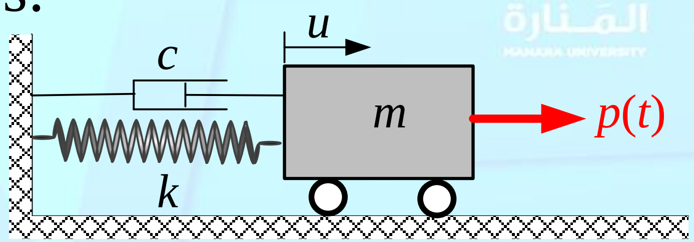
Free Response of a SDOF system (Undamped)

The equation of motion of a SDOF system has been written as follows:

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=p(t)$$



which is a non-homogeneous second-order linear ordinary differential equation with constant coefficients.

Its solution depends on the dynamic loading $p(t)$ and on the initial conditions.

Forced response is the solution of the equation, with $p(t) \neq 0$.

Free response is the solution of the homogeneous equation, with $p(t)=0$.

It describes the motion of a SDOF oscillator with non-zero initial conditions.

The *free response* is the solution of the homogeneous differential equation, with $p(t) = 0$.

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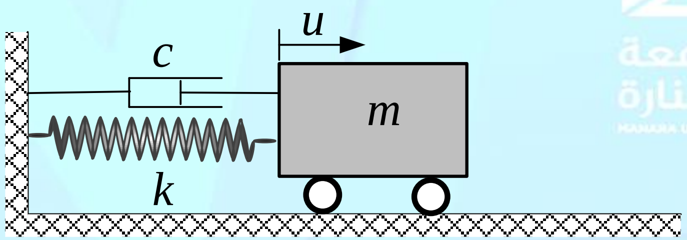
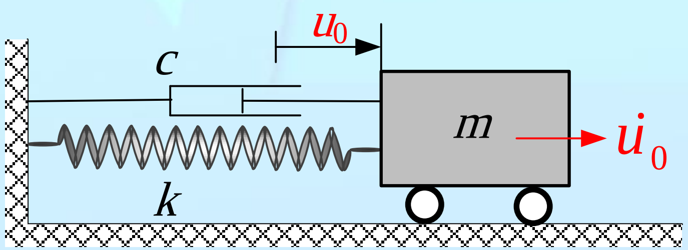


It describes the motion of a SDOF oscillator with non-zero initial conditions.

$$u_0 = u(0) \text{ and } \dot{u}_0 = \dot{u}(0)$$



First of all we write the equation of motion in its *canonical form.*

$$\ddot{u}(t)+\frac{c}{m}\dot{u}(t)+\frac{k}{m}u(t)=0$$

Putting: $$\omega^2=\frac{k}{m} \text{ and } 2\xi\omega=\frac{c}{m}$$

The canonical form of the equation of motion becomes $$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2u(t)=0$$

ω : is the angular frequency [rad/sec].

c : damping factor [kg/sec] or [N.sec/m] ξ : damping ration [?]

$$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2u(t)=0, \text{ with } u_0=u(0) \text{ and } \dot{u}_0=\dot{u}(0)$$

Characteristic equation

A particular solution for equation can be found considering that the variables $u, \dot{u}, and\ \ddot{u}$ are in some way linearly dependent for their sum to be zero. The exponential function has precisely that property and a solution could therefore be

$$u(t)=Ce^{st}$$

The constant C has dimension [L] and st has no dimension. Hence, constant s has dimension T^{-1}. Substituting this solution into the canonical equation we have

$$\left(s^2+2\xi\omega s+\omega^2\right)Ce^{st}=0$$

This equation is valid for all values of t, if

$$s^2+2\xi\omega s+\omega^2=0$$

which is known as the *characteristic equation.* It is an Algebraic Equation

$$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2u(t)=0, \text{ with } u_0=u(0) \text{ and } \dot{u}_0=\dot{u}(0)$$

$$u(t)=Ce^{st}$$

Undamped free response

The undamped free response is obtained with ($c = 0$, or $\xi = 0$) to reduces the characteristic equation to the form:

$$s^2+\omega^2=0$$

which has the following roots

$$s=\pm\omega\sqrt{-1}=\pm\omega i$$

Inserting these roots into equation of motion with $\xi=0$, we obtain the general solution

$$u(t)=C_1e^{i\omega t}+C_2e^{-i\omega t}$$

Now, from Euler's formulas: $e^{\pm i\theta}=\cos\theta\pm i\sin\theta$, the solution becomes

$$u(t)=(C_1+C_2)\cos\omega t+i(C_1-C_2)\sin\omega t$$

$$u(t) = (C_1 + C_2)\cos\omega t + i\,(C_1 - C_2)\sin\omega t$$

C_1 and C_2 are constants of integration and are both complex numbers. Since $u(t)$ *is* real, C_1 and C_2 must be complex conjugates because, if $C_1 = a + ib$ and $C_2 = a - ib$, then $C_1 + C_2 = 2a = A$, and $i(C_1 - C_2) = -2b = B$, hence real. We can rewrite the solution as

$$u(t) = A\cos\omega t + B\sin\omega t$$

where A and B are two new real arbitrary constants of integration.

