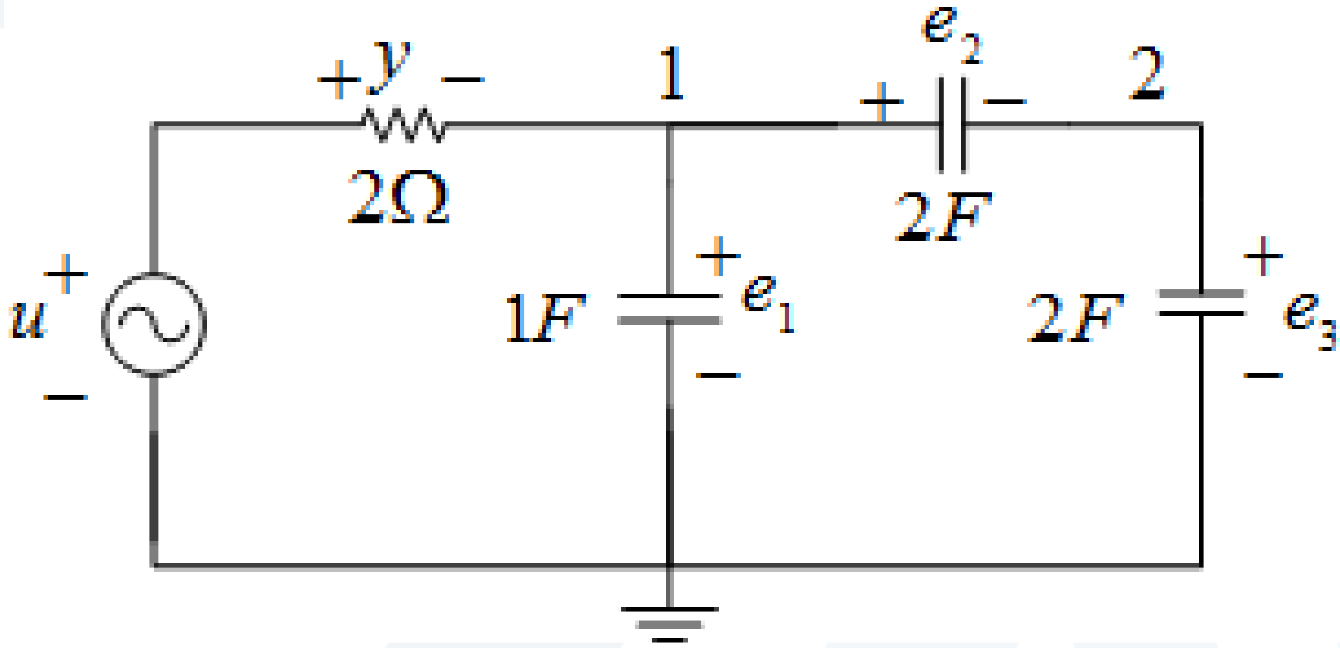
MODELING IN STATE SPACE



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$$\frac{de_1(t)}{dt} + 2\frac{de_2(t)}{dt} = \frac{u(t) - e_1(t)}{2}$$

$$2\frac{de_3(t)}{dt} = 2\frac{de_2(t)}{dt}$$

$$e_1(t) - e_2(t) = e_3(t)$$

$$\frac{de_1(t)}{dt} = -\frac{1}{4}e_1(t) + \frac{1}{4}u(t)$$

$$\frac{de_2(t)}{dt} = -\frac{1}{8}e_1(t) + \frac{1}{8}u(t)$$

$$y(t) = -e_1(t) + u(t)$$

Modern Control Theory

The modern trend in engineering systems is toward greater complexity, due mainly to the requirements of complex tasks and good accuracy.

Complex systems may have multiple inputs and multiple outputs and may be time varying.

Because of the necessity of meeting increasingly stringent requirements on the performance of control systems, the increase in system complexity, and easy access to large scale computers, modern control theory, which is a new approach to the analysis and design of complex control systems, has been developed since around 1960.

This new approach is based on the concept of state. The concept of state by itself is not new, since it has been in existence for a long time in the field of classical dynamics and other fields.

Modern Control Theory Versus Conventional Control Theory

Modern control theory is contrasted with conventional control theory in that the former is applicable to multiple-input, multiple-output systems, which may be linear or nonlinear, time invariant or time varying, while the latter is applicable only to linear time-invariant single-input, single-output systems.

Also, modern control theory is essentially time-domain approach and frequency domain approach, while conventional control theory is a complex frequency-domain approach.

Before we proceed further, we must define state, state variables, state vector, and state space.

State

The state of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

Note that the concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State Variables

The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least ***n*** variables x_1, x_2, ….. , x_n are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such ***n*** variables are a set of state variables.

Note that state variables need not be physically measurable or observable quantities.

Variables that do not represent physical quantities and those that are neither measurable nor observable can be chosen as state variables. Such freedom in choosing state variables is an advantage of the state-space methods. Practically, however, it is convenient to choose easily measurable quantities for the state variables, if this is possible at all, because optimal control laws will require the feedback of all state variables with suitable weighting.

State Vector

If ***n*** state variables are needed to completely describe the behavior of a given system, then these ***n*** state variables can be considered the ***n*** components of a vector **x**. Such a vector is called a *state vector*.

A state vector is thus a vector that determines uniquely the system state ***x(t)*** for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input ***u(t)*** for $t \geq t_0$ is specified.

State Space

The *n*-dimensional space whose coordinate axes consist of the $x_1$axis, x_2 axis, …. , x_n axis, where x_1, x_2, ….. , x_n are state variables, is called a *state space*. Any state can be represented by a point in the state space.

State-Space Equations

In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables. The state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

The dynamic system must involve elements that memorize the values of the input for $t \geq t_1$. Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system. Thus the outputs of integrators serve as state variables.

State-Space Equations

The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.

Assume that a multiple-input, multiple-output system involves ***n*** integrators. Assume also that there are ***r*** inputs $u_1(t)$, $u_2(t)$, … , $u_r(t)$, and ***m*** outputs $y_1(t)$, $y_2(t)$, … , $y_m(t)$.

Define n outputs of the integrators as state variables: x_1, x_2, ….. , x_n
Then the system may be described by

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}
$$

State-Space Equations

The outputs $y_1(t), y_2(t), \ldots, y_m(t)$ of the system may be given by

$$
\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\cdot \\
&\cdot \\
&\cdot \\
y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}
$$

State-Space Equations

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix},$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

State-Space Equations

Then

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$ is the state equation

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$$ is the output equation

If vector functions ***f*** and/or ***g*** involve time ***t*** explicitly, then the system is called a time-varying system.

If Equations are linearized about the operating state, then we have the following linearized state equation and output equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

