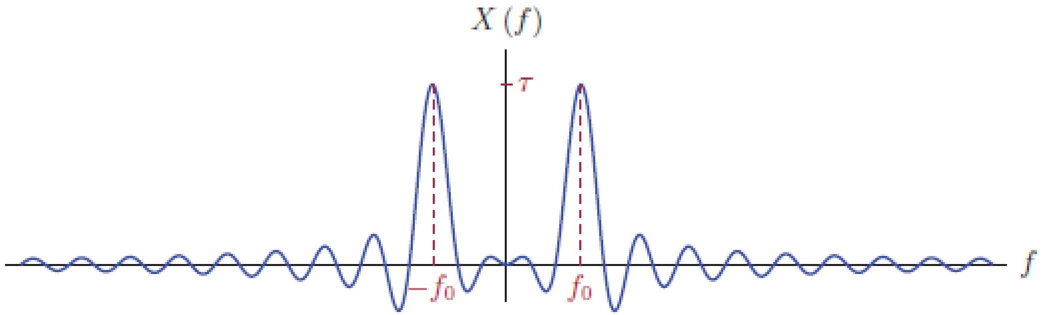
CECC507: Signals and Systems

Lecture Notes 4: Analyzing Discrete Time Systems in the Time Domain



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Chapter 3

Analyzing Discrete Time Systems in the Time Domain

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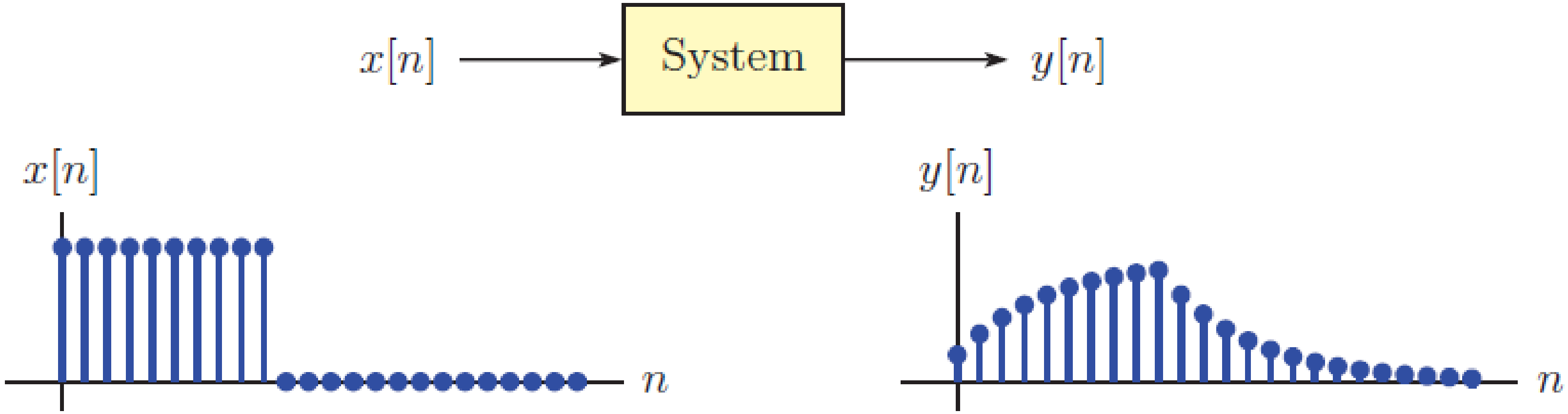
1. Introduction

- In general, a discrete-time (DT) system is a mathematical formula, method or algorithm that defines a cause-effect relationship between a set of discrete-time input signals and a set of discrete-time output signals.

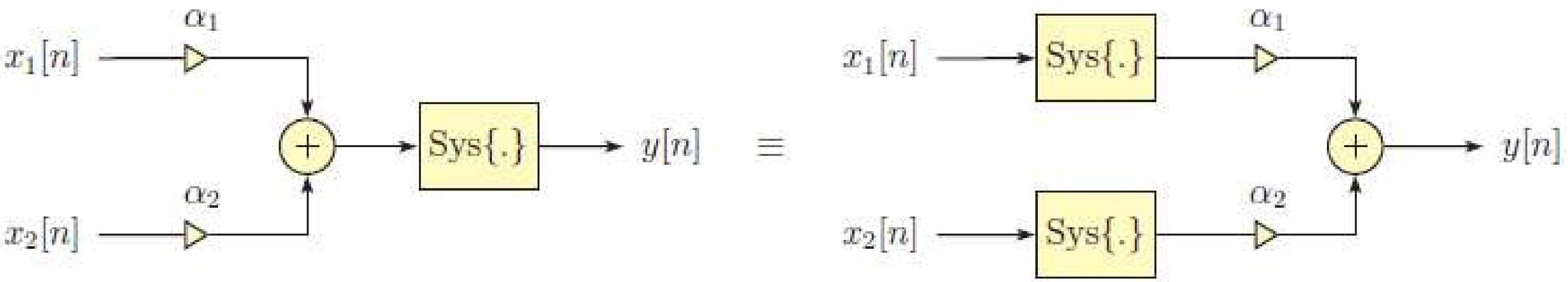


- The input signal is $x[n]$, and the output signal is $y[n]$. The system may be denoted by the equation $y[n] = T\{x[n]\}$, where $T\{.\}$ indicates a transformation.