The velocity can be expressed as

$$\dot{u}(t) = -\omega A\sin\omega t + \omega B\cos\omega t$$

A & B can be determined knowing the initial displacement and the initial velocity at time t = 0

$$u_0 = u(0) \text{ and } \dot{u}_0 = \dot{u}(0)$$

Substituting these initial conditions into the above two expressions and solving for A and B, we obtain

$$A = u_0 \text{ and } B = \frac{\dot{u}_0}{\omega}$$

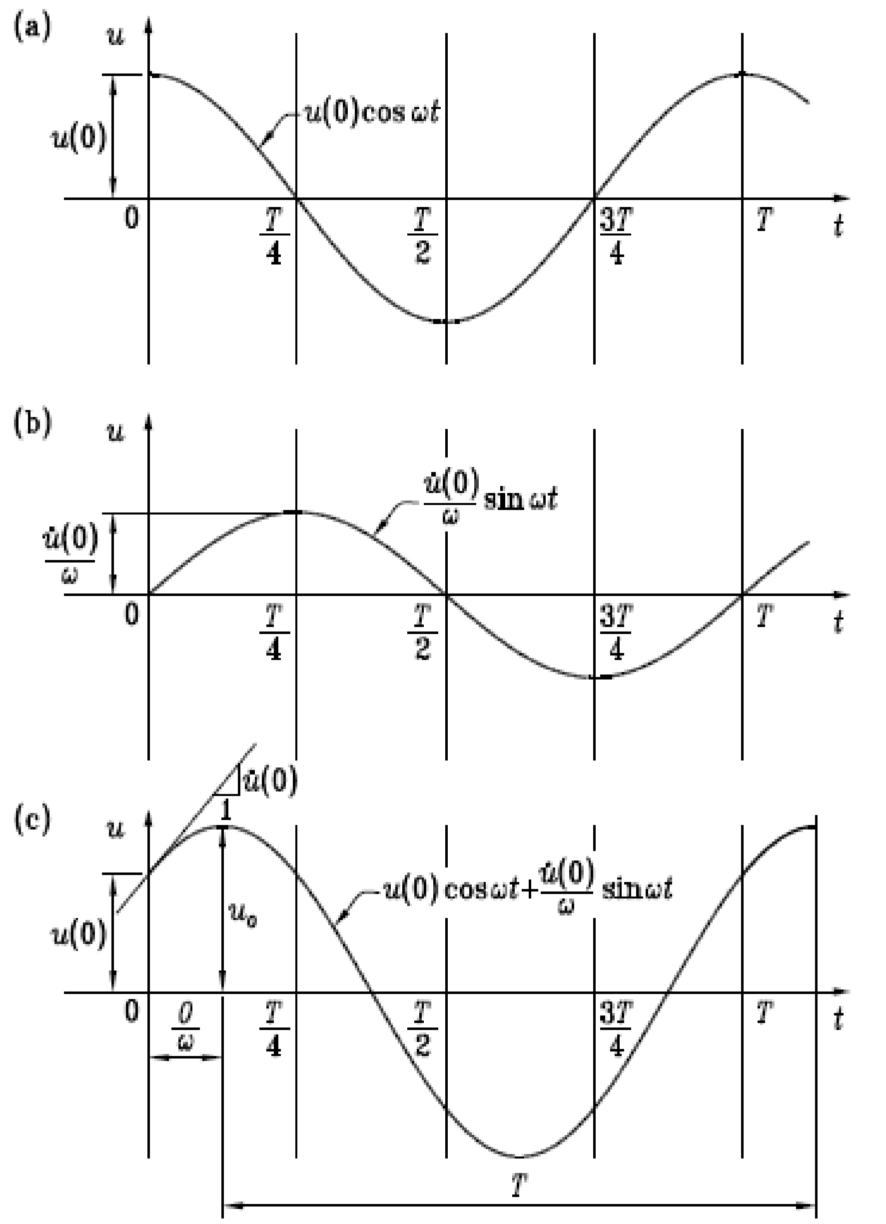
$$u(t) = u_0 \cos \omega t + \frac{\dot{u}_0}{\omega} \sin \omega t$$

The motion is defined by the sum of two harmonic functions as shown in Figures (a) and (b).