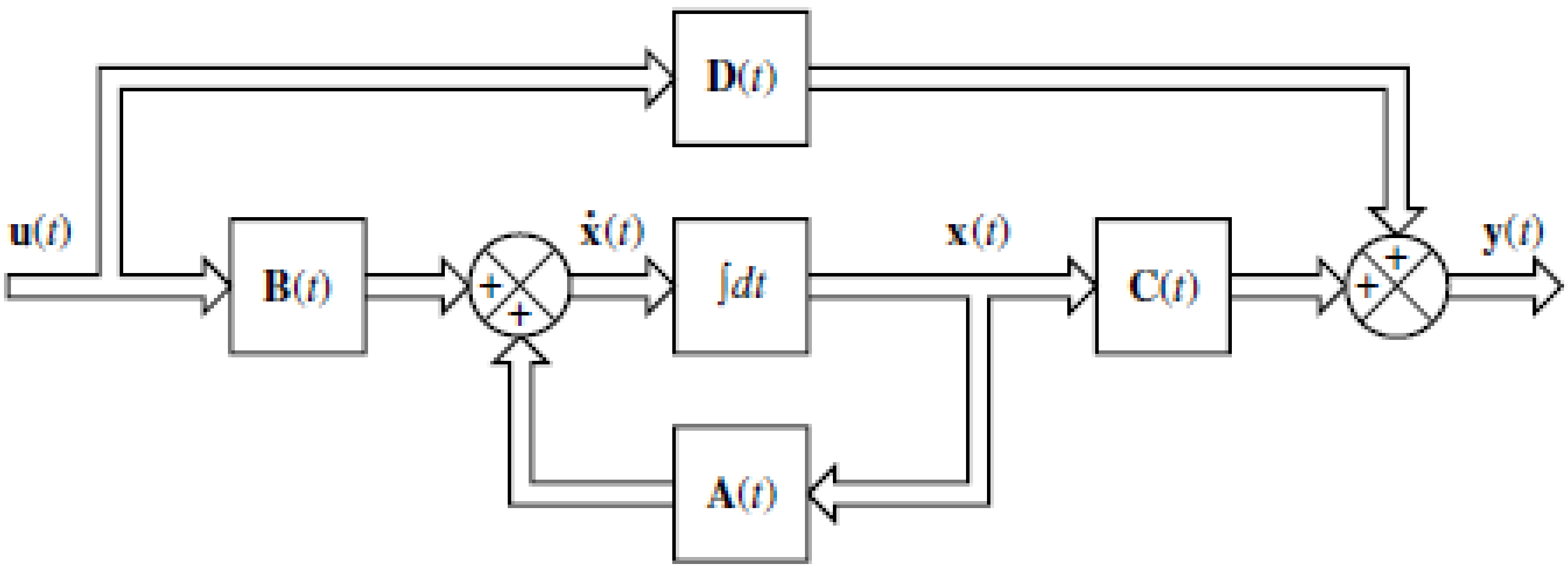
where **A(*t*)** is called the state matrix, **B(*t*)** the input matrix, **C(*t*)** the output matrix, and **D(*t*)** the direct transmission matrix.

State-Space Equations

If vector functions ***f*** and ***g*** do not involve time ***t*** explicitly then the system is called a time-invariant system. In this case, Equations can be simplified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



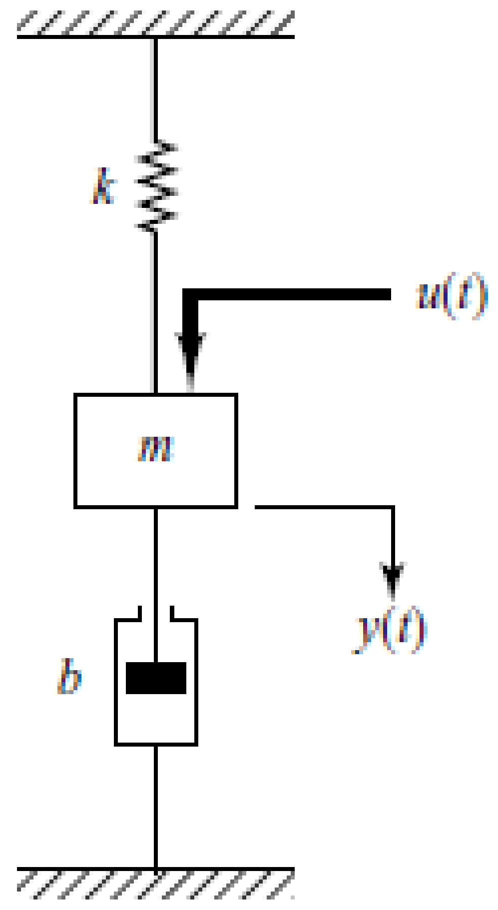
State-Space Equations

In what follows we shall present an example for deriving a state equation and output equation.

EXAMPLE:

Consider the mechanical system shown in Figure.We assume that the system is linear. The external force **u(t)** is the input to the system, and the displacement **y(t)** of the mass is the output.

The displacement **y(t)** is measured from the equilibrium position in the absence of the external force. This system is a single-input, single-output system.



EXAMPLE

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u$$

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The output equation is $y = x_1$

EXAMPLE

In a vector-matrix form, Equations can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

The output equation can be written as

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

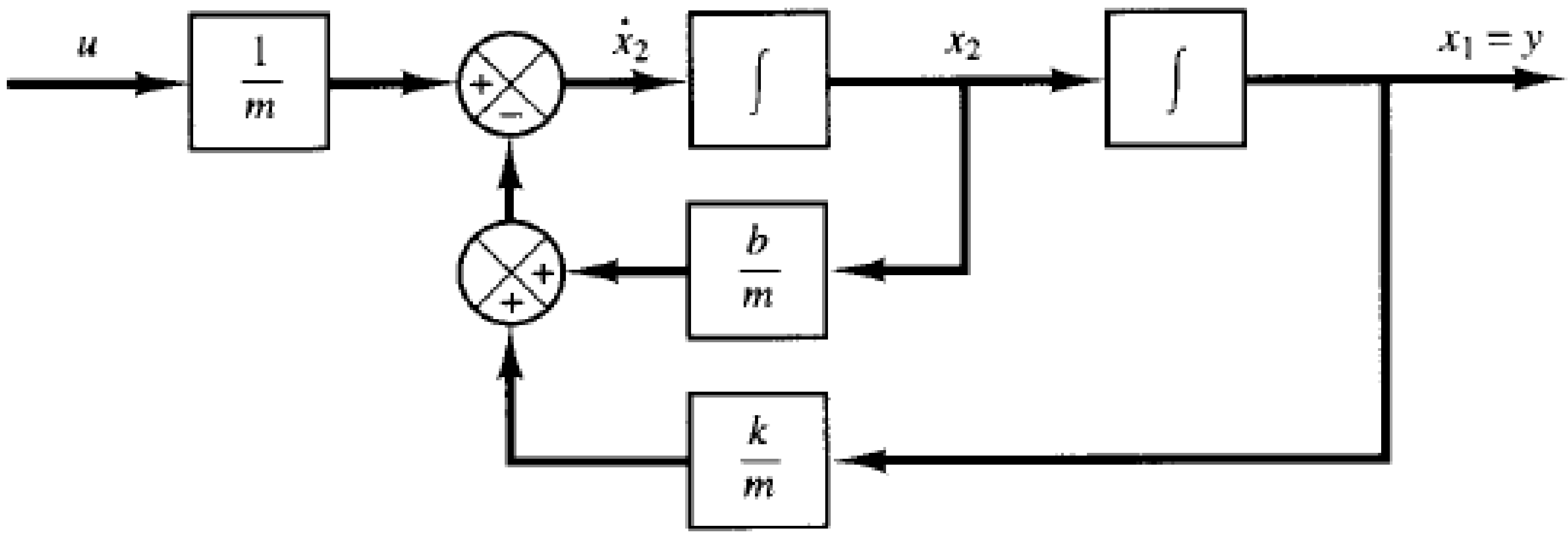
They are in the standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

Where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

Figure is a block diagram for the system. Notice that the outputs of the integrators are state variables.



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Correlation Between Transfer Functions and State-Space Equations

In what follows we shall show how to derive the transfer function of a single-input, single-output system from the state-space equations.

- Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s)$$

This system may be represented in state space by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

Correlation Between Transfer Functions and State-Space Equations

where ***x*** is the state vector, ***u*** is the input, and ***y*** is the output. The Laplace transforms of Equations are given by

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s)$$

Initial conditions were zero

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s)$$

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \qquad G(s) = \frac{Q(s)}{|s\mathbf{I} - \mathbf{A}|}$$

In other words, the eigenvalues of **A** are identical to the poles of **G(s)**.

EXAMPLE

We shall obtain the transfer function for the system (last example) from the state-space equations.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = [1 \quad 0]\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{ms^2 + bs + k}$$

Transfer Matrix.

Next, consider a multiple-input, multiple-output system. Assume that there are ***r*** inputs and ***m*** outputs Define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix}$$

The transfer matrix **G**(s) relates the output **Y**(s) to the input **U**(s), or $\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$

where **G**(s) is given by $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

Since the input vector **u** is **r** dimensional and the output vector **y** is **m** dimensional, the transfer matrix **G**(s) is an **m**x**r** matrix.