2. Linearity and Time Invariance

Linearity in continuous-time systems

- A system T is linear, if for all functions x_1 and x_2 and all constants α_1 and α_2, the following condition holds: $T\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 T\{x_1[n]\} + \alpha_2 T\{x_2[n]\}$.



- The linearity property is also referred to as the superposition property.
- Example 1: Testing linearity of discrete-time systems

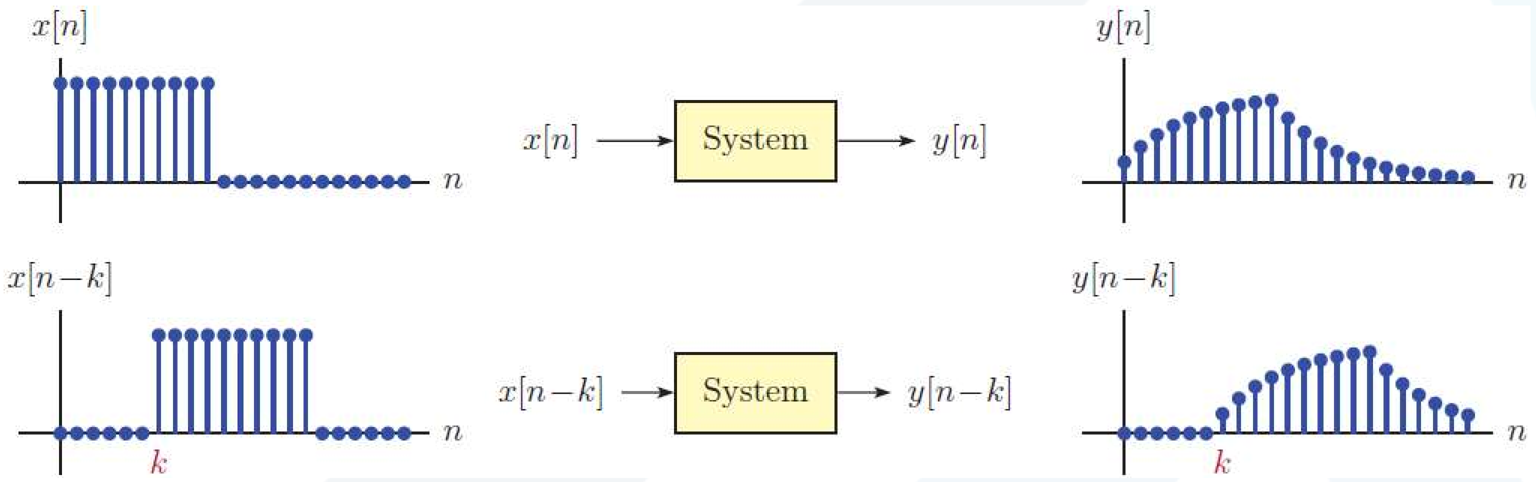
  a. $y[n] = 3x[n] + 2x[n-1]$ √ b. $y[n] = 3x[n] + 2x[n-1]\, x[n+1]$ X

  c. $y[n] = a^{-n}x[n]$ √

Time Invariance in discrete-time systems

- A system T is said to be time invariant (TI) (or shift invariant (SI)) if, for every function x and every integer constant k, the following condition holds:

$$T\{x[n]\} = y[n] \Rightarrow T\{x[n-k]\} = y[n-k]$$



- Example 2: Testing time invariance of discrete-time systems

a. $y[n] = y[n-1] + 3x[n]$ √ b. $y[n] = x[n]\, y[n-1]$ √ c. $y[n] = nx[n-1]$ ✗

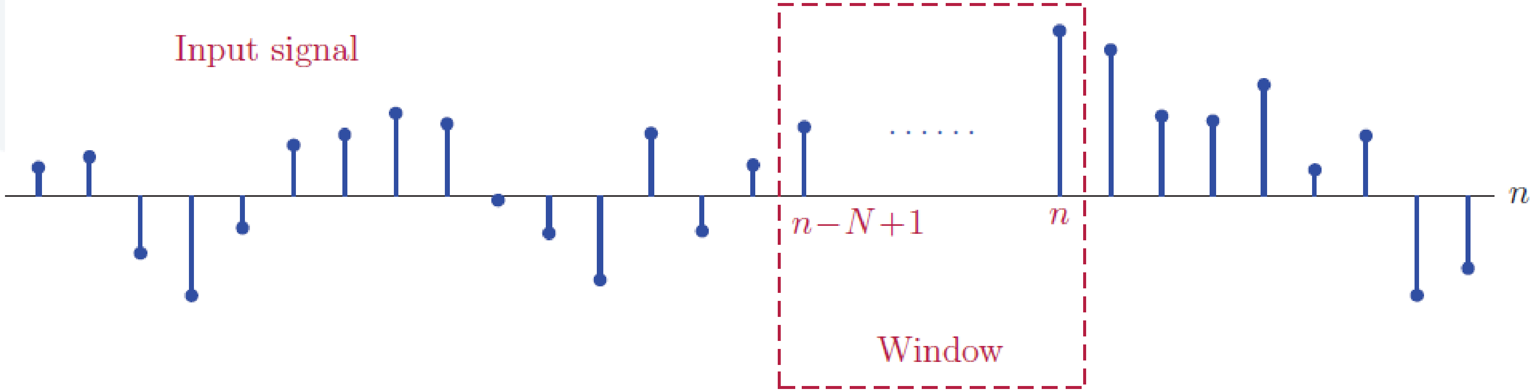
3. Difference Equations for Discrete-Time Systems

- One method of representing the relationship established by a system between its input and output signals is a difference equation (DE).
- A DT systems can be modeled with difference equations involving current, past, or future samples of input and output signals.
- Example 3: Moving-average filter

A length-N moving average filter is a simple system that produces an output equal to the arithmetic average of the most recent N samples of the input signal.

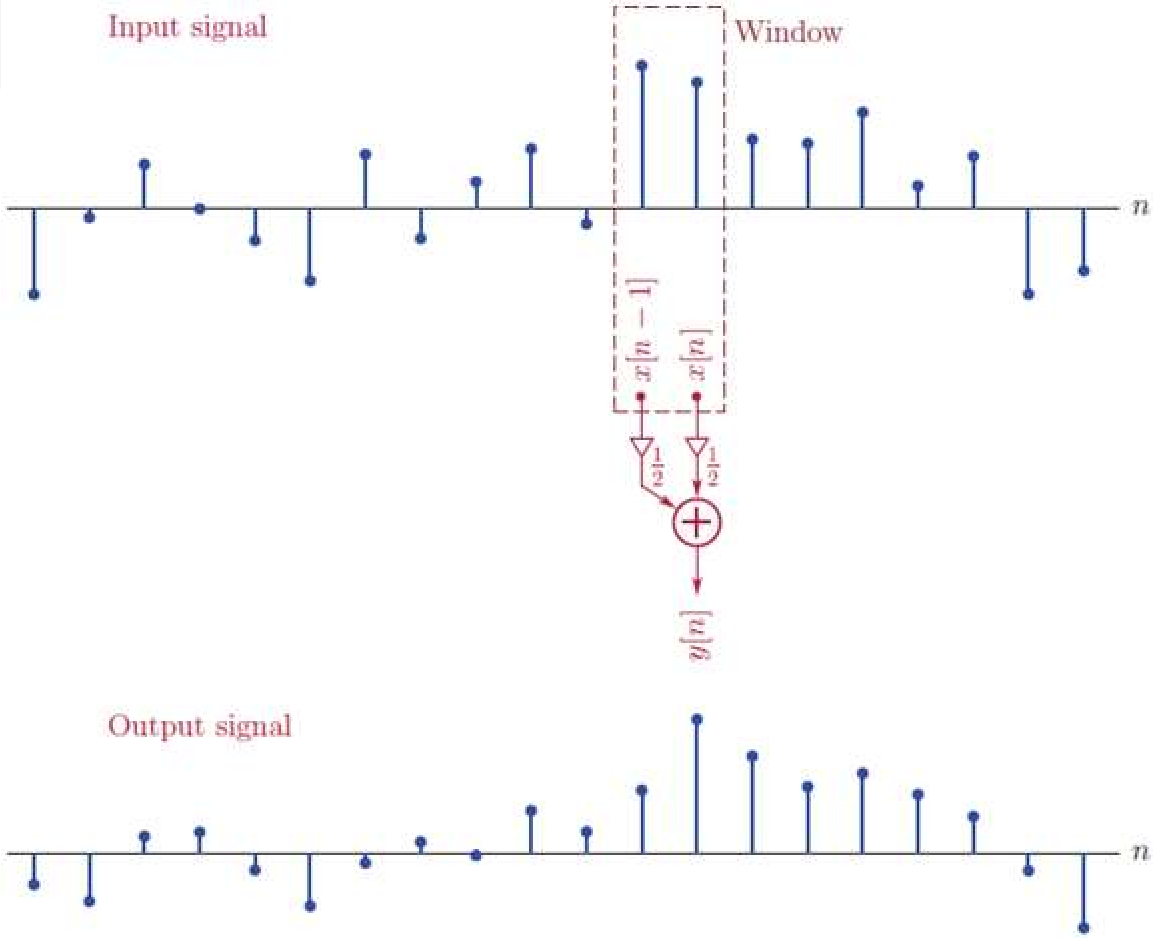
$$y[n] = \frac{x[n] + x[n-1] + \cdots + x[n-(N-1)]}{N} = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

