تحليل رياضي 2

المحاضرة 6

ميكاترونيكس

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٨. تمثيل التوابع بسلاسل القوى Representation of Functions by Power Series

سلاسل القوى الهندسية Geometric Power Series

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} = f(x) \; ; \; |x| < 1$$

$$f(x) = \frac{4}{x+2}$$

$$f(x) = \frac{4}{x+2} = \frac{4}{2\left(1-\left(\frac{-x}{2}\right)\right)} = \frac{2}{1-\left(\frac{-x}{2}\right)} = 2\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\cdots\right)$$

$$\left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2 \qquad \longrightarrow \qquad R = 2$$

$$f(x) = \frac{1}{x} \qquad c = 1$$

$$f(x) = \frac{1}{x} = \frac{1}{1-1+x} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (-(x-1))^n = 1-(x-1)+(x-1)^2+\cdots$$

$$|x-1| < 1 \longrightarrow R = 1 \ , \ ]0,2[$$

Operations with Power Series العمليات على سلاسل القوى

$$f(x)=\sum_{n=0}^{\infty}a_n x^n \quad , \quad g(x)=\sum_{n=0}^{\infty}b_n x^n$$

1. $$f(kx)=\sum_{n=0}^{\infty}a_n k^n x^n$$

2. $$f(x^N)=\sum_{n=0}^{\infty}a_n x^{nN}$$

3. $$f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n \pm b_n)x^n$$

$$\underbrace{\sum_{n=0}^{\infty}x^n}_{(-1,\,1)} + \underbrace{\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n}_{(-2,\,2)} = \underbrace{\sum_{n=0}^{\infty}\left(1+\frac{1}{2^n}\right)x^n}_{(-1,\,1)}$$

$$(-1,1)\cap(-2,2) = (-1,1)$$

Using Partition Fractions استخدام تفريق الكسور

$$f(x)=\frac{3x-1}{x^2-1} \qquad c=0$$

$$f(x)=\frac{3x-1}{x^2-1}=\frac{3x-1}{(x-1)(x+1)}=\frac{2}{x+1}+\frac{1}{x-1}$$

$$\left.\begin{array}{l} \dfrac{2}{x+1}=\dfrac{2}{1-(-x)}=\sum\limits_{n=0}^{\infty}2(-1)^n x^n \quad ; |x|<1 \\[2ex] \dfrac{1}{x-1}=-\dfrac{1}{1-x}=-\sum\limits_{n=0}^{\infty}x^n \quad ; |x|<1 \end{array}\right\} \frac{3x-1}{x^2-1}=\sum_{n=0}^{\infty}\left[2(-1)^n-1\right]x^n$$

Using Integral استخدام التكامل

$f(x) = \ln x \qquad c = 1$

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad]0,2[$$

$$f(x) = \ln x = \int \frac{1}{x} dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

$$x = 1 \Rightarrow C = 0$$

$$\longrightarrow \quad f(x) = \ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \qquad]0,2]$$

٩. سلاسل تايلورماكلورين Taylor and Maclaurin Series

ليكن f تابع قابل للاشتقاق عدد لا نهائي من المرات على مجال يحوي c ، عندئذ تعطى سلسلة تايلور للتابع f في النقطة $x=c$

Taylor Series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

وعندما $x=0$ نحصل على سلسلة ماكلورين

Maclaurin Series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

مثال أوجد سلسلة تايلور للتابع $f(x)=1/x$ في النقطة $a=2$

$$f(x)=x^{-1},\quad f'(x)=-x^{-2},\quad f''(x)=2!x^{-3},\ldots,f^{(n)}(x)=(-1)^n n!x^{-(n+1)},$$

$$f(2)=2^{-1}=\frac{1}{2},\quad f'(2)=-\frac{1}{2^2},\quad \frac{f''(2)}{2!}=2^{-3}=\frac{1}{2^3},\ldots,\frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{2^{n+1}}$$

$$f(2) + f'(2)(x-2) - \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

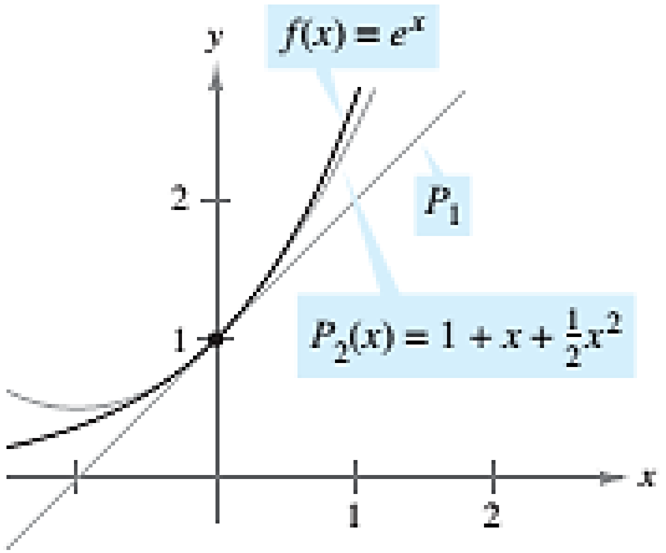
استخدام سلسلة ذي الحدين Using Binomial Series