STATE-SPACE REPRESENTATION OF SCALAR DIFFERENTIAL EQUATION SYSTEMS

A dynamic system consisting of a finite number of lumped elements may be described by ordinary differential equations in which time is the independent variable. By use of vector-matrix notation, an **n**th-order differential equation may be expressed by a first order vector-matrix differential equation. If **n** elements of the vector are a set of state variables, then the vector-matrix differential equation is a *state* equation. we shall present methods for obtaining state-space representations of continuous-time systems.

State-Space Representation of *n*th-Order Systems of Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms.

Consider the following nth-order system:

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

Noting that the knowledge of $y(0), \dot{y}(0) \ldots y^{(n-1)}(0)$ together with the input **u(t)** for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t), \dot{y}(t) \ldots y^{(n-1)}(t)$ as a set of **n** state variables.

Let us define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y}$$

Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u$$

Or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

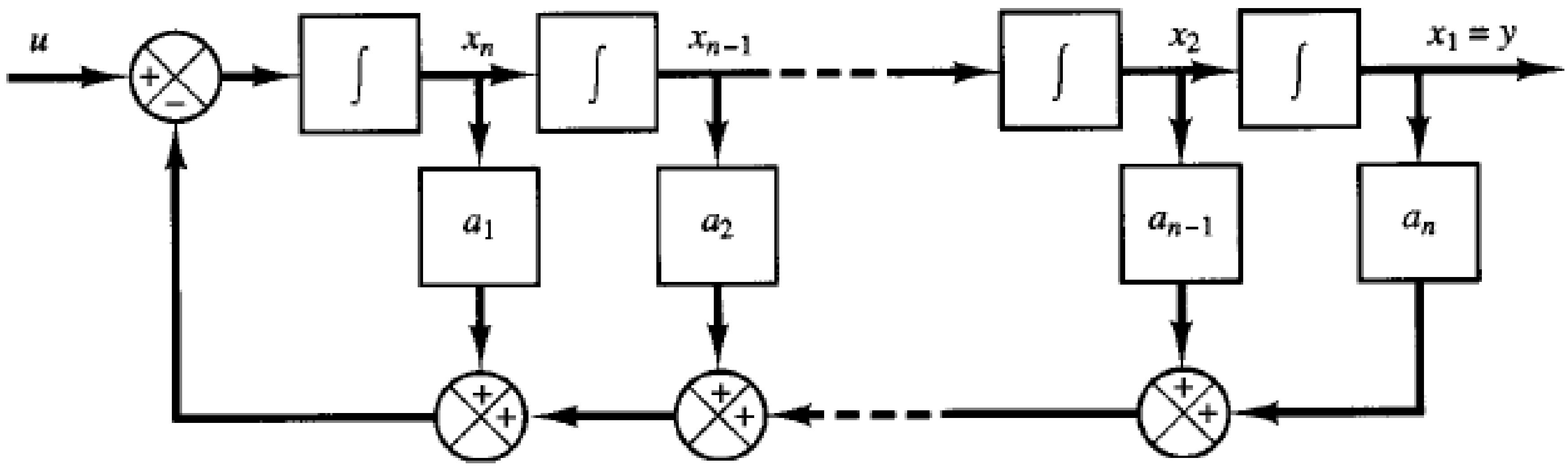
$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$



u
+
−
x_n
x_{n-1}
x_2
$x_1 = y$
a_1
a_2
a_{n-1}
a_n

Note that the state-space representation for the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by above Equations.

State-Space Representation of *n*th-Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms

Consider the differential equation system that involves derivatives of the forcing function, such as

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

The main problem in defining the state variables for this case lies in the derivative terms of the input **u**. The state variables must be such that they will eliminate the derivatives of **u** in the state equation.

One way to obtain a state equation and output equation for this case is to define the following **n** variables as a set of n state variables:

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are determined from

$$\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\ \ \vdots \\
\beta_{n-1} &= b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0
\end{aligned}$$

With this choice of state variables the existence and uniqueness of the solution of the state equation is guaranteed. With the present choice of state variables, we obtain:

$$\dot{x}_1 = x_2 + \beta_1 u$$
$$\dot{x}_2 = x_3 + \beta_2 u$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + \beta_{n-1} u$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u$$

where β_n is given by

$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_{n-1} \beta_0$$

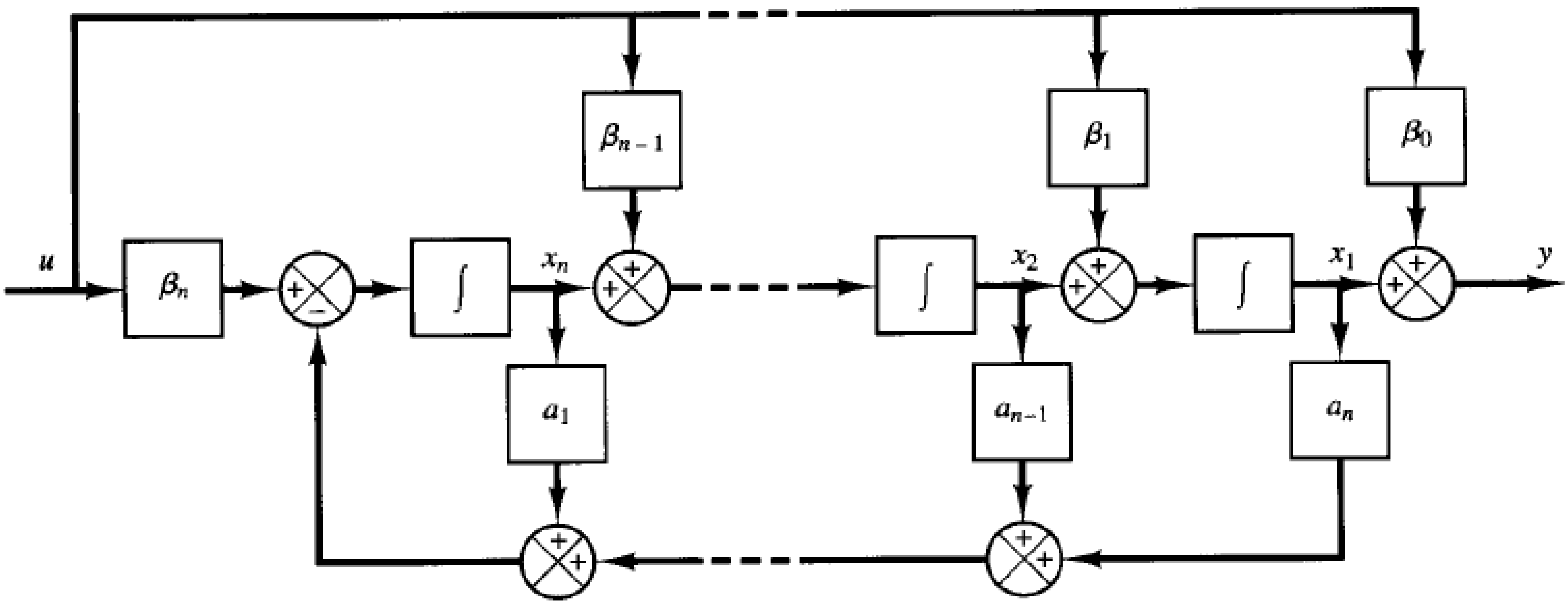
In terms of vector-matrix equations, and the output equation can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

Note that the state-space representation for the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$



Control Systems Analysis in State Space

INTRODUCTION

While conventional control theory is based on the input–output relationship, or transfer function, modern control theory is based on the description of system equations in terms of n first-order differential equations, which may be combined into a first-order vector-matrix differential equation. The use of vector-matrix notation greatly simplifies the mathematical representation of systems of equations.

