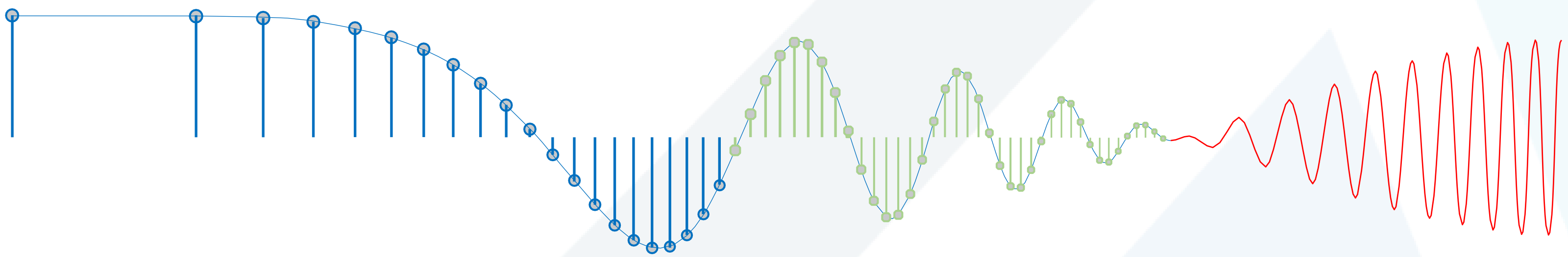
CEDC606: Digital Signal Processing

Lecture Notes 4: Fourier Representation of Signals



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

Chapter 4

Fourier representation of signals

1. Sinusoidal signals and their properties
2. Fourier representation of continuous-time signals
3. Fourier representation of discrete-time signals
4. Properties of the discrete-time Fourier transform

Jean-Baptiste Joseph Fourier
(1768-1830)

Born 21 March 1768 in Auxerre, Kingdom of France

Died 16 May 1830 (aged 62) in Paris, Kingdom of France

1. Sinusoidal signals and their properties

- Most signals of practical interest can be described as a sum or integral of sinusoidal signals.
- However, the exact form of the representation depends on whether the signal is continuous-time or discrete-time and whether it is periodic or aperiodic.
- For the class of periodic signals, such a decomposition is called a Fourier series. For the class of finite energy signals, the decomposition is called the Fourier transform.
- These decompositions are extremely important in the analysis of LTI systems because the response of an LTI system to a sinusoidal input signal is a sinusoid of the same frequency but of different amplitude and phase.

Continuous-time sinusoids

$$x(t) = A\cos(2\pi F_0 t + \phi), \qquad -\infty < t < \infty$$

where A is the amplitude, ϕ is the phase in radians, and F_0 is the frequency. The units of F_0 are cycles per second or Hertz (Hz). The angular frequency $\Omega_0 = 2\pi F_0$ measured in radians per second.

$$A\cos(\Omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\Omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\Omega_0 t}$$

Therefore, we can study the properties of the sinusoidal signal by studying the properties of the complex exponential $x(t) = e^{j\Omega_0 t}$.

- Let us determine the response $y(t)$ of LTI system to the input $x(t) = e^{j\Omega t}$ using the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$y(t)=\int_{-\infty}^{\infty}h(\tau)e^{j\Omega(t-\tau)}d\tau=\int_{-\infty}^{\infty}h(\tau)e^{j\Omega t}e^{-j\Omega\tau}d\tau=\left(\int_{-\infty}^{\infty}h(\tau)e^{-j\Omega\tau}d\tau\right)e^{j\Omega t}$$

$$y(t)=H(j\Omega)e^{j\Omega t},\qquad -\infty<t<\infty$$

- This implies that the complex exponentials are eigenfunctions of continuous-time LTI systems.
- For a specific value of Ω, the constant $H(j\Omega)$ is an eigenvalue associated with the eigenfunction $e^{j\Omega t}$.
- A set of harmonically related complex exponential signals, with fundamental frequency $\Omega_0=2\pi F_0=2\pi/T_0$, is defined by:

$$s_k(t)=e^{jk\Omega_0 t}=e^{j2\pi kF_0 t},\qquad k=0,\pm1,\pm2,\cdots$$

- We say that $s_1(t)$ is the fundamental harmonic of the set and $s_k(t)$ is the kth harmonic of the set. Clearly all harmonics $s_k(t)$ are periodic with period T_0.

- Furthermore a very important characteristic of harmonically related complex exponentials is the following orthogonality property:

$$\int_{T_0} s_m(t)s_n^*(t)dt = \int_{T_0} e^{jm\Omega_0 t}e^{-jn\Omega_0 t}dt = \begin{cases} T_0, & m = n \\ 0, & m \neq n \end{cases}$$

Discrete-time sinusoids

- A discrete-time sinusoidal signal is conveniently obtained by sampling the continuous-time sinusoid at equally spaced points $t = nT$:

$$x[n] = x(nT) = A\cos(2\pi F_0 nT + \phi) = A\cos(2\pi \frac{F_0}{F_s} n + \phi)$$

If we define the normalized frequency variable: $f = \frac{F}{F_s} = FT$

and the normalized angular frequency variable: $\omega = 2\pi f = 2\pi \frac{F}{F_s} = \Omega T$

$$x[n] = A\cos(2\pi f_0 n + \phi) = A\cos(\omega_0 n + \phi), \quad -\infty < n < \infty$$

- The response $y[n]$ of LTI system to the input $x[n] = e^{j\omega n}$:

$$x[n] = e^{j\omega n} \Rightarrow y[n] = H(e^{j\omega})e^{j\omega n}, \quad \text{for all } n$$

which is obtained from z-transform by setting $z = e^{j\omega}$. Thus, the complex exponentials $e^{j\omega n}$ are eigenfunctions of discrete-time LTI systems with eigenvalues given by the system function $H(z)$ evaluated at $z = e^{j\omega}$.

Periodicity in time: The sequence $x[n] = A\cos(\omega_0 n + \phi)$ is periodic of period N if and only if $f_0 = k/N$, that is, f_0 is a rational number. If k and N are a pair of prime numbers, then N is the fundamental period of $x[n]$. $f_0 = F_0/F_s = k/N = T/T_0$

Periodicity in frequency: The sequence $x[n] = A\cos(\omega_0 n + \phi)$ is periodic in ω_0 with fundamental period 2π and periodic in f_0 with fundamental period one.

