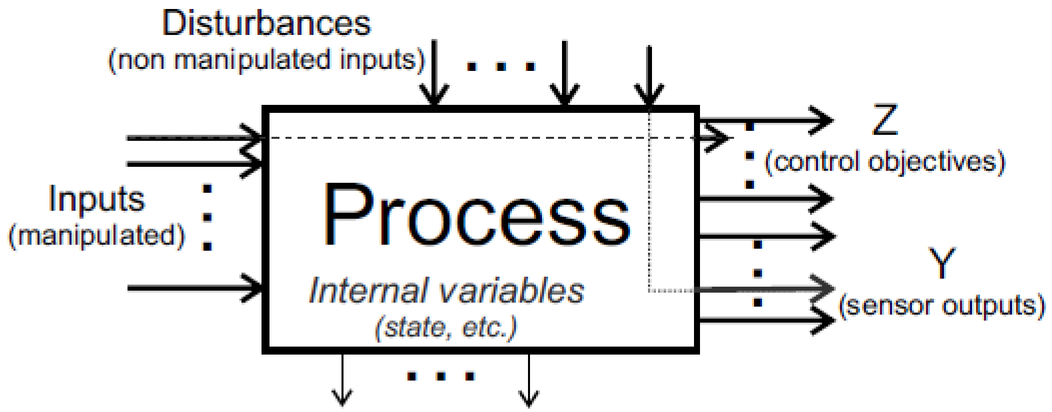
كلية الهندسة
جامعة المنارة
MANARA UNIVERSITY

نظم متعددة المتغيرات
Multivariable Systems



الروبوت والأنظمة الذكية

مدرس المقرر
د.بلال شيحا

المحاضرة الثانية

MODELING IN STATE SPACE

SOLVING THE TIME-INVARIANT STATE EQUATION

We shall first consider the homogeneous case and then the nonhomogeneous case.

Solution of Homogeneous State Equations.

Before we solve vector-matrix differential equations, let us review the solution of the scalar differential equation

$$\dot{x} = ax$$

In solving this equation, we may assume a solution ***x(t)*** of the form

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots$$

By substituting this assumed solution into last Equation, we obtain

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + kb_k t^{k-1} + \cdots$$
$$= a\left(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots\right)$$

Solution of Homogeneous State Equations

If the assumed solution is to be the true solution, last Equation must hold for any ***t***.

Hence, equating the coefficients of the equal powers of t, we obtain

$$b_1 = ab_0$$

$$b_2 = \frac{1}{2}ab_1 = \frac{1}{2}a^2b_0$$

$$b_3 = \frac{1}{3}ab_2 = \frac{1}{3 \times 2}a^3b_0$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$b_k = \frac{1}{k!}a^kb_0$$

Solution of Homogeneous State Equations

The value of ***b_0*** is determined by substituting ***t=0*** into Equation, or

$$x(0) = b_0$$

Hence, the solution ***x(t)*** can be written as

$$x(t) = \left(1 + at + \frac{1}{2!}a^2t^2 + \cdots + \frac{1}{k!}a^kt^k + \cdots\right)x(0)$$

$$= e^{at}x(0)$$

We shall now solve the vector-matrix differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Where **x** = *n*-vector

A = *n* X *n* constant matrix

Solution of Homogeneous State Equations

By analogy with the scalar case, we assume that the solution is in the form of a vector power series in t, or

$$\mathbf{x}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_k t^k + \cdots$$

By substituting this assumed solution into Equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ we obtain

$$\begin{aligned} &\mathbf{b}_1 + 2\mathbf{b}_2 t + 3\mathbf{b}_3 t^2 + \cdots + k\mathbf{b}_k t^{k-1} + \cdots \\ &= \mathbf{A}\left(\mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_k t^k + \cdots\right) \end{aligned}$$

If the assumed solution is to be the true solution, last Equation must hold for all ***t***. Thus, by equating the coefficients of like powers of t on both sides of Equation, we obtain