INTRODUCTION

The increase in the number of state variables, the number of inputs, or the number of outputs does not increase the complexity of the equations. In fact, the analysis of complicated multiple-input, multiple output systems can be carried out by procedures that are only slightly more complicated than those required for the analysis of systems of first-order scalar differential equations.

STATE-SPACE REPRESENTATIONS OF TRANSFER-FUNCTION SYSTEMS

Many techniques are available for obtaining state-pace representations of transfer function systems. This section presents state-space representations in the controllable, observable, diagonal, or Jordan canonical forms.

State-Space Representations in Canonical Forms

Consider a system defined by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

where **u** is the input and **y** is the output. This equation can also be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Controllable Canonical Form

The following state-space representation is called a controllable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Controllable Canonical Form

The controllable canonical form is important in discussing the pole-placement approach to control systems design.

Observable Canonical Form

The following state-space representation is called an observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

Observable Canonical Form

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Diagonal Canonical Form

Consider the transfer-function system defined by

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Here we consider the case where the denominator polynomial involves only distinct roots. For the distinct-roots case, the transfer-function can be written as

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

Diagonal Canonical Form

The diagonal canonical form of the state-space representation of this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form

Next we shall consider the case where the denominator polynomial of the transfer-function involves multiple roots. For this case, the preceding diagonal canonical form must be modified into the Jordan canonical form. Suppose, for example, that the p_i's are different from one another, except that the first three p_i's are equal, or $p_1 = p_2 = p_3$. Then the factored form of **Y(s)/U(s)** becomes

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s + p_1)^3 (s + p_4)(s + p_5) \cdots (s + p_n)}$$

Jordan Canonical Form

The partial-fraction expansion of this last equation becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{c_1}{(s + p_1)^3} + \frac{c_2}{(s + p_1)^2} + \frac{c_3}{s + p_1} + \frac{c_4}{s + p_4} + \cdots + \frac{c_n}{s + p_n}$$

A state-space representation of this system in the Jordan canonical form is given by

Jordan Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p_4 & & 0 \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & \\ 0 & \cdots & 0 & 0 & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Example

Consider the system given by

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Obtain state-space representations in the controllable canonical form, observable canonical form, and diagonal canonical form.

Controllable canonical form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Observable canonical form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonal canonical form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Eigenvalues of an $n \times n$ matrix A. The eigenvalues of an $n \times n$ matrix **A** are the roots of the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

The eigenvalues are also called the characteristic roots.

Consider, for example, the following matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

The characteristic equation is

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigenvalues of **A** are the roots of the characteristic equation, or -1, -2, and -3.

Diagonalization of *n X n* Matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

the transformation

x=Pz, where

$$\lambda_1, \lambda_2, \ldots, \lambda_n = n$$

distinct eigenvalues of **A**

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Diagonalization of *n X n* Matrix

will transform $\boldsymbol{P^{-1}A\,P}$ into the diagonal matrix, or

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_n \end{bmatrix}$$

If the matrix **A** involves multiple eigenvalues, then diagonalization is impossible. For example, if the 3X3 matrix **A**, where

Diagonalization of *n X n* Matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

has the eigenvalues $\lambda_1, \lambda_1, \lambda_3$ then the transformation **x=Sz**, where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix}$$

will yield

(Jordan canonical form)

$$\mathbf{S}^{-1}\mathbf{AS} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

1	0	0	$\cdots$	1
λ_1	1	0	$\cdots$	λ_n
λ_1^2	$2\lambda_1$	1	$\cdots$	λ_n^2
λ_1^3	$3\lambda_1^2$	$3\lambda_1$	$\cdots$	λ_n^3
$\vdots$	$\vdots$	$\vdots$	$\vdots \cdots$	$\vdots$
λ_1^{n-1}	$(n-1)\lambda_1^{n-2}$	$\dfrac{(n-1)(n-2)}{2}\lambda_1^{n-3}$	$\cdots$	λ_n^{n-1}

Consider the following state-space representation of a system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Equations can be put in a standard form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

Where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad 0]$$

The eigenvalues of matrix **A** are

$$\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$$

Thus, three eigenvalues are distinct. If we define a set of new state variables z_1, z_2, and z_3, by the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Or

$$\mathbf{x} = \mathbf{P}\mathbf{z}$$

Where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Then we obtain

$$\mathbf{P}\dot{\mathbf{z}} = \mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{B}u$$

By premultiplying both sides of this last equation by P^{-1}, we get

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}u$$

Or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

Simplifying gives

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u$$

Equation is also a state equation that describes the same system.

The output equation, is modified to $y = \mathbf{CPz}$

Or

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Notice that the transformation matrix **P** modifies the coefficient matrix of **z** into the diagonal matrix. Notice also that the diagonal elements of the matrix $\boldsymbol{P^{-1} A\, P}$ are identical with the three eigenvalues of **A**. It is very important to note that the eigenvalues of **A** and those of $\boldsymbol{P^{-1} A\, P}$ are identical.

Invariance of Eigenvalues

To prove the invariance of the eigenvalues under a linear transformation, we must show that the characteristic polynomials $|\lambda I - A|$ and $|\lambda I - P^{-1} A P|$ are identical.

Since the determinant of a product is the product of the determinants, we obtain

$$\begin{aligned} |\lambda \mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| &= |\lambda \mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| \\ &= |\mathbf{P}^{-1}(\lambda \mathbf{I} - \mathbf{A})\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\lambda \mathbf{I} - \mathbf{A}||\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\mathbf{P}||\lambda \mathbf{I} - \mathbf{A}| \end{aligned}$$

Invariance of Eigenvalues

Noting that the product of the determinants $|P^{-1}|$ and $|P|$ is the determinant of the product $|P^{-1}\,P|$, we obtain

$$\begin{aligned}|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| &= |\mathbf{P}^{-1}\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \\ &= |\lambda\mathbf{I} - \mathbf{A}|\end{aligned}$$

Thus, we have proved that the eigenvalues of **A** are invariant under a linear transformation.

Nonuniqueness of a Set of State Variables

It has been stated that a set of state variables is not unique for a given system. Suppose that $x_1, x_2, \ldots x_n$ are a set of state variables.

Then we may take as another set of state variables any set of functions

$$\begin{aligned}
\hat{x}_1 &= X_1(x_1, x_2, \ldots, x_n) \\
\hat{x}_2 &= X_2(x_1, x_2, \ldots, x_n) \\
&\cdot \\
&\cdot \\
&\cdot \\
\hat{x}_n &= X_n(x_1, x_2, \ldots, x_n)
\end{aligned}$$

Nonuniqueness of a Set of State Variables

provided that, for every set of values $\hat{x}_1$, $\hat{x}_2$, … $\hat{x}_n$ there corresponds a unique set of valuesx_1, x_2, … x_n, and vice versa. Thus, if **x** is a state vector, then $\hat{x}$ where

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

is also a state vector, provided the matrix **P** is nonsingular. Different state vectors convey the same information about the system behavior.

examples

Show that for the differential equation system

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

state and output equations can be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u$$

where state variables are defined by

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0\dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

and $\beta_0 = b_0$

$$\beta_1 = b_1 - a_1\beta_0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

Solution. From the definition of state variables x_2 and x_3, we have

$$\dot{x}_1 = x_2 + \beta_1 u$$

$$\dot{x}_2 = x_3 + \beta_2 u$$

To derive the equation for $\dot{x}_3$, we first note from Equation (3–99) that

$$\dddot{y} = -a_1 \ddot{y} - a_2 \dot{y} - a_3 y + b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