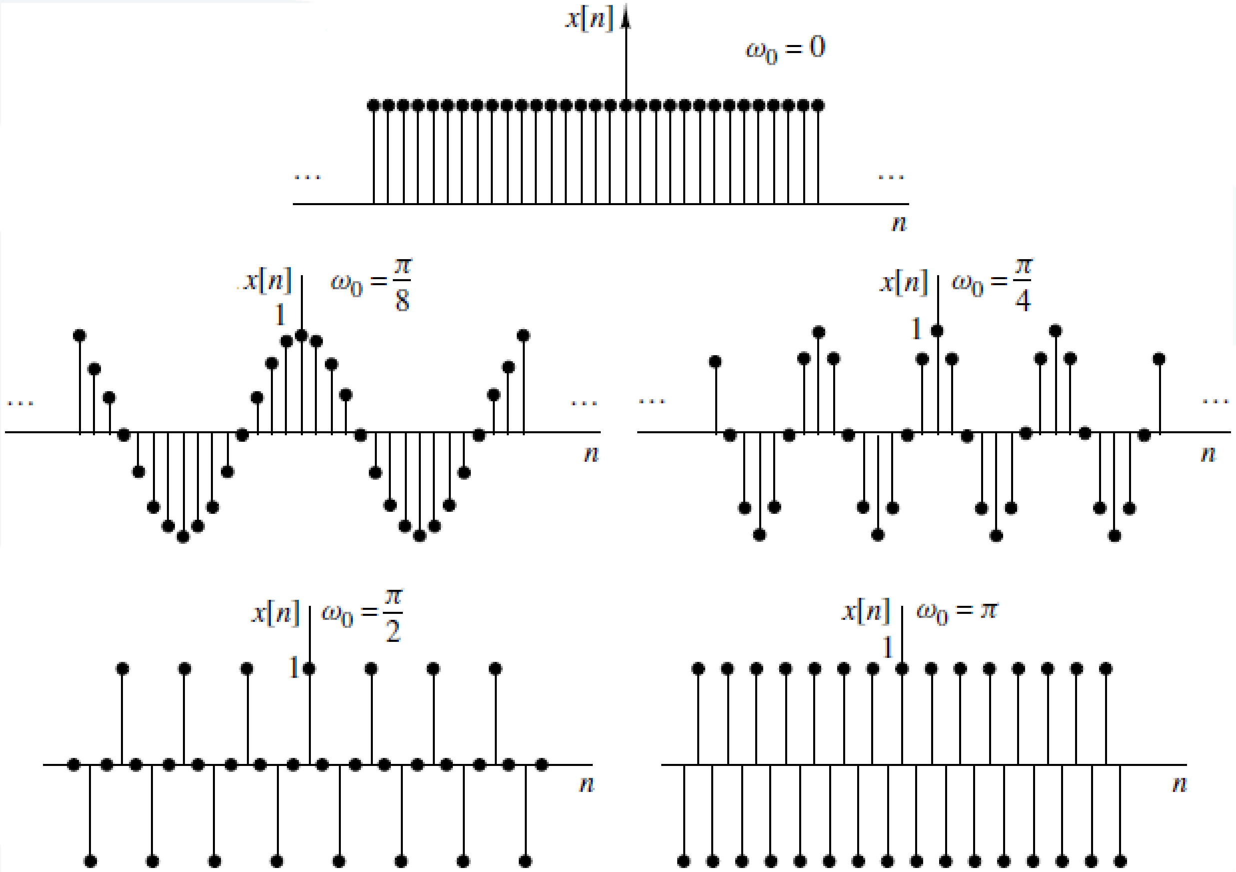
- All distinct sinusoidal sequences have frequencies within an interval of 2π rads. We shall use the fundamental frequency ranges $-\pi < \omega \leq \pi$ or $0 \leq \omega < 2\pi$.

- The rate of oscillation of a DT sinusoid increases as ω_0 increases from $\omega_0 = 0$ to $\omega_0 = \pi$. However, as ω_0 increases from $\omega_0 = \pi$ to $\omega_0 = 2\pi$, the oscillations become slower.

- Similar properties hold for the DT complex exponentials:

$$s_k[n] = A_k e^{j\omega_k n}, -\infty < n < \infty$$

- For $s_k[n]$ to be periodic with fundamental period N, the frequency ω_k should be a rational multiple of 2π, that is, $\omega_k = 2\pi k/N$.



- Therefore, all distinct complex exponentials with period N and frequency in the fundamental range, have frequencies given by $\omega_k = 2\pi k/N$, $k = 0, 1, ..., N-1$. The set of sequences:

$$s_k[n] = e^{j2\pi kn/N}, \quad -\infty < k, n < \infty$$

are periodic both in n (time) and k (frequency) with fundamental period N.

- There are only N distinct harmonically related complex exponentials with fundamental frequency $f_0 = 1/N$ and harmonics at frequencies $f_k = k/N$, $0 \le k \le N-1$.

$$s_k[n+N] = s_k[n], \qquad \text{(periodic in time)}$$

$$s_{k+N}[n] = s_k[n]. \qquad \text{(periodic in frequency)}$$

$$\sum_{n=\langle N\rangle} s_k[n]s_m^*[n] = \sum_{n=\langle N\rangle} e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}mn} = \begin{cases} N, & k = m \\ 0, & k \neq m \end{cases} \quad \text{(orthogonality property)}$$

2. Fourier representation of continuous-time signals

Fourier series for continuous-time periodic signals

- The continuous-time Fourier series representation (CTFS) of a periodic signal, when it exists, is defines as follows:

  1. Synthesis equation: $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$

  2. Analysis equation: $c_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t)\, e^{-jk\Omega_0 t} dt$

- The set of coefficients $\{c_k\}$ are known as the Fourier series coefficients.
- The plot of c_k as a function of frequency $F = kF_0$ (spectrum) constitutes a description of the signal in the frequency-domain.