Solution of Homogeneous State Equations

$$\mathbf{b}_1 = \mathbf{A}\mathbf{b}_0$$

$$\mathbf{b}_2 = \frac{1}{2}\mathbf{A}\mathbf{b}_1 = \frac{1}{2}\mathbf{A}^2\mathbf{b}_0$$

$$\mathbf{b}_3 = \frac{1}{3}\mathbf{A}\mathbf{b}_2 = \frac{1}{3 \times 2}\mathbf{A}^3\mathbf{b}_0$$

$$\vdots$$

$$\mathbf{b}_k = \frac{1}{k!}\mathbf{A}^k\mathbf{b}_0$$

By substituting ***t=0*** into Equation

$$\mathbf{x}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_k t^k + \cdots$$

we obtain

$$\mathbf{x}(0) = \mathbf{b}_0$$

Solution of Homogeneous State Equations

Thus, the solution ***x(t)*** can be written as

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots\right)\mathbf{x}(0)$$

The expression in the parentheses on the right-hand side of this last equation is an **nxn** matrix. Because of its similarity to the infinite power series for a scalar exponential, we call it the matrix exponential and write

$$\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots = e^{\mathbf{A}t}$$

In terms of the matrix exponential, the solution of Equation ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$) can be written a

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

Solution of Homogeneous State Equations

Since the matrix exponential is very important in the state-space analysis of linear systems, we shall next examine its properties.

- **Matrix Exponential.** It can be proved that the matrix exponential of an **nxn** matrix **A**,

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$$

converges absolutely for all finite $\boldsymbol{t}$. (Hence, computer calculations for evaluating the elements of $\boldsymbol{e^{At}}$ by using the series expansion can be easily carried out.)

Solution of Homogeneous State Equations

Because of the convergence of the infinite series the series can be differentiated term by term to give

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

The matrix exponential has the property that

$$e^{\mathbf{A}(t+s)} = e^{\mathbf{A}t}e^{\mathbf{A}s}$$

In particular, if ***s=–t***, then

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{-\mathbf{A}t}e^{\mathbf{A}t} = e^{\mathbf{A}(t-t)} = \mathbf{I}$$

Thus, the inverse of $\boldsymbol{e^{At}}$ is $\boldsymbol{e^{-At}}$ Since the inverse of $\boldsymbol{e^{At}}$ always exists, $\boldsymbol{e^{At}}$ is nonsingular.

Laplace Transform Approach to the Solution of Homogeneous State Equations

Let us first consider the scalar case: $\dot{x} = ax$

Taking the Laplace transform of Equation, we obtain

$$sX(s) - x(0) = aX(s)$$

Where $X(s) = \mathscr{L}[x]$

Solving Equation for ***X(s)*** gives $X(s) = \dfrac{x(0)}{s - a} = (s - a)^{-1}x(0)$

The inverse Laplace transform of this last equation gives the solution $x(t) = e^{at}x(0)$

The foregoing approach to the solution of the homogeneous scalar differential equation can be extended to the homogeneous state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Laplace Transform Approach to the Solution of Homogeneous State Equations

Taking the Laplace transform of both sides of Equation we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

Where $\mathbf{X}(s) = \mathscr{L}[\mathbf{x}]$. Hence, $(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0)$

Premultiplying both sides of this last equation by $(\mathrm{SI} - \mathrm{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

The inverse Laplace transform of $\boldsymbol{X}(\boldsymbol{s})$ gives the solution $\boldsymbol{x}(\boldsymbol{t})$, Thus,

$$\mathbf{x}(t) = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0)$$

Note that

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \cdots$$

Laplace Transform Approach to the Solution of Homogeneous State Equations

Hence, the inverse Laplace transform of $(\boldsymbol{SI} - \boldsymbol{A})^{-1}$ gives

$$\mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

(The inverse Laplace transform of a matrix is the matrix consisting of the inverse Laplace transforms of all elements.)