It is also a *simple harmonic motion* (SHM) shown in Figure (c).

The quantity ω is the *angular frequency* of the undamped SDOF system also known as harmonic oscillator.

The quantity ω is also called *circular frequency* and represents an angular velocity measured in radians per second (rad/s).



$$u(t) = u_0 \cos \omega t + \frac{\dot{u}_0}{\omega} \sin \omega t$$

An undamped SDOF system oscillates indefinitely in simple harmonic motion when the mass is released with initial conditions.

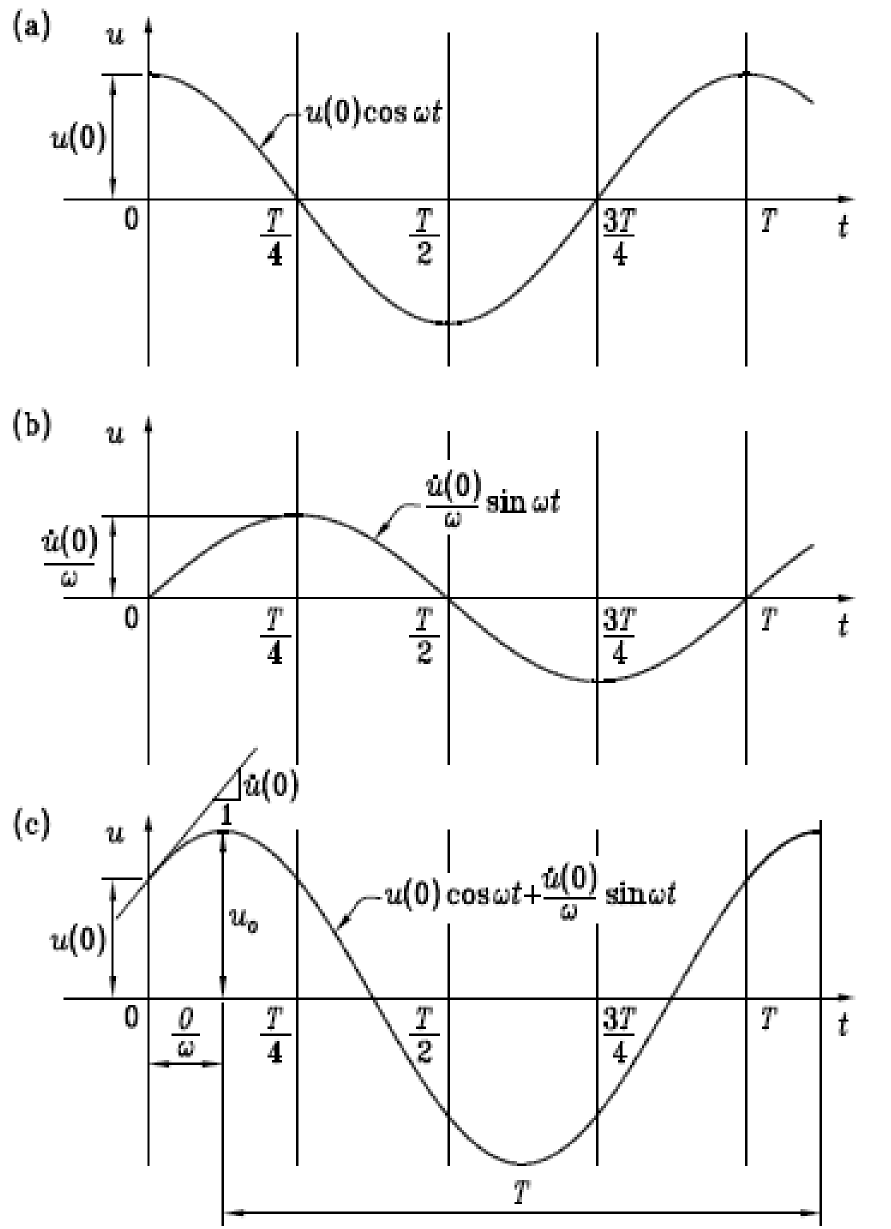
The *natural period is the* time T, that the system takes to complete one cycle of oscillation and is equal to the difference between two successive instants when both the displacement and the velocity repeat themselves.

$$\omega(t+T) = \omega t + 2\pi \Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{m/k}$$

The unit of T, is the second [s].