- The plot of $|c_k|$ ($c_k = |c_k| e^{j\phi_k}$) is known as the magnitude spectrum of $\tilde{x}(t)$, while the plot of ϕ_k is known as the phase spectrum of $\tilde{x}(t)$.
- Parseval's relation: The average power in one period of $\tilde{x}(t)$ can be expressed in terms of the Fourier coefficients using Parseval's relation:

$$\mathcal{P}_{av} = \frac{1}{T_0} \int_{T_0} |\tilde{x}(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

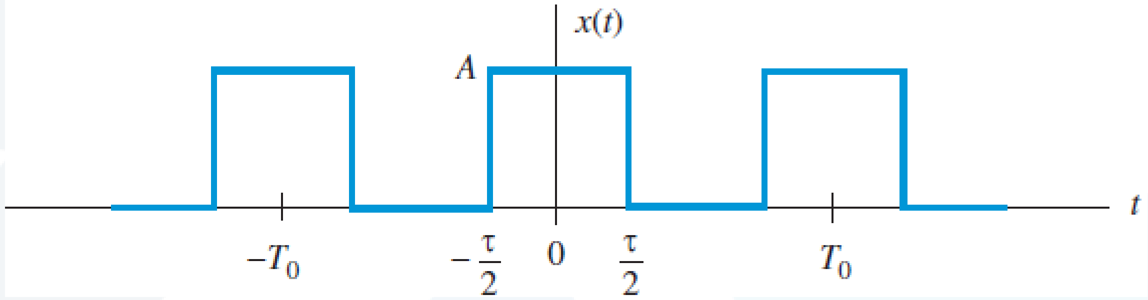
- The value of $|c_k|^2$ provides the portion of the average power of signal $\tilde{x}(t)$ that is contributed by the k^{th} harmonic.
- The graph of $|c_k|^2$ is known as the power spectrum of $\tilde{x}(t)$. (discrete spectra).

Fourier transforms for continuous-time aperiodic signals

- We can think of an aperiodic signal as a periodic signal with infinite period.

- We can think of an aperiodic signal as a periodic signal with infinite period. For example let us start with the rectangular pulse train:

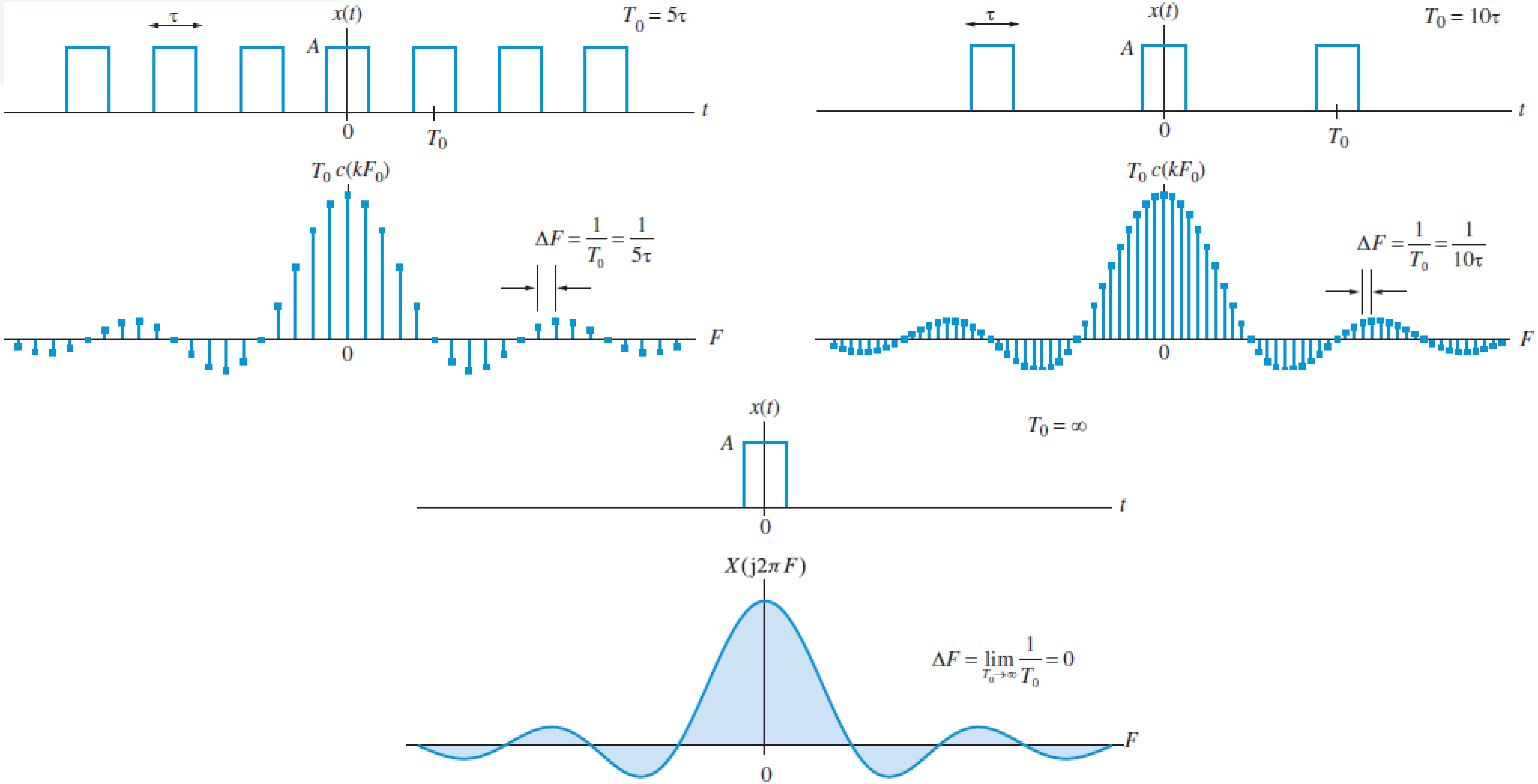
$$x(t) = \begin{cases} A, & |t| < \tau/2 \\ 0, & \tau/2 < |t| < T_0/2 \end{cases}$$



and periodically repeats with period $T_0 > \tau$

$$c_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = \frac{A\tau}{T_0}\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}, \; k = 0, \pm 1, \pm 2, \cdots$$

- As $T_0 \rightarrow \infty$ (a) in the time domain the result is an aperiodic signal corresponding to one period of the rectangular pulse train, and (b) in the frequency domain the result is a "continuum" of spectral lines.