The solution of Equation is obtained as $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$

The importance of above Equation lies in the fact that it provides a convenient means for finding the closed solution for the matrix exponential.

Laplace Transform Approach to the Solution of Homogeneous State Equations

State-Transition Matrix. We can write the solution of the homogeneous stat equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

As $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$

If the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of the matrix **A** are distinct, than $\boldsymbol{\Phi}(\boldsymbol{t})$ will contain the ***n*** exponentials $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}$

In particular, if the matrix **A** is diagonal, then

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix} \qquad (\mathbf{A}\text{: diagonal})$$

Laplace Transform Approach to the Solution of Homogeneous State Equations

If there is a multiplicity in the eigenvalues—for example, if the eigenvalues of **A** are $\lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_5, \ldots, \lambda_n,$

then $\boldsymbol{\Phi}(\boldsymbol{t})$ will contain, in addition to the exponentials $e^{\lambda_1 t}$,$e^{\lambda_4 t}$,$e^{\lambda_5 t}$, …$e^{\lambda_n t}$ terms like $te^{\lambda_1 t}$ and $t^2 e^{\lambda_1 t}$.

Properties of State-Transition Matrices.

We shall now summarize the important properties of the state-transition matrix $\boldsymbol{\Phi}(\boldsymbol{t})$. For the time-invariant system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

for which

$$\boldsymbol{\Phi}(t) = e^{\mathbf{A}t}$$

Properties of State-Transition Matrices

we have the following:

1. $\mathbf{\Phi}(0) = e^{\mathbf{A}0} = \mathbf{I}$
2. $\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \left(e^{-\mathbf{A}t}\right)^{-1} = \left[\mathbf{\Phi}(-t)\right]^{-1}$ or $\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t)$
3. $\mathbf{\Phi}(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} = \mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_2)\mathbf{\Phi}(t_1)$
4. $\left[\mathbf{\Phi}(t)\right]^n = \mathbf{\Phi}(nt)$
5. $\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1 - t_0) = \mathbf{\Phi}(t_2 - t_0) = \mathbf{\Phi}(t_1 - t_0)\mathbf{\Phi}(t_2 - t_1)$

EXAMPLE

Obtain the state-transition matrix $\boldsymbol{\Phi}(\boldsymbol{t})$ of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain also the inverse of the state-transition matrix. $\boldsymbol{\Phi}^{-1}(\boldsymbol{t})$. For this system,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The state-transition matrix $\boldsymbol{\Phi}(\boldsymbol{t})$ is given by

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

Since

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

EXAMPLE

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

Hence,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Noting that $\boldsymbol{\Phi}^{-1}(\boldsymbol{t}) = \boldsymbol{\Phi}(-\boldsymbol{t})$ we obtain the inverse of the state-transition matrix as follows:

$$\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

Solution of Nonhomogeneous State Equations

We shall begin by considering the scalar case

$$\dot{x} = ax + bu$$

Let us rewrite Equation as $$\dot{x} - ax = bu$$

Multiplying both sides of this equation by e^{-at}, we obtain

$$e^{-at}[\dot{x}(t) - ax(t)] = \frac{d}{dt}[e^{-at}x(t)] = e^{-at}bu(t)$$

Integrating this equation between 0 and t gives

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)\,d\tau$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)\,d\tau$$

Solution of Nonhomogeneous State Equations

The first term on the right-hand side is the response to the initial condition and the second term is the response to the input ***u(t)***.

Let us now consider the nonhomogeneous state equation described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x} = n\text{-vector}$

$\mathbf{u} = r\text{-vector}$

$\mathbf{A} = n \times n \text{ constant matrix}$

$\mathbf{B} = n \times r \text{ constant matrix}$

By writing Equation as

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

Solution of Nonhomogeneous State Equations

and premultiplying both sides of this equation by e^{-at}, we obtain

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = \frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Integrating the preceding equation between **0** and ***t*** gives

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Equation can also be written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Solution of Nonhomogeneous State Equations

The solution **x(t)** is clearly the sum of a term consisting of the transition of the initial state and a term arising from the input vector.