Since

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u$$

we have

$$\begin{aligned}
\dot{x}_3 &= \dddot{y} - \beta_0 \dddot{u} - \beta_1 \ddot{u} - \beta_2 \dot{u} \\
&= (-a_1 \ddot{y} - a_2 \dot{y} - a_3 y) + b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u - \beta_0 \dddot{u} - \beta_1 \ddot{u} - \beta_2 \dot{u} \\
&= -a_1(\ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u) - a_1 \beta_0 \ddot{u} - a_1 \beta_1 \dot{u} - a_1 \beta_2 u \\
&\quad -a_2(\dot{y} - \beta_0 \dot{u} - \beta_1 u) - a_2 \beta_0 \dot{u} - a_2 \beta_1 u - a_3(y - \beta_0 u) - a_3 \beta_0 u \\
&\quad + b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u - \beta_0 \dddot{u} - \beta_1 \ddot{u} - \beta_2 \dot{u} \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_0 - \beta_0)\dddot{u} + (b_1 - \beta_1 - a_1 \beta_0)\ddot{u} \\
&\quad + (b_2 - \beta_2 - a_1 \beta_1 - a_2 \beta_0)\dot{u} + (b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0)u \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0)u \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + \beta_3 u
\end{aligned}$$

Hence, we get

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u$$

Obtain a state-space equation and output equation for the system defined by

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + s^2 + s + 2}{s^3 + 4s^2 + 5s + 2}$$

Solution. From the given transfer function, the differential equation for the system is

$$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\dddot{u} + \ddot{u} + \dot{u} + 2u$$

Comparing this equation with the standard equation given by Equation (3–33), rewritten

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we find

$$a_1 = 4, \quad a_2 = 5, \quad a_3 = 2$$

$$b_0 = 2, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 2$$

Referring to Equation (3–35), we have

$$\beta_0 = b_0 = 2$$

$$\beta_1 = b_1 - a_1\beta_0 = 1 - 4 \times 2 = -7$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 1 - 4 \times (-7) - 5 \times 2 = 19$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$

$$= 2 - 4 \times 19 - 5 \times (-7) - 2 \times 2 = -43$$

Referring to Equation (3–34), we define

$$x_1 = y - \beta_0 u = y - 2u$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 + 7u$$

$$x_3 = \dot{x}_2 - \beta_2 u = \dot{x}_2 - 19u$$

Then referring to Equation (3–36),

$$\begin{aligned}
\dot{x}_1 &= x_2 - 7u \\
\dot{x}_2 &= x_3 + 19u \\
\dot{x}_3 &= -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u \\
&= -2x_1 - 5x_2 - 4x_3 - 43u
\end{aligned}$$

Hence, the state-space representation of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -7 \\ 19 \\ -43 \end{bmatrix} u$$

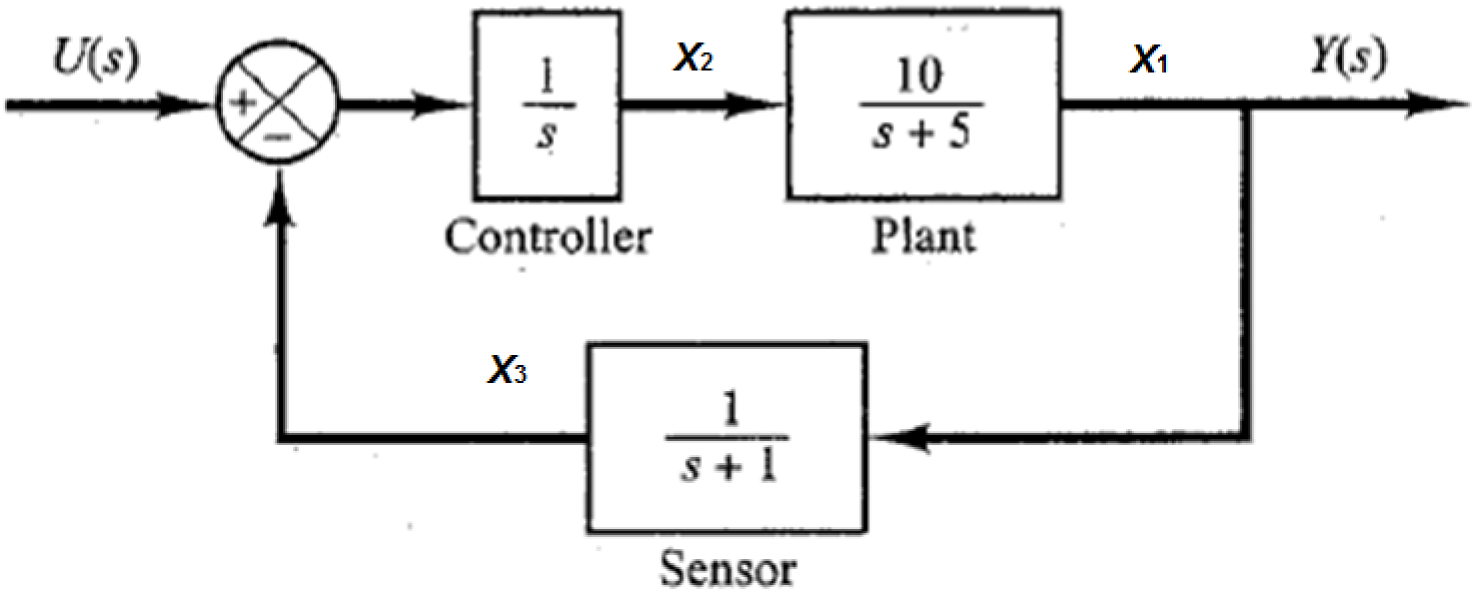
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u$$

This is one possible state-space representation of the system. There are many (infinitely many) others. If we use MATLAB, it produces the following state-space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & -5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -7 & -9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2u$$

Obtain a state-space model of the system shown in Figure



Solution. The system involves one integrator and two delayed integrators. The output of each integrator or delayed integrator can be a state variable. Let us define the output of the plant as x_1, the output of the controller as x_2, and the output of the sensor as x_3. Then we obtain

$$\frac{X_1(s)}{X_2(s)} = \frac{10}{s+5}$$

$$\frac{X_2(s)}{U(s) - X_3(s)} = \frac{1}{s}$$

$$\frac{X_3(s)}{X_1(s)} = \frac{1}{s+1}$$

$$Y(s) = X_1(s)$$

which can be rewritten as

$$
\begin{aligned}
sX_1(s) &= -5X_1(s) + 10X_2(s) \\
sX_2(s) &= -X_3(s) + U(s) \\
sX_3(s) &= X_1(s) - X_3(s) \\
Y(s) &= X_1(s)
\end{aligned}
$$

By taking the inverse Laplace transforms of the preceding four equations, we obtain

$$
\begin{aligned}
\dot{x}_1 &= -5x_1 + 10x_2 \\
\dot{x}_2 &= -x_3 + u \\
\dot{x}_3 &= x_1 - x_3 \\
y &= x_1
\end{aligned}
$$

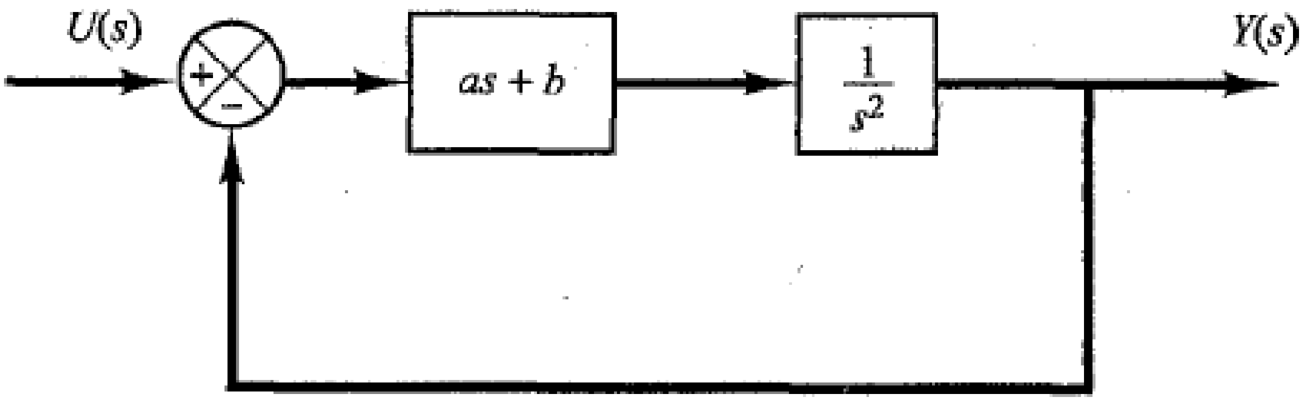
Thus, a state-space model of the system in the standard form is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 10 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It is important to note that this is not the only state-space representation of the system. Many other state-space representations are possible. However, the number of state variables is the same in any state-space representation of the same system. In the present system, the number of state variables is three, regardless of what variables are chosen as state variables.