The number of cycles that the oscillator takes during one second is called *natural frequency* of the system, f, and is the reciprocal of the natural period T.

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$ unit is [Hz]



$$u(t) = A\cos\omega t + B\sin\omega t$$

$$u(t) = u_0\cos\omega t + \frac{\dot{u}_0}{\omega}\sin\omega t$$

A simpler and more instructive form of this equation can be obtained in replacing the arbitrary constants A and B, by two new constants.

$$A = C\cos\theta \text{ and } B = C\sin\theta.$$

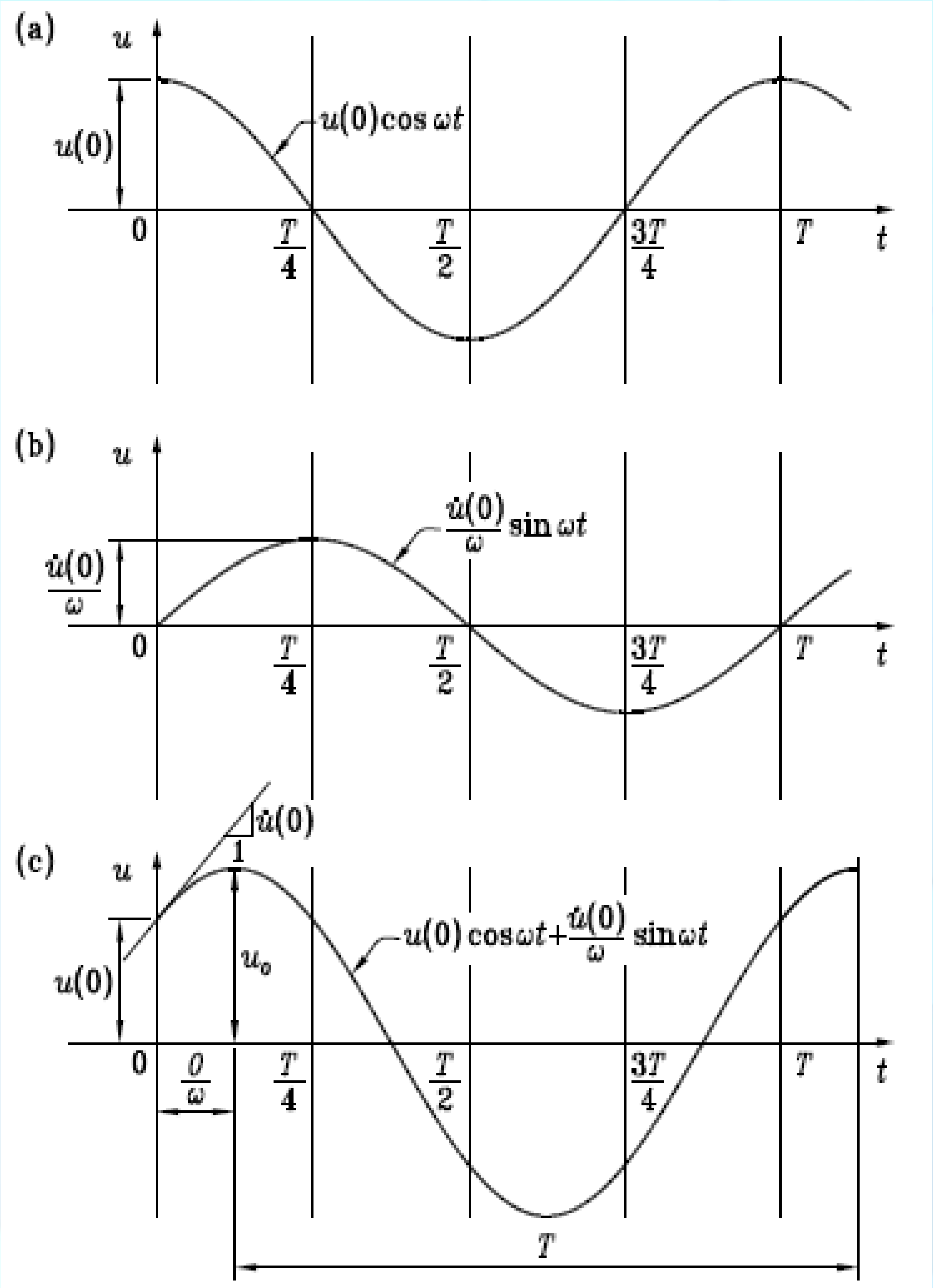
$$u(t) = C(\cos\omega t\cos\theta + \sin\omega t\sin\theta) \Rightarrow$$