τ
x(t)
T0 = 5τ
A
t
0
T0
T0 c(kF0)
ΔF = 1/T0 = 1/5τ
F
0
τ
x(t)
T0 = 10τ
A
t
0
T0
T0 c(kF0)
ΔF = 1/T0 = 1/10τ
F
0
x(t)
T0 = ∞
A
t
0
X(j2π F)
ΔF = lim T0→∞ 1/T0 = 0
F
0

- The continuous-time Fourier transform representation (CTFT) of an aperiodic signal, when it exists, is defines as follows:

1. Synthesis equation: $x(t) = \mathcal{F}^{-1}\{X(j2\pi F)\} = \int_{-\infty}^{\infty} X(j2\pi F)\, e^{j2\pi Ft} dF$

$$x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t} d\Omega$$

2. Analysis equation: $X(j2\pi F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft} dt$

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t} dt$$

- Parseval's relation: For aperiodic signals with finite energy, Parseval's relation is given by:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(j2\pi F)|^2\, dF$$

- The quantity $|X(j2\pi F)|^2$ is known as the energy-density spectrum of $x(t)$.

3. Fourier representation of discrete-time signals

Fourier series for discrete-time periodic signals

- The discrete Fourier series representation (DTFS) of a periodic signal, is defines as follows:

1. Synthesis equation: $\tilde{x}[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$

2. Analysis equation: $\tilde{c}_k = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$

- Parseval's relation: The average power in one period of $\tilde{x}[n]$ can be expressed as:

$$\mathcal{P}_{av} = \frac{1}{N}\sum_{n=0}^{N-1} \left|\tilde{x}[n]\right|^2 = \sum_{k=0}^{N-1} \left|\tilde{c}_k\right|^2$$

- The value of $|\tilde{c}_k|^2$ provides the portion of the average power of signal $\tilde{x}[n]$ that is contributed by the k^{th} harmonic.
- There are only N distinct harmonic components.
- The graph of $|\tilde{c}_k|^2$ as a function of $f = k/N$, $\omega = 2\pi k/N$, or simply k, is known as the power spectrum of the periodic signal $\tilde{x}[n]$.

Fourier transforms for discrete-time aperiodic signals

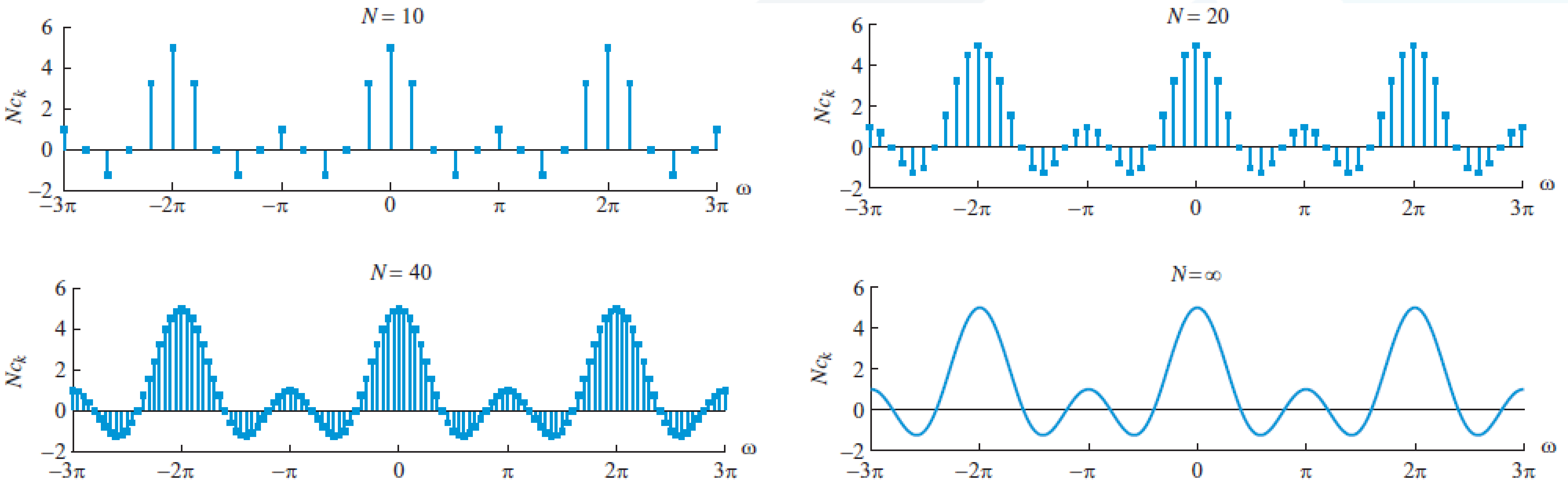
- An aperiodic signal as a periodic signal with infinite period, we could obtain its Fourier representation by taking the limit of DTFS as the period increases indefinitely. For example let us start with the rectangular pulse train:

$$x[n] = \begin{cases} 1, & |n| \le L \\ 0, & L < |n| < N/2 \end{cases}$$

and periodically repeats with period $N > 2L + 1$

$$c_k = \frac{1}{N} \frac{\sin\left[\frac{2\pi}{N} k \left(L + \frac{1}{2}\right)\right]}{\sin\left(\frac{2\pi}{N} k \frac{1}{2}\right)}$$

- As $N \to \infty$, $x[n]$ becomes an aperiodic sequence and its Fourier representation becomes a continuous function of ω.