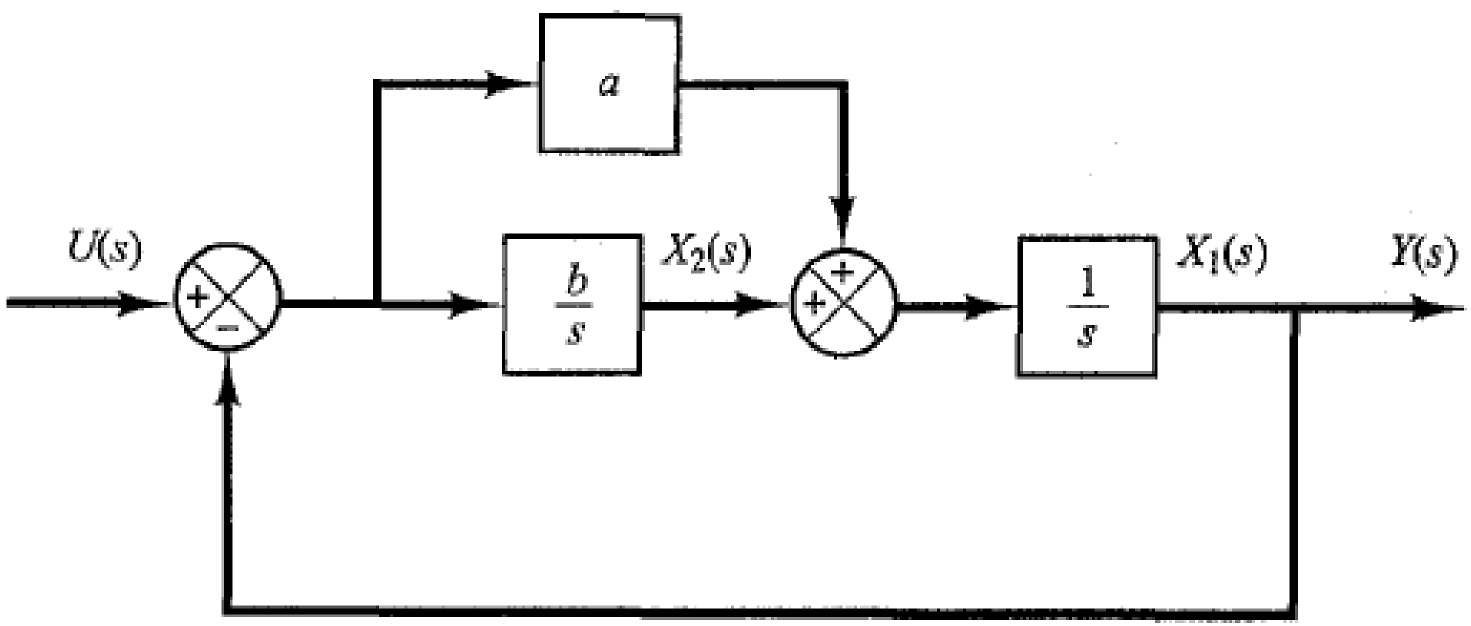
Obtain a state-space model for the system shown in Figure



Solution. First, notice that $(as + b)/s^2$ involves a derivative term. Such a derivative term may be avoided if we modify $(as + b)/s^2$ as

$$\frac{as + b}{s^2} = \left(a + \frac{b}{s}\right)\frac{1}{s}$$

Using this modification, the block diagram of Figure 3–52(a) can be modified to that shown in Figure 3–52(b).



(b)

Define the outputs of the integrators as state variables, as shown in Figure . Then we obtain

$$\frac{X_1(s)}{X_2(s) + a[U(s) - X_1(s)]} = \frac{1}{s}$$

$$\frac{X_2(s)}{U(s) - X_1(s)} = \frac{b}{s}$$

$$Y(s) = X_1(s)$$

which may be modified to

$$sX_1(s) = X_2(s) + a[U(s) - X_1(s)]$$

$$sX_2(s) = -bX_1(s) + bU(s)$$

$$Y(s) = X_1(s)$$

Taking the inverse Laplace transforms of the preceding three equations, we obtain

$$\dot{x}_1 = -ax_1 + x_2 + au$$

$$\dot{x}_2 = -bx_1 + bu$$

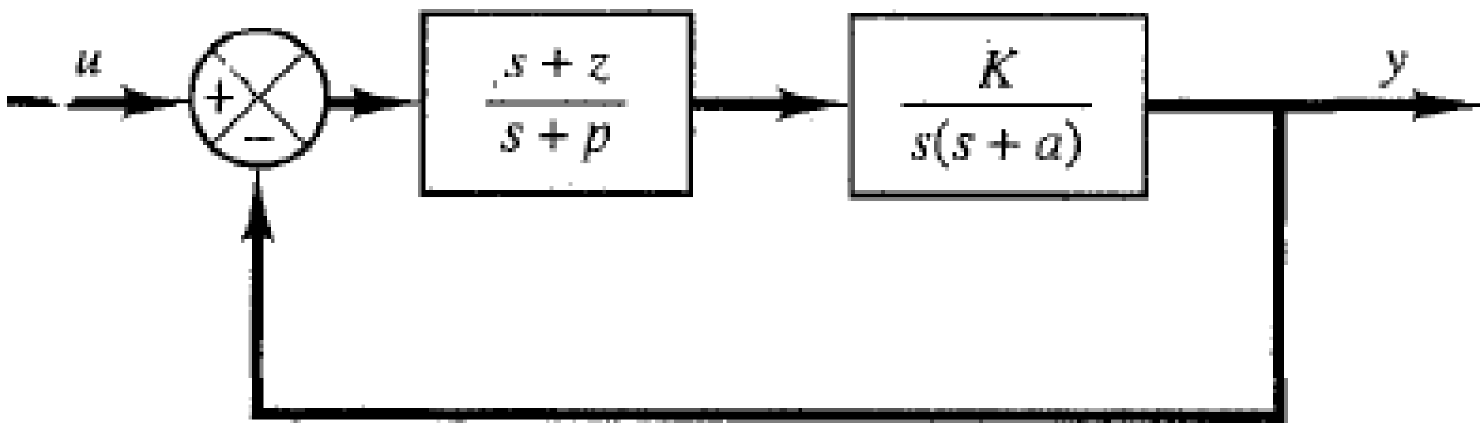
$$y = x_1$$

Rewriting the state and output equations in the standard vector-matrix form, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain a state-space representation of the system shown in Figure