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \quad |x|<1 \quad k \in \mathbb{Q}$$

$$f(x) = \sqrt[3]{1+x}$$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}x^3 + \cdots = 1 + \frac{1}{3}x - \frac{2}{3^2 2!}x^2 + \frac{10}{3^3 3!}x^3 + \cdots \qquad |x|<1$$

Taylor and Maclaurin Polynomials كثيرات حدود تايلوروماكلورين



كثيرة حدود تايلورللتابع f من الدرجة n في النقطة c

nth Taylor polynomial for f at c

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

nth Maclaurin polynomial for f كثيرة حدود ماكلورين للتابع f من الدرجة n

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n$$

أمثلة

$f(x)=\ln x \qquad P_0(x),\ P_1(x),\ P_2(x),\ P_3(x)\ ? \qquad c=1$

$f(x)=\ln x \qquad f(1)=\ln 1=0$

$f'(x)=\dfrac{1}{x} \qquad f'(1)=\dfrac{1}{1}=1$

$f''(x)=\dfrac{-1}{x^2} \qquad f''(1)=\dfrac{-1}{1}=-1$

$f'''(x)=\dfrac{2}{x^3} \qquad f'''(1)=\dfrac{2}{1}=2$

$$P_0(x)=f(1)=0$$

$$P_1(x)=f(1)+f'(1)(x-1)=(x-1)$$

$$P_2(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2=(x-1)-\frac{1}{2}(x-1)^2$$

$$P_3(x)=f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3$$

$f(x)=\cos x \quad P_0(x), P_2(x), P_4(x), P_6(x)$ **?** $c=0 \qquad \cos(0.1)$ **?**

$f(x)=\cos x \quad f(0)=\cos 0=1 \qquad f''(x)=-\cos x \quad f''(1)=-1$

$f'(x)=-\sin x \quad f'(0)=\sin 0=0 \qquad f'''(x)=\sin x \quad f'''(0)=0$

$f^{(4)}(x)=\cos x \quad f^{(4)}(0)=1 \qquad f^{(5)}(x)=-\sin x \quad f^{(5)}(0)=0$

$f^{(6)}(x)=-\cos x \quad f^{(6)}(0)=-1$

$P_0(x)=1 \qquad P_2(x)=1-\frac{1}{2!}x^2$

$P_4(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4 \qquad P_6(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6$

$$\cos(0.1)=1-\frac{1}{2!}(0.1)^2+\frac{1}{4!}(0.1)^4-\frac{1}{6!}(0.1)^6=0.995004165$$

$f(x)=\sin x \qquad P_3(x) \quad ? \quad c=\pi/6$

$$f(x)=\sin x \qquad f(\pi/6)=1/2$$

$$f'(x)=\cos x \qquad f'(\pi/6)=\sqrt{3}/2$$

$$f''(x)=-\sin x \qquad f''(\pi/6)=-1/2$$

$$f'''(x)=-\cos x \qquad f'''(\pi/6)=-\sqrt{3}/2$$

$$P_3(x)=f\left(\frac{\pi}{6}\right)+f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)+\frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2+\cdots+\frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3$$

$$=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

$\ln(1.1) \approx$ **?** $c = 0$ $P_4(x)$

$f(x) = \ln(1+x)$ $f(0) = 0$

$f'(x) = (1+x)^{-1}$ $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $f'''(0) = 2$

$f^{(4)}(x) = -6(1+x)^{-4}$ $f^{(4)}(0) = -6$

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

$$\ln(1.1) = \ln(1+0.1) \approx P_4(0.1) = (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 = 0.0953083$$

تمارين

1 أوجد كثيرات حدود تايلور $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ لكل من التوابع الآتية:

- $f(x) = e^{2x}, \quad a = 0$
- $f(x) = \sqrt{x}, \quad a = 4$
- $f(x) = \ln(1 + x), \quad a = 0$
- $f(x) = \sqrt{1 - x}, \quad a = 0$

الحل:

$f(x) = e^{2x}, \quad a = 0$

$$f(x)=e^{2x},\ f'(x)=2e^{2x},\ f''(x)=4e^{2x},\ f'''(x)=8e^{2x};\quad f(0)=e^{2(0)}=1, f'(0)=2,\ f''(0)=4,\ f'''(0)=8$$

$$\Rightarrow P_0(x)=1,\ P_1(x)=1+2x,\ P_2(x)=1+2x+2x^2,\ P_3(x)=1+2x+2x^2+\tfrac{4}{3}x^3$$