- The discrete Fourier transform representation (DTFT) of aperiodic signal, is defines as follows:

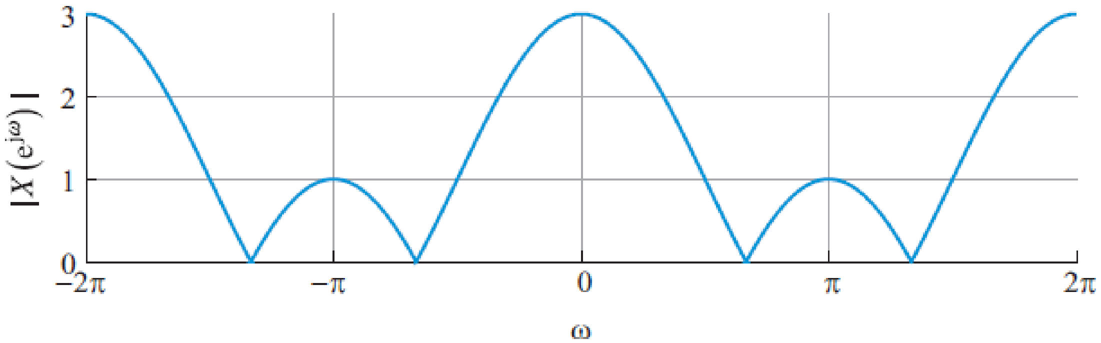
1. Synthesis equation: $x[n] = \mathcal{F}^{-1}\{\tilde{X}(e^{j\omega})\} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{X}(e^{j\omega})\, e^{j\omega n} d\omega$

2. Analysis equation: $\tilde{X}(e^{j\omega}) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

- Example 1: Finite length pulse

  Evaluate and plot the magnitude and phase of the DTFT of the sequence $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$

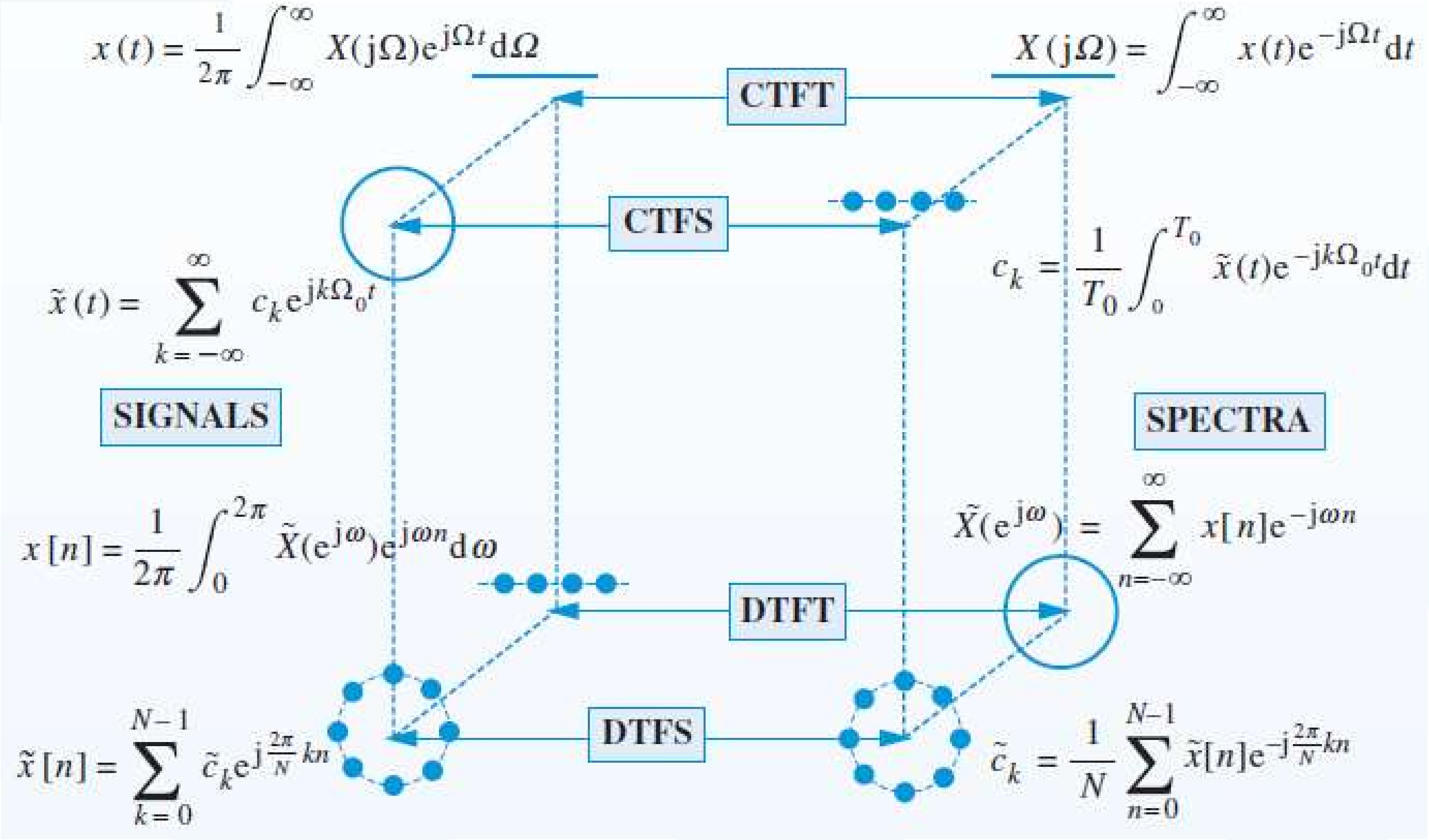
$$X(e^{j\omega}) = \sum_{n=-1}^{1} x[n] e^{-j\omega n} = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos(\omega)$$



- Parseval's relation: If $x[n]$ has finite energy, we have the following Parseval's relation:

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|\tilde{X}(e^{j\omega})\right|^2 d\omega$$

- The quantity $|\widetilde{X}(e^{j\omega})/2\pi|^2$ or $|\widetilde{X}(e^{j2\pi f})|^2$ is known as the energy-density spectrum of $x[n]$.
- There are four types of signal and related Fourier transform and series representations which are summarized in the figure below:


$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$
CTFT
CTFS
$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$
$c_k = \frac{1}{T_0}\int_0^{T_0} \tilde{x}(t)e^{-jk\Omega_0 t}dt$
SIGNALS
SPECTRA
$x[n] = \frac{1}{2\pi}\int_0^{2\pi} \tilde{X}(e^{j\omega})e^{j\omega n}d\omega$
$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
DTFT
DTFS
$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$

Summary of Fourier series and Fourier transforms

- Continuous-time periodic signals have discrete aperiodic spectrum. The spectrum exists only at $F = 0, \pm F_0, \pm 2F_0, ...$, that is, at discrete values of F. The spacing between the lines of this discrete or line spectrum is $F_0 = 1/T_0$.
- Continuous-time aperiodic signals have continuous aperiodic spectrum over the entire frequency axis. The spectrum exists for all F, $-\infty < F < \infty$.
- Discrete-time periodic signals have discrete periodic spectrum. The spacing between the lines of the resulting discrete spectrum is $\omega = 2\pi/N$.
- Discrete-time aperiodic signals have continuous periodic spectrum of period 2π.
- Note: Periodicity with "period" α in one domain implies discretization with "spacing" of $1/\alpha$ in the other domain, and vice versa.