$$u(t) = C\cos(\omega t - \theta)$$

where the two new arbitrary constants C and θ are linked to A and B as follows:

$$(C\cos\theta)^2 + (C\sin\theta)^2 = C^2 = A^2 + B^2$$

$$\tan\theta = C\sin\theta / C\cos\theta = B/A$$



$$u(t) = A\cos\omega t + B\sin\omega t$$

$$u(t) = u_0\cos\omega t + \frac{\dot{u}_0}{\omega}\sin\omega t$$

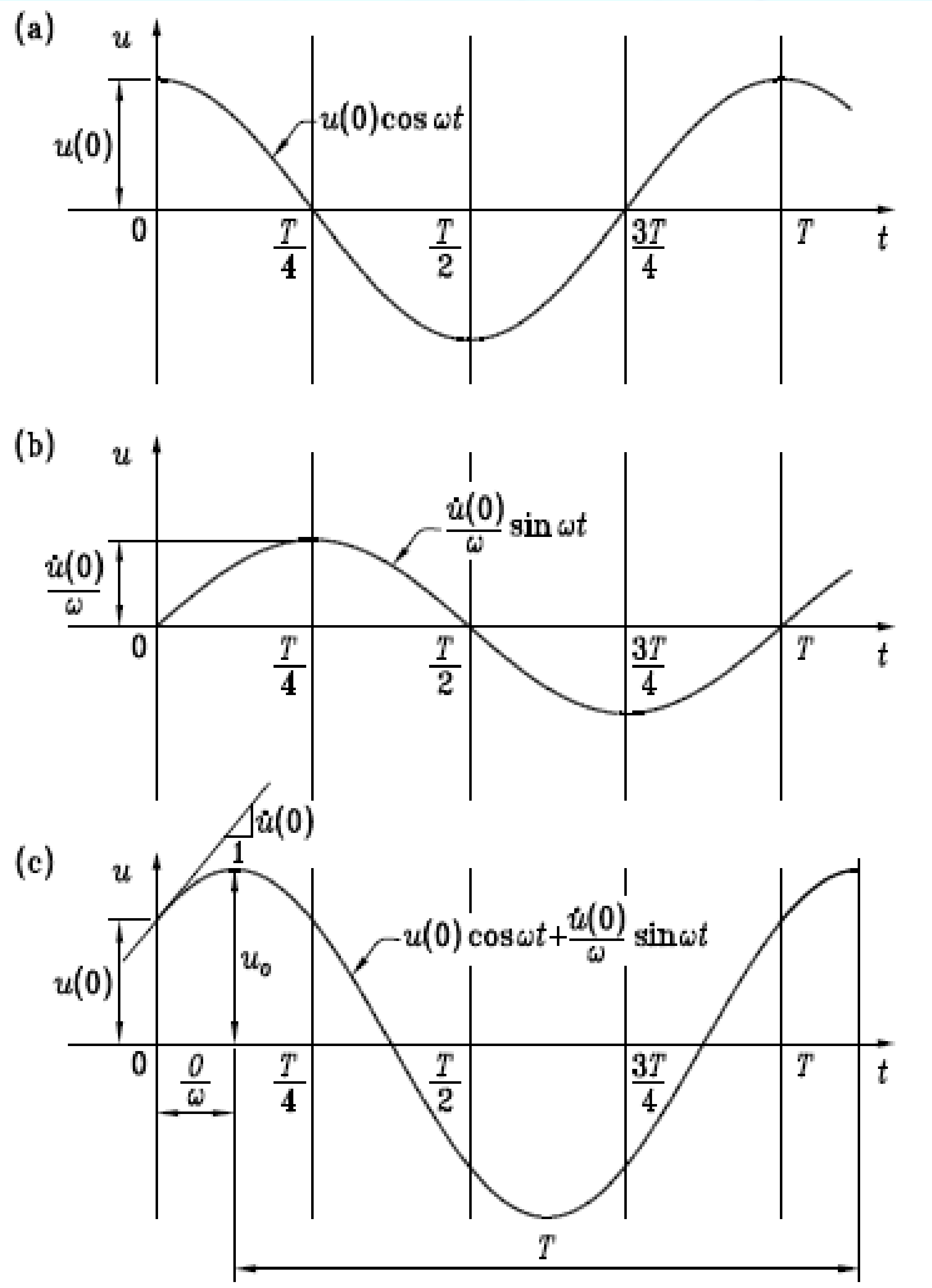
$$u(t) = C\cos(\omega t - \theta)$$

$C^2 = A^2 + B^2$ $\tan\theta = B/A$

$$u(t) = U\cos(\omega t - \theta)$$

$$U = \sqrt{(u_0)^2 + \left(\frac{\dot{u}_0}{\omega}\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{\dot{u}_0}{\omega u_0}\right)$$