$f(x) = \sqrt{x}, \quad a = 4$

$$f(x)=\sqrt{x}=x^{1/2},\ f'(x)=\left(\tfrac{1}{2}\right)x^{-1/2},\ f''(x)=\left(-\tfrac{1}{4}\right)x^{-3/2},\ f'''(x)=\left(\tfrac{3}{8}\right)x^{-5/2};f(4)=\sqrt{4}=2,$$

$$f'(4)=\left(\tfrac{1}{2}\right)4^{-1/2}=\tfrac{1}{4},\ f''(4)=\left(-\tfrac{1}{4}\right)4^{-3/2}=-\tfrac{1}{32},\ f'''(4)=\left(\tfrac{3}{8}\right)4^{-5/2}=\tfrac{3}{256}$$

$$\Rightarrow P_0(x)=2,\ P_1(x)=2+\tfrac{1}{4}(x-4),$$

$$P_2(x)=2+\tfrac{1}{4}(x-4)-\tfrac{1}{64}(x-4)^2,\ P_3(x)=2+\tfrac{1}{4}(x-4)-\tfrac{1}{64}(x-4)^2+\tfrac{1}{512}(x-4)^3$$

تمارين

$f(x) = \ln(1 + x), \quad a = 0$

$$f(x) = \ln(1+x),\ f'(x) = \tfrac{1}{1+x} = (1+x)^{-1},\ f''(x) = -(1+x)^{-2},\ f'''(x) = 2(1+x)^{-3};$$

$$f(0) = \ln 1 = 0,\ f'(0) = \tfrac{1}{1} = 1,\ f''(0) = -(1)^{-2} = -1,\ f'''(0) = 2(1)^{-3} = 2$$

$$\Rightarrow P_0(x) = 0, P_1(x) = x, P_2(x) = x - \frac{x^2}{2}, P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$f(x) = \sqrt{1 - x}, \quad a = 0$

$$f(x) = (1-x)^{1/2},\ f'(x) = -\tfrac{1}{2}(1-x)^{-1/2},\ f''(x) = -\tfrac{1}{4}(1-x)^{-3/2},\ f'''(x) = -\tfrac{3}{8}(1-x)^{-5/2};$$

$$f(0) = (1)^{1/2} = 1,\ f'(0) = -\tfrac{1}{2}(1)^{-1/2} = -\tfrac{1}{2},\ f''(0) = -\tfrac{1}{4}(1)^{-3/2} = -\tfrac{1}{4},\ f'''(0) = -\tfrac{3}{8}(1)^{-5/2} = -\tfrac{3}{8}$$

$$\Rightarrow P_0(x) = 1,\ P_1(x) = 1 - \tfrac{1}{2}x, P_2(x) = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2,\ P_3(x) = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3$$

تمارين

2 أوجد سلسلة تايلور أو ماكلورين لكل من التوابع الآتية:

- $f(x) = 1/x^2, \quad a = 1$
- $f(x) = 1/(1-x)^3, \quad a = 0$
- $f(x) = 2^x, \quad a = 1$

الحل:

$f(x) = 1/x^2, \quad a = 1$

$$f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3},\ f''(x) = 3!x^{-4},\ f'''(x) = -4!x^{-5} \Rightarrow f^{(n)}(x) = (-1)^n (n+1)!x^{-n-2};$$

$$f(1) = 1,\ f'(1) = -2,\ f''(1) = 3!,\ f'''(1) = -4!,\ f^{(n)}(1) = (-1)^n (n+1)!$$

$$\Rightarrow \frac{1}{x^2} = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

$f(x) = 1/(1-x)^3, \quad a = 0$

$$f(x) = \frac{1}{(1-x)^3} \Rightarrow f'(x) = 3(1-x)^{-4},\ f''(x) = 12(1-x)^{-5},\ f'''(x) = 60(1-x)^{-6} \Rightarrow f^{(n)}(x) = \frac{(n+2)!}{2}(1-x)^{-n-3};$$

$$f(0) = 1,\ f'(0) = 3,\ f''(0) = 12,\ f'''(0) = 60, \ldots,\ f^{(n)}(0) = \frac{(n+2)!}{2}$$

تمارين

$$\Rightarrow \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \ldots = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$$

$f(x) = 2^x, \quad a = 1$

$$f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2,\ f''(x) = 2^x (\ln 2)^2,\ f'''(x) = 2^x (\ln 2)^3 \Rightarrow f^{(n)}(x) = 2^x (\ln 2)^n$$

$$f(1) = 2,\ f'(1) = 2\ln 2,\ f''(1) = 2(\ln 2)^2,\ f'''(1) = 2(\ln 2)^3, \ldots, f^{(n)}(1) = 2(\ln 2)^n$$

$$\Rightarrow 2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \ldots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n (x-1)^n}{n!}$$