- Bandlimited signals: Signals whose frequency components are zero or "small" outside a finite interval $0 \leq B_1 \leq |F| \leq B_2 < \infty$ are said to be bandlimited. The quantity $B = B_2 - B_1$ is known as the bandwidth of the signal.
- Bandlimited signals cannot be time-limited, and vice versa; Time-limited signals cannot be bandlimited.

4. Properties of the discrete-time Fourier transform

Relationship to the z-transform and periodicity

- If the ROC of $X(z)$ includes the unit circle, defined by $z = e^{j\omega}$ or equivalently $|z| = 1$, we obtain:

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \tilde{X}(e^{j\omega})$$

the z-transform reduces to the Fourier transform.

Symmetry properties

- Suppose that both the signal $x[n]$ and its DTFT $X(e^{j\omega})$ are complex-valued functions. Then: $x[n] = x_R[n] + jx_I[n]$, and $X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \begin{cases} X_R(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \{x_R[n]\cos(\omega n) + x_I[n]\sin(\omega n)\} \\ X_I(e^{j\omega}) = -\sum_{n=-\infty}^{\infty} \{x_R[n]\sin(\omega n) - x_I[n]\cos(\omega n)\} \end{cases}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{X}(e^{j\omega})\, e^{j\omega n} d\omega \qquad \begin{cases} x_R[n] = \frac{1}{2\pi}\int_{2\pi} [X_R(e^{j\omega})\cos(\omega n) - X_I(e^{j\omega})\sin(\omega n)]d\omega \\ x_I[n] = \frac{1}{2\pi}\int_{2\pi} [X_R(e^{j\omega})\sin(\omega n) + X_I(e^{j\omega})\cos(\omega n)]d\omega \end{cases}$$

- Real signals If $x[n]$ is real, then $x_R[n] = x[n]$ and $x_I[n] = 0$

$$X_R(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n) \text{ and } X_I(e^{j\omega}) = -\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$$

$X_R(e^{-j\omega}) = X_R(e^{j\omega})$ (even symmetry)

$X_I(e^{-j\omega}) = -X_I(e^{j\omega})$ (odd symmetry)

$\Leftrightarrow$

$X^*(e^{j\omega}) = X(e^{-j\omega})$

(Hermitian symmetry)

Thus, the DTFT of a real signal has Hermitian (or complex-conjugate) symmetry.

The inverse DTFT is given by: $x[n] = \frac{1}{2\pi}\int_{2\pi}[X_R(e^{j\omega})\cos(\omega n) - X_I(e^{j\omega})\sin(\omega n)]d\omega$

Since $X_R(e^{j\omega})\cos(\omega n)$ and $X_I(e^{j\omega})\sin(\omega n)$ are even functions of ω, we have

$$x[n] = \frac{1}{\pi}\int_0^{\pi}[X_R(e^{j\omega})\cos(\omega n) - X_I(e^{j\omega})\sin(\omega n)]d\omega$$

- Real and even signals If $x[n]$ is real and even, that is, $x[-n] = x[n]$, then:

$$\begin{cases} X_R(e^{j\omega}) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n) & \text{(even symmetry)} \\ X_I(e^{j\omega}) = 0 & \\ x[n] = \frac{1}{\pi}\int_0^{\pi} X_R(e^{j\omega})\cos(\omega n)d\omega & \text{(even symmetry)} \end{cases}$$

Thus, real signals with even symmetry have real spectra with even symmetry.

- Real and odd signals If $x[n]$ is real and odd, that is, $x[-n] = -x[n]$, then:

$$\begin{cases} X_R(e^{j\omega}) = 0 & \\ X_I(e^{j\omega}) = -2\sum_{n=1}^{\infty} x[n]\sin(\omega n) & \text{(odd symmetry)} \\ x[n] = -\frac{1}{\pi}\int_0^{\pi} X_I(e^{j\omega})\sin(\omega n)d\omega & \text{(odd symmetry)} \end{cases}$$

Thus, real signals with odd symmetry have purely imaginary spectra with odd symmetry.

Property	$x[n]$	$X(e^{j\omega})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Time shifting	$x[n-k]$	$X(e^{j\omega})e^{-jk\omega}$
Frequency shifting	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$
Time reversal (Folding)	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Modulation	$x[n]\cos(\omega_0 n)$	$\frac{1}{2}[X(e^{j(\omega+\omega_0)}) + X(e^{j(\omega-\omega_0)})]$
Differentiation	$nx[n]$	$-j\,dX(e^{j\omega})/d\omega$
Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})\,X_2(e^{j\omega})$
Windowing	$x[n]w[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]$	$\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})\,X_2^*(e^{j\omega})d\omega$

- Example 2: Causal exponential sequence

Consider the sequence $x[n] = a^n u[n]$. For $|a| < 1$, the sequence is absolutely summable, that is:

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} < \infty$$

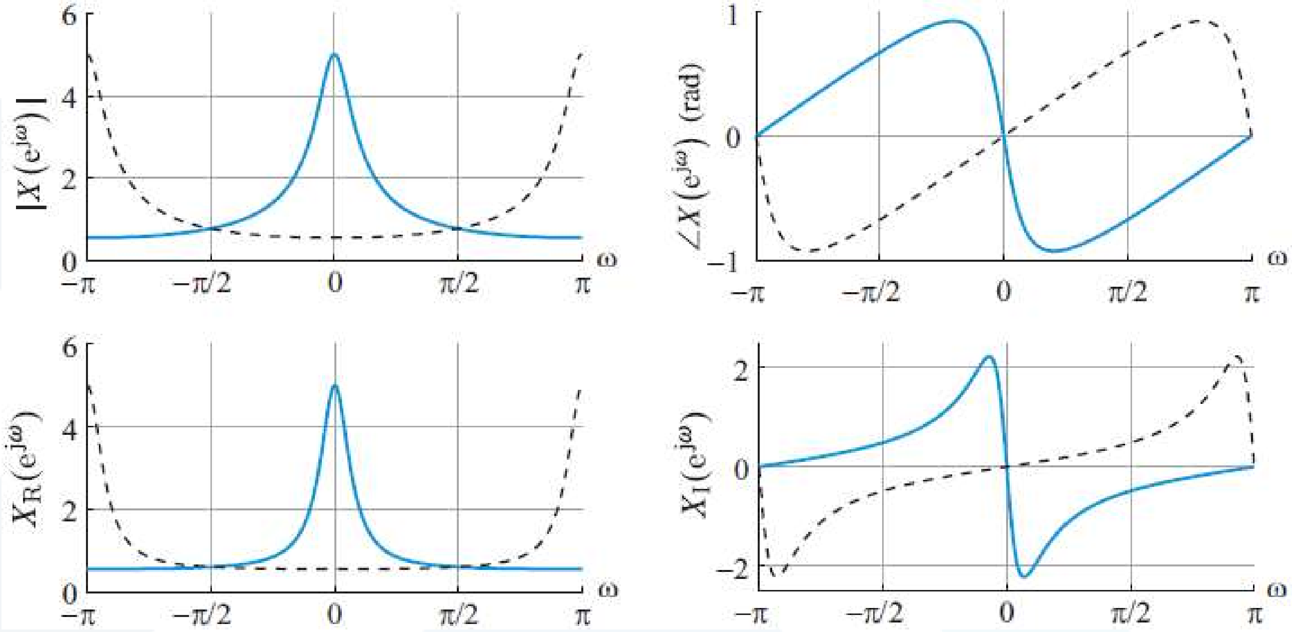
The DTFT exists and is given by: $X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}, \quad |a| < 1$

If $x[n]$ is real: $X_R(e^{j\omega}) = \frac{1-a\cos(\omega)}{1-2a\cos(\omega)+a^2} = X_R(e^{-j\omega})$ (even)

$$X_I(e^{j\omega}) = \frac{-a\sin(\omega)}{1-2a\cos(\omega)+a^2} = -X_I(e^{-j\omega}) \quad \text{(odd)}$$

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1-2a\cos\omega+a^2}} = \left|X(e^{-j\omega})\right|, \quad \angle X(e^{j\omega}) = \tan^{-1}\frac{-a\sin\omega}{1-a\cos\omega} = -\angle X(e^{-j\omega})$$

- Note: Lowpass sequence for ($0 < a < 1$) and highpass sequence ($-1 < a < 0$).

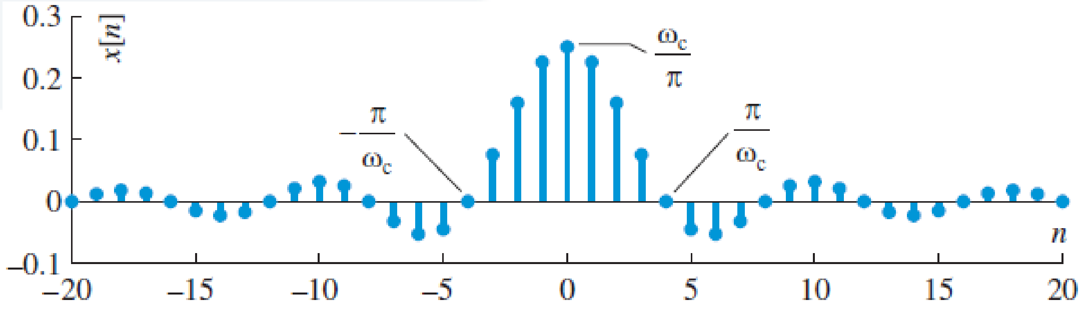


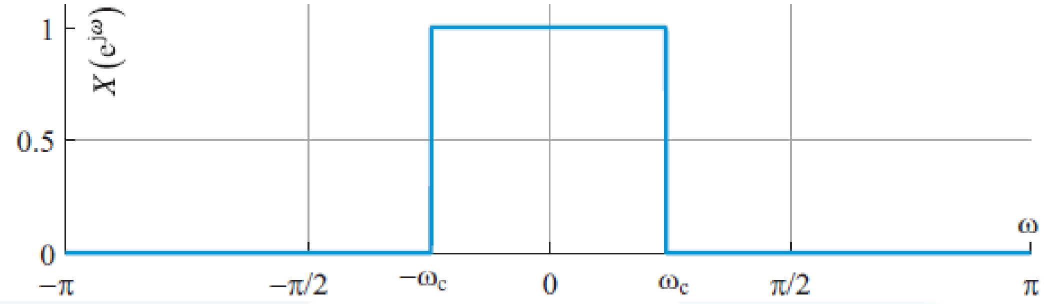
The solid lines correspond to a lowpass sequence (a = 0.8) and the dashed lines to a highpass sequence (a = -0.8)

- Example 3: Ideal lowpass sequence

A sequence $x[n]$ with DTFT over one period: $X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < \omega < \pi \end{cases}$

$$x[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi jn} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{\sin(\omega_c n)}{\pi n}, \qquad -\infty < n < \infty$$





Correlation of signals

- To measure the similarity between a signal of interest and a reference signal, we use the correlation sequence of two real-valued signals, $x[n]$ and $y[n]$ each of which has finite energy, defined by:

$$\left.\begin{aligned} r_{xy}[l] &= \sum_{n=-\infty}^{\infty} x[n]y[n-l] = \sum_{n=-\infty}^{\infty} x[n+l]y[n], \quad -\infty < l < \infty \\ r_{yx}[l] &= \sum_{n=-\infty}^{\infty} y[n]x[n-l] = \sum_{n=-\infty}^{\infty} y[n+l]x[n], \quad -\infty < l < \infty \end{aligned}\right\} \Rightarrow r_{xy}[l] = r_{yx}[-l]$$