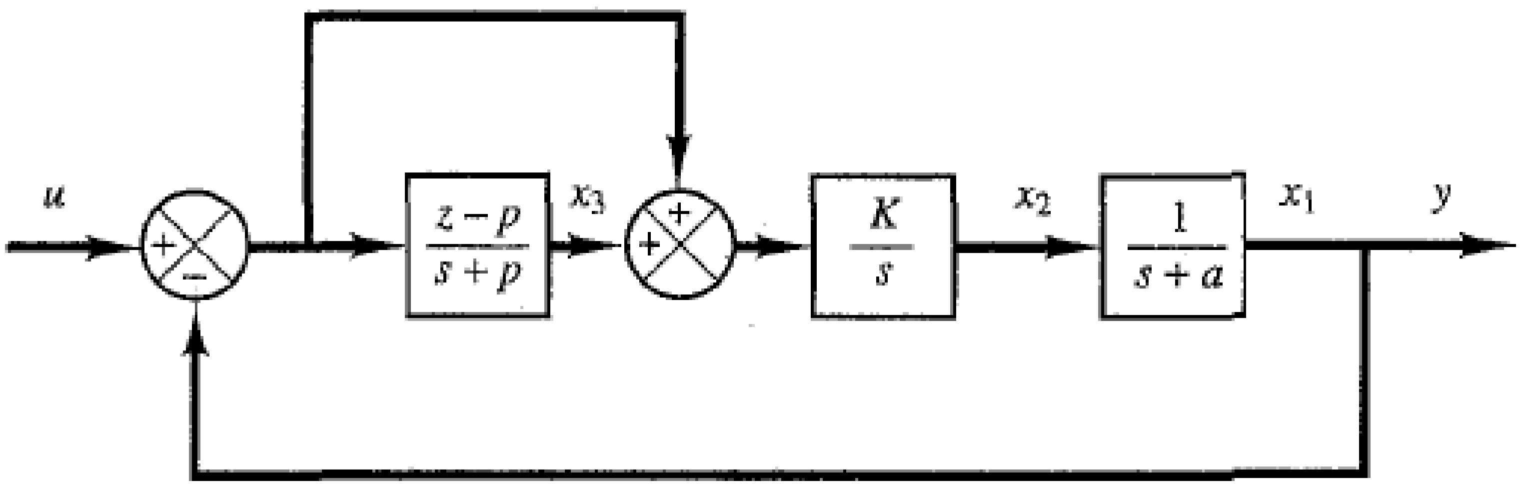


(a)

Solution. In this problem, first expand $(s+z)/(s+p)$ into partial fractions.

$$\frac{s+z}{s+p} = 1 + \frac{z-p}{s+p}$$

Next, convert $K/[s(s+a)]$ into the product of K/s and $1/(s+a)$. Then redraw the block diagram, as shown in Figure 3–53(b).



(b)

Defining a set of state variables, as shown in Figure , we obtain the following equations:

$$\dot{x}_1 = -ax_1 + x_2$$

$$\dot{x}_2 = -Kx_1 + Kx_3 + Ku$$

$$\dot{x}_3 = -(z - p)x_1 - px_3 + (z - p)u$$

$$y = x_1$$

Rewriting gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a & 1 & 0 \\ -K & 0 & K \\ -(z - p) & 0 & -p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ z - p \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Notice that the output of the integrator and the outputs of the first-order delayed integrators $[1/(s + a)$ and $(z - p)/(s + p)]$ are chosen as state variables. It is important to remember that the output of the block $(s + z)/(s + p)$ in Figure cannot be a state variable, because this block involves a derivative term, $s + z$.

EXAMPLE

Obtain the transfer function of the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution. Referring to Equation (3–29), the transfer function $G(s)$ is given by

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

In this problem, matrices **A**, **B**, **C**, and D are

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0], \quad D = 0$$

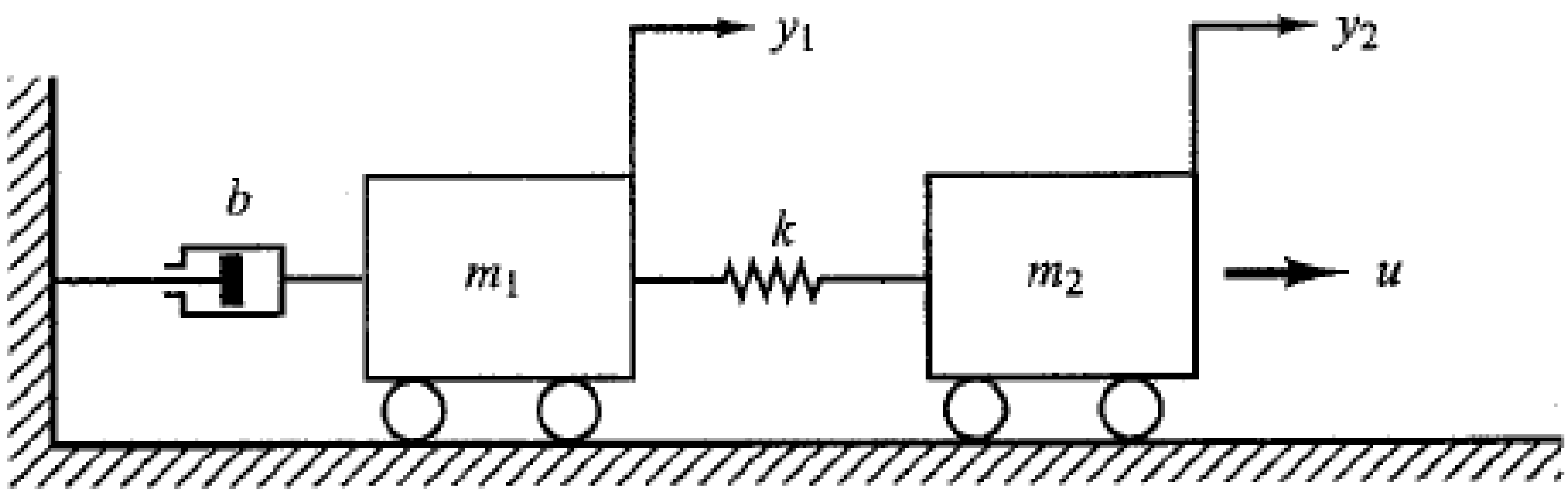
Hence

$$G(s) = [1 \quad 0 \quad 0]\begin{bmatrix} s+1 & -1 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= [1 \quad 0 \quad 0]\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{(s+1)^2} & \dfrac{1}{(s+1)^2(s+2)} \\ 0 & \dfrac{1}{s+1} & \dfrac{1}{(s+1)(s+2)} \\ 0 & 0 & \dfrac{1}{s+2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(s+1)^2(s+2)} = \frac{1}{s^3 + 4s^2 + 5s + 2}$$

Obtain a state-space representation of the system shown in Figure



Solution. The system equations are

$$m_1\ddot{y}_1 + b\dot{y}_1 + k(y_1 - y_2) = 0$$
$$m_2\ddot{y}_2 + k(y_2 - y_1) = u$$

The output variables for this system are y_1 and y_2. Define state variables as

$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$
$$x_3 = y_2$$
$$x_4 = \dot{y}_2$$

Then we obtain the following equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m_1}\left[-b\dot{y}_1 - k(y_1 - y_2)\right] = -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{1}{m_2}\left[-k(y_2 - y_1) + u\right] = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3 + \frac{1}{m_2}u$$

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k}{m_1} & -\dfrac{b}{m_1} & \dfrac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{m_2} & 0 & -\dfrac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix} u$$

and the output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Derive the following controllable canonical form

$$\frac{Y(s)}{U(s)} = b_0 + \frac{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1}b_0)s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

which can be modified to

$$Y(s) = b_0 U(s) + \hat{Y}(s)$$

where

$$\hat{Y}(s) = \frac{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1}b_0)s + (b_n - a_n b_0)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} U(s)$$

Let us rewrite this last equation in the following form:

$$\frac{\hat{Y}(s)}{(b_1 - a_1 b_0)s^{n-1} + \cdots + (b_{n-1} - a_{n-1}b_0)s + (b_n - a_n b_0)}$$

$$= \frac{U(s)}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = Q(s)$$

Derive the following controllable canonical form

From this last equation, the following two equations may be obtained:

$$s^n Q(s) = -a_1 s^{n-1} Q(s) - \cdots - a_{n-1} s Q(s) - a_n Q(s) + U(s)$$

$$\hat{Y}(s) = (b_1 - a_1 b_0) s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) + (b_n - a_n b_0) Q(s)$$