Laplace Transform Approach to the Solution of Nonhomogeneous State Equations

The solution of the nonhomogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

can also be obtained by the Laplace transform approach. The Laplace transform of this last equation yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Premultiplying both sides of this last equation by $(\mathbf{SI} - \boldsymbol{A})^{-\mathbf{1}}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Laplace Transform Approach to the Solution of Nonhomogeneous State Equations

Using the relationship given by Equation

$$\mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

Gives

$$\mathbf{X}(s) = \mathscr{L}\left[e^{\mathbf{A}t}\right]\mathbf{x}(0) + \mathscr{L}\left[e^{\mathbf{A}t}\right]\mathbf{B}\mathbf{U}(s)$$

The inverse Laplace transform of this last equation can be obtained by use of the convolution integral as follows:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

- **Solution in Terms of $X(t_0)$** Thus far we have assumed the initial time to be zero. If, however, the initial time is given by t_0 instead of **0**, then the solution to Equation must be modified to

Laplace Transform Approach to the Solution of Nonhomogeneous State Equations

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

EXAMPLE

Obtain the time response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where ***u(t)*** is the unit-step function occurring at ***t=0***, or ***u(t)=1(t).*** For this system,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

EXAMPLE

The state-transition matrix $\boldsymbol{\phi(t) = e^{at}}$ was obtained in last Example as

$$\Phi(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

The response to the unit-step input is then obtained as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

If the initial state is zero, or ***x(0)=0***, then ***x(t)*** can be simplified to

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

SOME USEFUL RESULTS IN VECTOR-MATRIX ANALYSIS

In this section we present some useful results in vector-matrix analysis that we use. Specifically, we present the **Cayley–Hamilton** theorem, the minimal polynomial, Sylvester's interpolation method for calculating and the linear independence of vectors.

Cayley–Hamilton Theorem

The Cayley–Hamilton theorem is very useful in proving theorems involving matrix equations or solving problems involving matrix equations.

Consider an **nxn** matrix ***A*** and its characteristic equation:

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

Cayley–Hamilton theorem

The Cayley–Hamilton theorem states that the matrix **A** satisfies its own characteristic equation, or that

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}$$

To prove this theorem, note that $\boldsymbol{adj(\lambda I - A)}$ is a polynomial in $\boldsymbol{\lambda}$ of degree **n-1**. That is,

$$\mathrm{adj}(\lambda\mathbf{I} - \mathbf{A}) = \mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \cdots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n$$

where $\mathbf{B}_1 = \mathbf{I}$. Since

$$(\lambda\mathbf{I} - \mathbf{A})\,\mathrm{adj}(\lambda\mathbf{I} - \mathbf{A}) = \left[\mathrm{adj}(\lambda\mathbf{I} - \mathbf{A})\right](\lambda\mathbf{I} - \mathbf{A}) = |\lambda\mathbf{I} - \mathbf{A}|\mathbf{I}$$

we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}|\mathbf{I} &= \mathbf{I}\lambda^n + a_1\mathbf{I}\lambda^{n-1} + \cdots + a_{n-1}\mathbf{I}\lambda + a_n\mathbf{I} \\ &= (\lambda\mathbf{I} - \mathbf{A})\left(\mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \cdots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n\right) \\ &= \left(\mathbf{B}_1\lambda^{n-1} + \mathbf{B}_2\lambda^{n-2} + \cdots + \mathbf{B}_{n-1}\lambda + \mathbf{B}_n\right)(\lambda\mathbf{I} - \mathbf{A}) \end{aligned}$$