- Note: We can compute correlation using convolution: $r_{xy}[l] = x[l] * y[-l]$

- To understand the meaning of correlation we first note that the energy E_z of the sequence $z[n] = ax[n] + y[n-l]$, which is nonnegative, can be expressed as:

$$E_z = a^2 E_x + 2ar_{xy}[l] + E_y \geq 0 \Rightarrow 4r_{xy}^2[l] - 4E_x E_y \leq 0$$

$$-1 \leq \rho_{xy}[l] = \frac{r_{xy}[l]}{\sqrt{E_x}\sqrt{E_y}} \leq 1$$

The sequence $\rho_{xy}[l]$, which is known as the normalized correlation coefficient:

- If $x[n] = cy[n-n_0]$, $c > 0$, we obtain $\rho_{xy}[n_0] = 1$ (maximum correlation);
- If $x[n] = -cy[n-n_0]$, $c > 0$, we obtain $\rho_{xy}[n_0] = -1$ (maximum negative correlation).
- If $\rho_{xy}[l] = 0$ for all lags, the two sequences are said to be uncorrelated.

$$r_{xy}[l] = x[l] * y[-l] \Rightarrow \text{using DTFT } R_{xy}(\omega) = X(e^{j\omega})\,Y(e^{-j\omega})$$

when $y[n] = x[n]$ we obtain the autocorrelation sequence $r_{xx}[l]$ or $r_x[l]$.

Since $x[n]$ is a real sequence, $X^*(e^{j\omega}) = X(e^{-j\omega})$ therefore the DTFT of $r_x[l]$ is:

$$r_x[l] = x[l] * x[-l] \Rightarrow R_x(\omega) = |X(e^{j\omega})|^2$$ Wiener-Khintchine theorem

- Example 4: Autocorrelation of exponential sequence

Let $x[n] = a^n u[n]$, $-1 < a < 1$. For $l > 0$, the product $x[n]u[n]x[n-l]u[n-l]$ is zero for $n < l$.

$$r_x[l] = \sum_{n=l}^{\infty} x[n]x[n-l] = \sum_{n=l}^{\infty} a^n a^{n-l} = a^l(1 + a^2 + a^4 + \cdots) = \frac{a^l}{1-a^2}$$

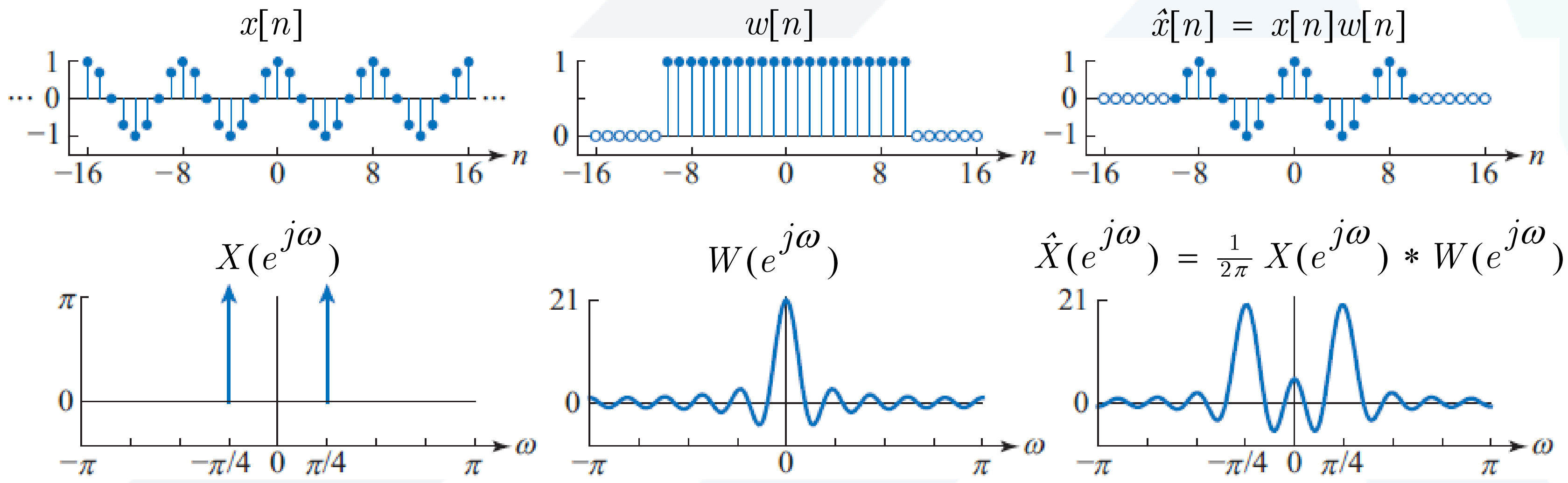
Since $r_x[l] = r_x[-l] \Rightarrow r_x[l] = \dfrac{a^{|l|}}{1-a^2}, \quad -1 < a < 1$ The Fourier transform is

$$R_x(\omega) = X(e^{j\omega})X(e^{-j\omega}) = \frac{1}{1-ae^{-j\omega}}\frac{1}{1-ae^{j\omega}} = \frac{1}{1-2a\cos(\omega)+a^2}$$

Since $r_x[l]$ is real and even, its Fourier transform $R_x(\omega)$ is also real and even.

Spectral and temporal ambiguity

- Spectral analysis is one of the most important applications of DSP. It is the process of measuring, estimating and characterizing the frequency content of signals.
- Let $x[n] = \cos \pi n/4$, which might represent the signal whose spectrum we are trying to measure.



- $x[n]$ is completely unlocalized in time (extends from $-\infty < n < \infty$); however, its spectrum $X(e^{j\omega})$ is highly localized to exactly two frequencies, $\omega = \pm\pi/4$.
- The result of this convolution (windowing of a cosine) is a signal whose spectrum is "smeared out" in frequency; it has two relatively broad peaks and energy spread over all frequencies.
- There is a trade-off between resolution in the time and the frequency domains; as we increase the localization of the signal in the time domain, we reduce the localization in the frequency domain.