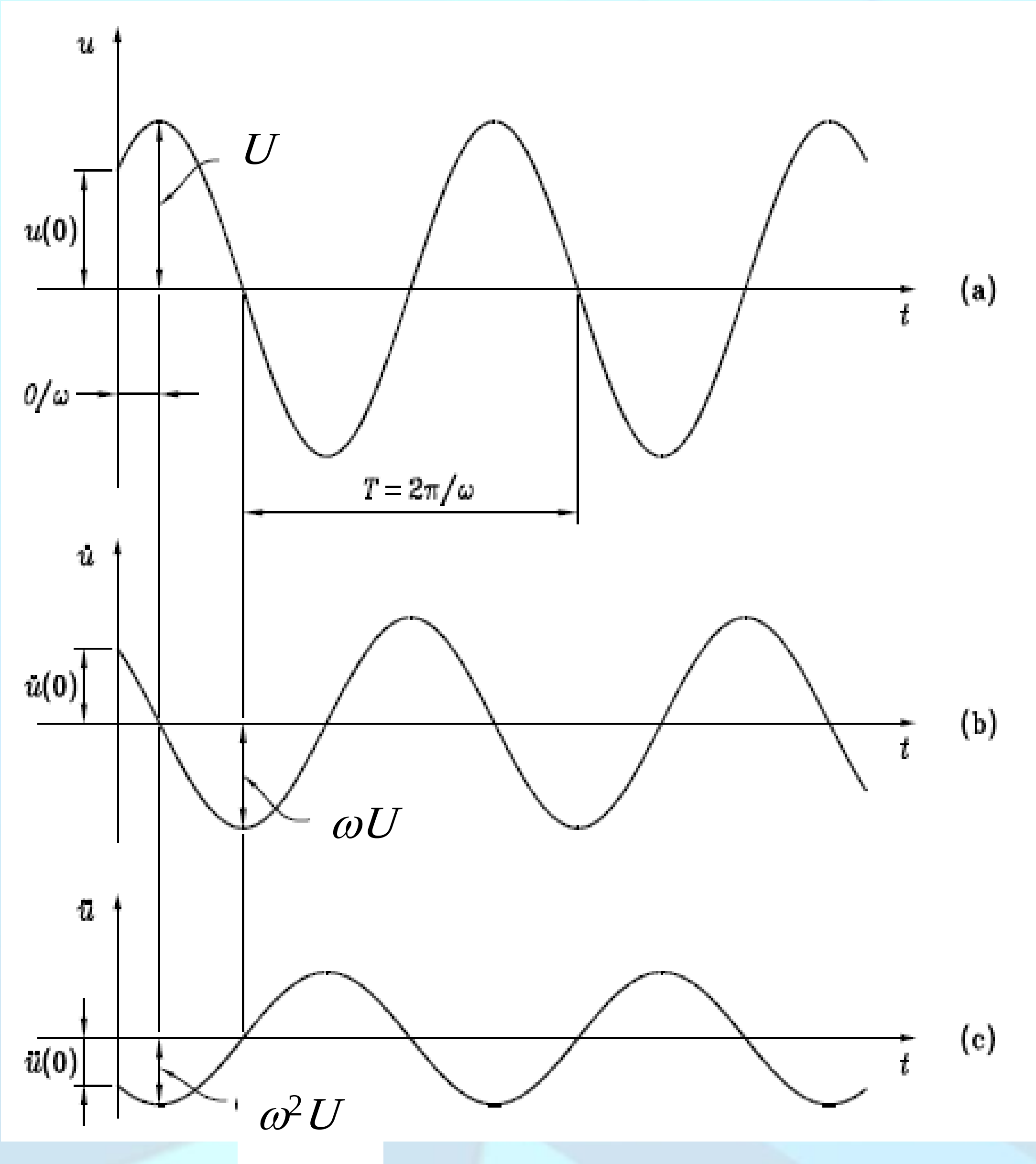
$$u(t) = U\cos(\omega t - \theta)$$

$$U = \sqrt{(u_0)^2 + \left(\frac{\dot{u}_0}{\omega}\right)^2}$$

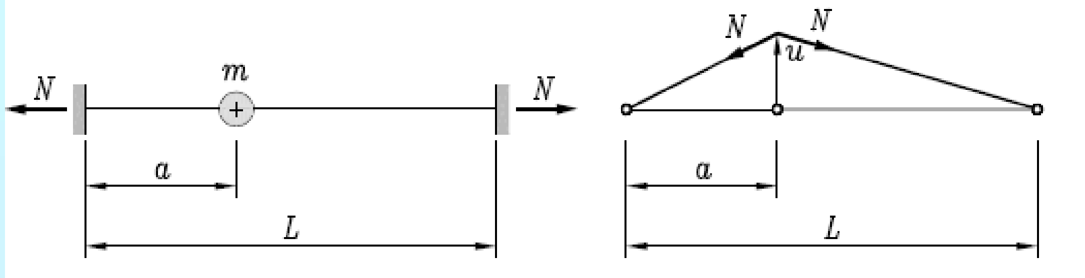
$$\theta = \tan^{-1}\left(\frac{\dot{u}_0}{\omega u_0}\right)$$

$$\dot{u}(t) = -\omega U\sin(\omega t - \theta)$$

$$\ddot{u}(t) = -\omega^2 U\cos(\omega t - \theta)$$



EX.1– A string of length L *is under tension* N *between two fixed supports* at its ends as shown in figure. The string has negligible mass and supports a concentrated mass m *at a distance* a *from the left support. Write the equation of* motion for small transverse oscillations of m *from the equilibrium position and find* the angular frequency.