Cayley–Hamilton theorem

From this equation, we see that $\boldsymbol{A}$ and $\boldsymbol{B_i}$**(i=1, 2, … , n)** commute. Hence, the product of $(\boldsymbol{\lambda I}-\boldsymbol{A})$ and $\boldsymbol{adj}(\boldsymbol{\lambda I}-\boldsymbol{A})$ becomes zero if either of these is zero. If **A** is substituted for $\boldsymbol{\lambda}$ in this last equation, then clearly $(\boldsymbol{\lambda I}-\boldsymbol{A})$ becomes zero. Hence, we obtain

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}$$

This proves the Cayley–Hamilton theorem, or Equation

$$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{0}$$

Minimal Polynomial

Referring to the Cayley–Hamilton theorem, every **nxn** matrix **A** satisfies its own characteristic equation. The characteristic equation is not, however, necessarily the scalar equation of least degree that **A** satisfies. The least-degree polynomial having **A** as a root is called the *minimal polynomial*. That is, the minimal polynomial of an **nxn** matrix **A** is defined as the polynomial $\boldsymbol{\Phi}(\boldsymbol{\lambda})$ of least degree,

$$\phi(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m, \qquad m \leq n$$

such that $\boldsymbol{\Phi}(\boldsymbol{A}) = \mathbf{0}$**,** or

$$\phi(\mathbf{A}) = \mathbf{A}^m + a_1\mathbf{A}^{m-1} + \cdots + a_{m-1}\mathbf{A} + a_m\mathbf{I} = \mathbf{0}$$

The minimal polynomial plays an important role in the computation of polynomials in an **nxn** matrix.

Minimal Polynomial

Let us suppose that $\boldsymbol{d(\lambda)}$ a polynomial in $\boldsymbol{\lambda}$, is the greatest common divisor of all the elements of $\boldsymbol{adj(\lambda I - A)}$. We can show that if the coefficient of the highest-degree term in $\boldsymbol{\lambda}$ of $\boldsymbol{d(\lambda)}$ is chosen as 1, then the minimal polynomial $\boldsymbol{\Phi(\lambda)}$ is given by

$$\phi(\lambda) = \frac{|\lambda \mathbf{I} - \mathbf{A}|}{d(\lambda)}$$

It is noted that the minimal polynomial $\boldsymbol{\Phi(\lambda)}$ of an **nxn** matrix **A** can be determined by the following procedure:

1. Form $\boldsymbol{adj(\lambda I - A)}$and write the elements of $\boldsymbol{adj(\lambda I - A)}$ as factored polynomials in $\boldsymbol{\lambda}$.

2. Determine $\boldsymbol{d(\lambda)}$ as the greatest common divisor of all the elements of $\boldsymbol{adj(\lambda I - A)}$ Choose the coefficient of the highest-degree term in $\boldsymbol{\lambda}$ of $\boldsymbol{d(\lambda)}$ to be 1. If there is no common divisor $\boldsymbol{d(\lambda) = 0}$.

3. The minimal polynomial $\boldsymbol{\Phi(\lambda)}$ is then given as $|\boldsymbol{\lambda I - A}|$ divided by$\boldsymbol{d(\lambda)}$.

Matrix Exponential e^{At}

In solving control engineering problems, it often becomes necessary to compute e^{At}. If matrix **A** is given with all elements in numerical values, MATLAB provides a simple way to compute e^{AT}, where ***T*** is a constant.

Aside from computational methods, several analytical methods are available for the computation of e^{At}. We shall present three methods here.

Computation of e^{At}: Method 1

If matrix **A** can be transformed into a diagonal form, then e^{At} can be given by

Computation of e^{At}: Method 1

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}$$

where **P** is a diagonalizing matrix for **A**.

If matrix **A** can be transformed into a Jordan canonical form, then $\boldsymbol{e^{At}}$ can be given by

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

where **S** is a transformation matrix that transforms matrix **A** into a Jordan canonical form **J**.