Now define state variables as follows:

$$\begin{aligned}
X_1(s) &= Q(s) \\
X_2(s) &= sQ(s) \\
&\vdots \\
X_{n-1}(s) &= s^{n-2} Q(s) \\
X_n(s) &= s^{n-1} Q(s)
\end{aligned}$$

Derive the following controllable canonical form

Then, clearly,

$$\begin{aligned} sX_1(s) &= X_2(s) \\ sX_2(s) &= X_3(s) \\ &\cdot \\ &\cdot \\ &\cdot \\ sX_{n-1}(s) &= X_n(s) \end{aligned}$$

which may be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{x}_{n-1} &= x_n \end{aligned}$$

Derive the following controllable canonical form

Noting that $s^n Q(s) = sX_n(s)$, we can rewrite Equation (9–72) as

$$sX_n(s) = -a_1 X_n(s) - \cdots - a_{n-1} X_2(s) - a_n X_1(s) + U(s)$$

or

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + u$$

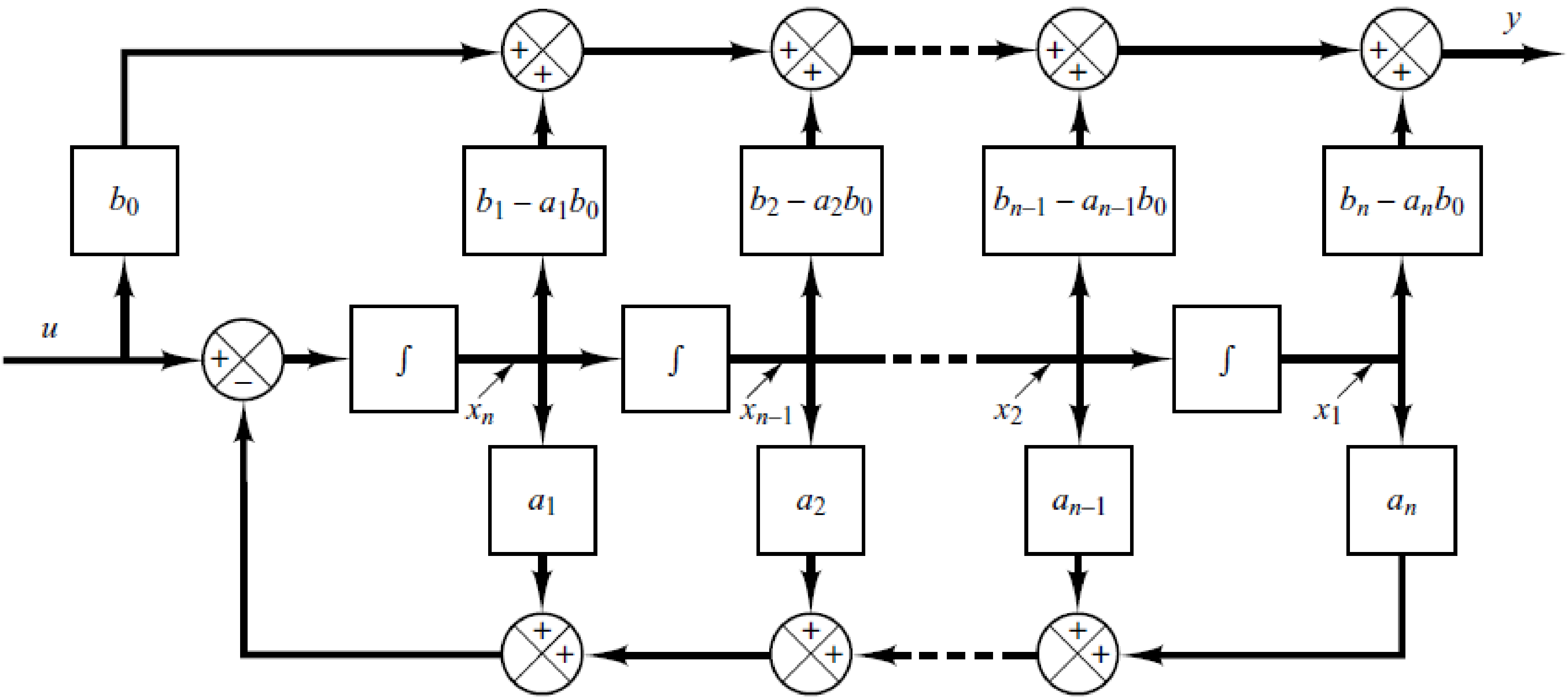
Also, from Equations (9–71) and (9–73), we obtain

$$\begin{aligned} Y(s) &= b_0 U(s) + (b_1 - a_1 b_0) s^{n-1} Q(s) + \cdots + (b_{n-1} - a_{n-1} b_0) s Q(s) \\ &\quad + (b_n - a_n b_0) Q(s) \\ &= b_0 U(s) + (b_1 - a_1 b_0) X_n(s) + \cdots + (b_{n-1} - a_{n-1} b_0) X_2(s) \\ &\quad + (b_n - a_n b_0) X_1(s) \end{aligned}$$

Derive the following controllable canonical form

The inverse Laplace transform of this output equation becomes

$$y = (b_n - a_n b_0)x_1 + (b_{n-1} - a_{n-1}b_0)x_2 + \cdots + (b_1 - a_1 b_0)x_n + b_0 u$$



Derive the following observable canonical form

Equation c

an be modified into the following form:

$$s^n\left[Y(s) - b_0U(s)\right] + s^{n-1}\left[a_1Y(s) - b_1U(s)\right] + \cdots$$
$$+ s\left[a_{n-1}Y(s) - b_{n-1}U(s)\right] + a_nY(s) - b_nU(s) = 0$$

By dividing the entire equation by s^n and rearranging, we obtain

$$Y(s) = b_0U(s) + \frac{1}{s}\left[b_1U(s) - a_1Y(s)\right] + \cdots$$

$$+ \frac{1}{s^{n-1}}\left[b_{n-1}U(s) - a_{n-1}Y(s)\right] + \frac{1}{s^n}\left[b_nU(s) - a_nY(s)\right]$$

Derive the following observable canonical form

Now define state variables as follows:

$$X_n(s) = \frac{1}{s}\left[b_1 U(s) - a_1 Y(s) + X_{n-1}(s)\right]$$

$$X_{n-1}(s) = \frac{1}{s}\left[b_2 U(s) - a_2 Y(s) + X_{n-2}(s)\right]$$

$$\vdots$$

$$X_2(s) = \frac{1}{s}\left[b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s)\right]$$

$$X_1(s) = \frac{1}{s}\left[b_n U(s) - a_n Y(s)\right]$$

Derive the following observable canonical form

Then Equation can be written as $Y(s) = b_0U(s) + X_n(s)$

By substituting and multiplying both sides of the equations by s, we obtain

$$sX_n(s) = X_{n-1}(s) - a_1X_n(s) + (b_1 - a_1b_0)U(s)$$

$$sX_{n-1}(s) = X_{n-2}(s) - a_2X_n(s) + (b_2 - a_2b_0)U(s)$$

$$\vdots$$

$$sX_2(s) = X_1(s) - a_{n-1}X_n(s) + (b_{n-1} - a_{n-1}b_0)U(s)$$

$$sX_1(s) = -a_nX_n(s) + (b_n - a_nb_0)U(s)$$

Derive the following observable canonical form

Taking the inverse Laplace transforms of the preceding n equations and writing them in the reverse order, we get

$$\dot{x}_1 = -a_n x_n + (b_n - a_n b_0)u$$

$$\dot{x}_2 = x_1 - a_{n-1} x_n + (b_{n-1} - a_{n-1} b_0)u$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\dot{x}_{n-1} = x_{n-2} - a_2 x_n + (b_2 - a_2 b_0)u$$

$$\dot{x}_n = x_{n-1} - a_1 x_n + (b_1 - a_1 b_0)u$$

Derive the following observable canonical form

Also, the inverse Laplace transform of Equation gives

$$y = x_n + b_0 u$$