SOLUTION: Let $u \ll L$ *be the displacement of the mass* m*. For small displacements,* the tension N can be considered constant. The mass m is subjected to a force F, which tends to move it toward the equilibrium position and whose value is

$$F = N\sin\theta_A + N\sin\theta_B.$$

Angles θ_A and θ_B are small because $u \ll L$. The sinus of these angles can then be taken as the tangent. We can write

$$\sin\theta_A = u/a, \quad \sin\theta_B = u/(L-a)$$

$$F = N\left(\frac{u}{a} + \frac{u}{L-a}\right) = N\left(\frac{1}{a} + \frac{1}{L-a}\right)u.$$

$$m\ddot{u} + ku = 0$$

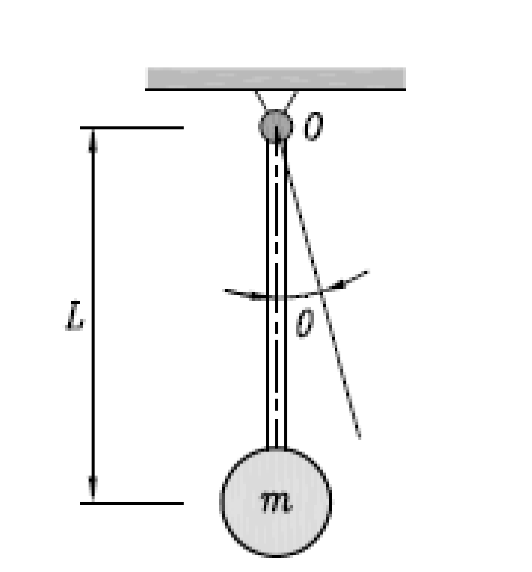
$$k = N\left(\frac{1}{a} + \frac{1}{L-a}\right)$$