Computation of e^{At}*: Method 1*

As an example, consider the following matrix **A:**

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$|\lambda\mathbf{I} - \mathbf{A}| = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$$

Thus, matrix **A** has a multiple eigenvalue of order 3 at $\boldsymbol{\lambda} = \mathbf{1}$ .It can be shown that matrix **A** has a multiple eigenvector of order 3.The transformation matrix that will transform matrix **A** into a Jordan canonical form can be given by

Computation of e^{At}: Method 1

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

The inverse of matrix $\mathbf{S}$ is

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Then it can be seen that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{J}$$

Computation of e^{At}: Method 1

Noting that

$$e^{\mathbf{J}t} = \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

we find

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e^t & te^t & \frac{1}{2}t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t - te^t + \frac{1}{2}t^2e^t & te^t - t^2e^t & \frac{1}{2}t^2e^t \\ \frac{1}{2}t^2e^t & e^t - te^t - t^2e^t & te^t + \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t & -3te^t - t^2e^t & e^t + 2te^t + \frac{1}{2}t^2e^t \end{bmatrix} \end{aligned}$$

Computation of e^{At}: Method 2

The second method of computing e^{At} uses the Laplace transform approach. Referring to Equation

$$\mathscr{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots = e^{\mathbf{A}t}$$

e^{At} can be given as follows:

$$e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]$$

Thus, to obtain e^{At} first invert the matrix$(sI - A)$ This results in a matrix whose elements are rational functions of **s**. Then take the inverse Laplace transform of each element of the matrix.

EXAMPLE Consider the following matrix **A:**

Compute e^{At} by use of the two analytical methods presented previously.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

EXAMPLE

Method 1. The eigenvalues of **A** are **0** and **–2**$(\boldsymbol{\lambda_1 = 0, \lambda_2 = -2})$ A necessary transformation matrix **P** may be obtained as

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

Then, from Equation

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}$$

is obtained as follows:

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^0 & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

EXAMPLE

Method 2. Since

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}$$

we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+2)} \\ 0 & \dfrac{1}{s+2} \end{bmatrix}$$

Hence,

$$e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Computation of $e^{\mathbf{A}t}$: Method 3

The third method is based on Sylvester's interpolation method. We shall first consider the case where the roots of the minimal polynomial $\boldsymbol{\Phi}(\boldsymbol{\lambda})$ of **A** are distinct. Then we shall deal with the case of multiple roots.

Case 1: *Minimal Polynomial of* **A** *Involves Only Distinct Roots.* We shall assume that the degree of the minimal polynomial of **A** is m. By using Sylvester's interpolation formula, it can be shown that e^{At} can be obtained by solving the following determinant equation

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & e^{\lambda_2 t} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \cdots & \mathbf{A}^{m-1} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Computation of e^{At}: Method 3

By solving last Equation for e^{At}, e^{At} can be obtained in terms of the A^k(k = 0, 1, 2,… , m-1) and the $e^{\lambda_i t}$ (i=1, 2, 3, … , m). [Equation may be expanded, for example, about the last column.]

Notice that solving Equation for e^{At} is the same as writing

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots + \alpha_{m-1}(t)\mathbf{A}^{m-1}$$

and determining the $\alpha_k(t)$ (k = 0, 1, 2,… , m-1) by solving the following set of m equations for the $\alpha_k(t)$:

$$\begin{aligned}
\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} &= e^{\lambda_1 t} \\
\alpha_0(t) + \alpha_1(t)\lambda_2 + \alpha_2(t)\lambda_2^2 + \cdots + \alpha_{m-1}(t)\lambda_2^{m-1} &= e^{\lambda_2 t} \\
&\;\;\vdots \\
\alpha_0(t) + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} &= e^{\lambda_m t}
\end{aligned}$$

Computation of e^{At}: Method 3

If **A** is an **nxn** matrix and has distinct eigenvalues, then the number of $\alpha_k(t)$'s to be determined is ***m=n***. If **A** involves multiple eigenvalues, but its minimal polynomial has only simple roots, however, then the number ***m*** of $\alpha_k(t)$'s to be determined is less than ***n***.

Case 2: *Minimal Polynomial of* **A** *Involves Multiple Roots.* As an example, consider the case where the minimal polynomial of **A** involves three equal roots ($\lambda_1 = \lambda_2 = \lambda_3$) and has other roots ($\lambda_4, \lambda_5, \dots \lambda_m$) that are all distinct. By applying Sylvester's interpolation formula, it can be shown that e^{At} can be obtained from the following determinant equation:

Computation of e^{At}: Method 3

$$\begin{vmatrix} 0 & 0 & 1 & 3\lambda_1 & \cdots & \frac{(m-1)(m-2)}{2}\lambda_1^{m-3} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (m-1)\lambda_1^{m-2} & te^{\lambda_1 t} \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 & \cdots & \lambda_4^{m-1} & e^{\lambda_4 t} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \lambda_m^3 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \mathbf{A}^3 & \cdots & \mathbf{A}^{m-1} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Equation can be solved for by expanding it about the last column.

Computation of e^{At}: Method 3

It is noted that, just as in case 1, solving Equation for $\boldsymbol{e^{At}}$ is the same as writing

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots + \alpha_{m-1}(t)\mathbf{A}^{m-1}$$

and determining the $\boldsymbol{\alpha_k(t)'s}$ ($\boldsymbol{k}$ = 0, 1, 2,… , m-1) from

$$\alpha_2(t) + 3\alpha_3(t)\lambda_1 + \cdots + \frac{(m-1)(m-2)}{2}\alpha_{m-1}(t)\lambda_1^{m-3} = \frac{t^2}{2}e^{\lambda_1 t}$$

$$\alpha_1(t) + 2\alpha_2(t)\lambda_1 + 3\alpha_3(t)\lambda_1^2 + \cdots + (m-1)\alpha_{m-1}(t)\lambda_1^{m-2} = te^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots + \alpha_{m-1}(t)\lambda_1^{m-1} = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_4 + \alpha_2(t)\lambda_4^2 + \cdots + \alpha_{m-1}(t)\lambda_4^{m-1} = e^{\lambda_4 t}$$

$$\vdots$$

$$\alpha_0(t) + \alpha_1(t)\lambda_m + \alpha_2(t)\lambda_m^2 + \cdots + \alpha_{m-1}(t)\lambda_m^{m-1} = e^{\lambda_m t}$$

Computation of e^{At}: Method 3

The extension to other cases where, for example, there are two or more sets of multiple roots will be apparent. Note that if the minimal polynomial of **A** is not found, it is possible to substitute the characteristic polynomial for the minimal polynomial. The number of computations may, of course, be increased.

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

Compute e^{At} using Sylvester's interpolation formula.

From Equation

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{m-1} & e^{\lambda_1 t} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{m-1} & e^{\lambda_2 t} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \lambda_m & \lambda_m^2 & \cdots & \lambda_m^{m-1} & e^{\lambda_m t} \\ \mathbf{I} & \mathbf{A} & \mathbf{A}^2 & \cdots & \mathbf{A}^{m-1} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Computation of $e^{\mathbf{A}t}$: Method 3

we get

$$\begin{vmatrix} 1 & \lambda_1 & e^{\lambda_1 t} \\ 1 & \lambda_2 & e^{\lambda_2 t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Substituting 0 for λ_1 and –2 for λ_2 in this last equation, we obtain

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -2 & e^{-2t} \\ \mathbf{I} & \mathbf{A} & e^{\mathbf{A}t} \end{vmatrix} = \mathbf{0}$$