$$\omega = \sqrt{\frac{NL}{ma(L-a)}} \quad (\text{rad/s}).$$

For the mass m *located at* $a = L/2$ *we have*

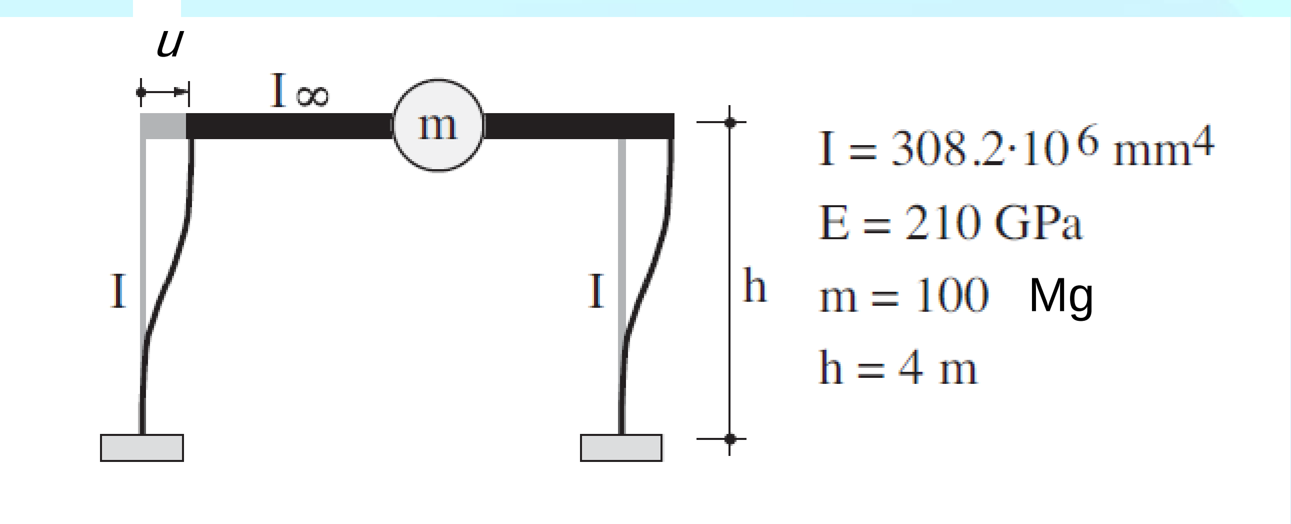
$$\omega = \sqrt{\frac{4N}{mL}} \quad (\text{rad/s})$$

EX. 2.– Find the natural period of the pendulum shown in figure for small oscillations about a vertical line passing through the hinge *O.*

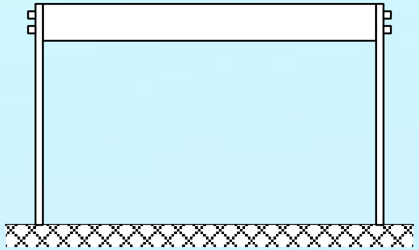


$$T = 2\pi\sqrt{L/g}.$$

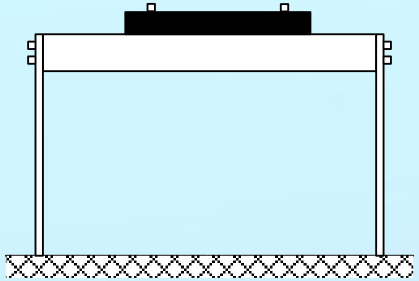
Ex. 3. Compute the natural period of each of the SDOF systems shown in figure.



Ex. 4. A Heavy table is supported by flat steel legs as on the figure. Its natural period in lateral vibration is 0.5 sec. When an additional mass of 25kg is clamped to its surface, the natural period in lateral vibration is lengthened to 0.75 sec. What are the table mass and the effective lateral stiffness of the table?

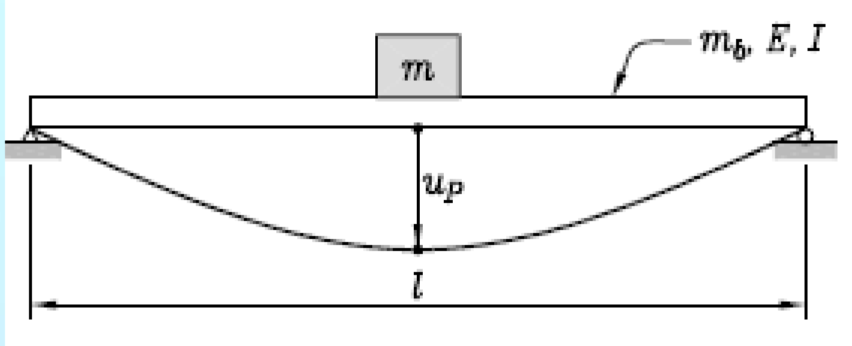


$T_n = 0.5$ sec.



$T_n = 0.75$ sec.

EX.5.– Find the natural period of the simply supported beam shown in figure, supporting a concentrated mass located at mid-span. Assume that the mass m_b of the beam is negligible compared to the mass *m.*



$$\omega = \sqrt{\frac{48EI}{ml^3}}.$$

A better approximation of the angular frequency of the beam supporting a mass at mid-span would be obtained by concentrating half of the mass m_b *of the* beam at mid-span.

The previous expression would still be applicable by replacing m by $(m + 0.5m_b)$.

Ex. 6. Compute the equivalent stiffness and the natural period of each of the SDOF systems shown in figure.