Expanding the determinant, we obtain

Computation of e^{At}: Method 3

$$-2e^{\mathbf{A}t} + \mathbf{A} + 2\mathbf{I} - \mathbf{A}e^{-2t} = \mathbf{0}$$

$$\begin{aligned} e^{\mathbf{A}t} &= \tfrac{1}{2}(\mathbf{A} + 2\mathbf{I} - \mathbf{A}e^{-2t}) \\ &= \frac{1}{2}\left\{\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} e^{-2t}\right\} \\ &= \begin{bmatrix} 1 & \tfrac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

An alternative approach is to use Equation

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots + \alpha_{m-1}(t)\mathbf{A}^{m-1}$$

We first determine $\boldsymbol{\alpha_0(t)}$ and $\boldsymbol{\alpha_1(t)}$ from

Computation of e^{At}: Method 3

$$\alpha_0(t) + \alpha_1(t)\lambda_1 = e^{\lambda_1 t}$$

$$\alpha_0(t) + \alpha_1(t)\lambda_2 = e^{\lambda_2 t}$$

Since $\lambda_1 = 0$ and $\lambda_2 = -2$, the last two equations become

$$\alpha_0(t) = 1$$

$$\alpha_0(t) - 2\alpha_1(t) = e^{-2t}$$

Solving for $\boldsymbol{\alpha_0(t)}$ and $\boldsymbol{\alpha_1(t)}$ gives

$$\alpha_0(t) = 1, \qquad \alpha_1(t) = \frac{1}{2}(1 - e^{-2t})$$

Computation of e^{At}: Method 3

Then e^{At} can be written as

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} = \mathbf{I} + \frac{1}{2}\left(1 - e^{-2t}\right)\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2}\left(1 - e^{-2t}\right) \\ 0 & e^{-2t} \end{bmatrix}$$

Linear Independence of Vectors

The vectors $X_1, X_2, \dots X_n$ are said to be linearly independent if

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$$

where $c_1, c_2, \dots c_n$ are constants, implies that

$$c_1 = c_2 = \cdots = c_n = 0$$

Conversely, the vectors $X_1, X_2, \dots X_n$ are said to be linearly dependent if and only if X_i can be expressed as a linear combination of X_j (*j*=1, 2, … , n; *j* ≠ *i*), or

$$\mathbf{x}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} c_j \mathbf{x}_j$$

Linear Independence of Vectors

for some set of constants c_j .This means that if X_i can be expressed as a linear combination of the other vectors in the set, it is linearly dependent on them or it is not an independent member of the set.

The vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

are linearly dependent since $\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$

The vectors

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Linear Independence of Vectors

are linearly independent since

$$c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + c_3 \mathbf{y}_3 = \mathbf{0}$$

implies that

$$c_1 = c_2 = c_3 = 0$$

Note that if an **nxn** matrix is nonsingular (that is, the matrix is of rank ***n*** or the determinant is nonzero) then ***n*** column (or row) vectors are linearly independent. If the **nxn** matrix is singular (that is, the rank of the matrix is less than ***n*** or the determinant is zero), then ***n*** column (or row) vectors are linearly dependent. To demonstrate this, notice that

Linear Independence of Vectors

$$[\mathbf{x}_1 \mid \mathbf{x}_2 \mid \mathbf{x}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \text{singular}$$

$$[\mathbf{y}_1 \mid \mathbf{y}_2 \mid \mathbf{y}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \text{nonsingular}$$

1	0	0	$\cdots$	1
λ_1	1	0	$\cdots$	λ_n
λ_1^2	$2\lambda_1$	1	$\cdots$	λ_n^2
λ_1^3	$3\lambda_1^2$	$3\lambda_1$	$\cdots$	λ_n^3
$\vdots$	$\vdots$	$\vdots$	$\vdots \cdots$	$\vdots$
λ_1^{n-1}	$(n-1)\lambda_1^{n-2}$	$\frac{(n-1)(n-2)}{2}\lambda_1^{n-3}$	$\cdots$	λ_n^{n-1}