- Moving average filters are used in to smooth the variations in a signal.

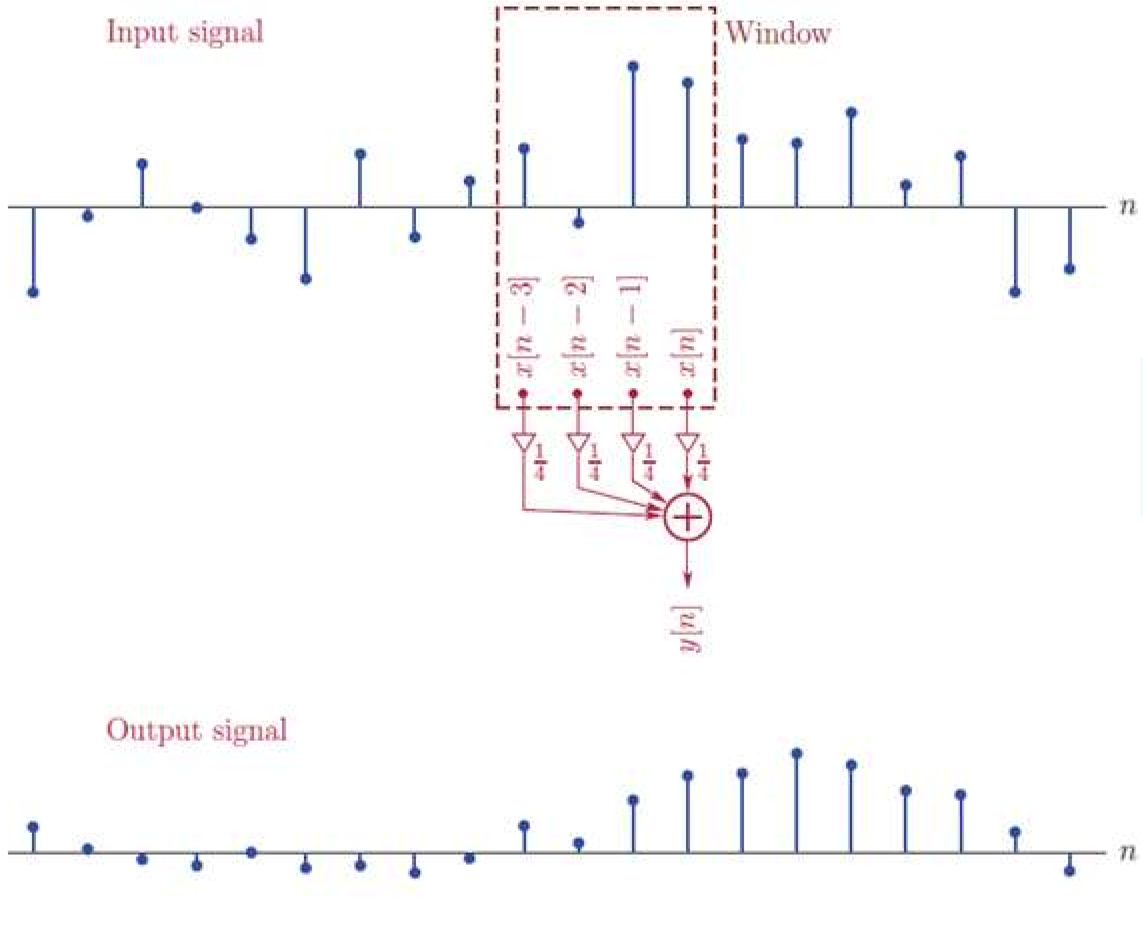


- One example is in analyzing the changes in a financial index such as the Dow Jones Industrial Average.
- The degree of smoothing is dependent on N, the size of the window.
- Example 4: Length-2 and Length-4 moving-average filter

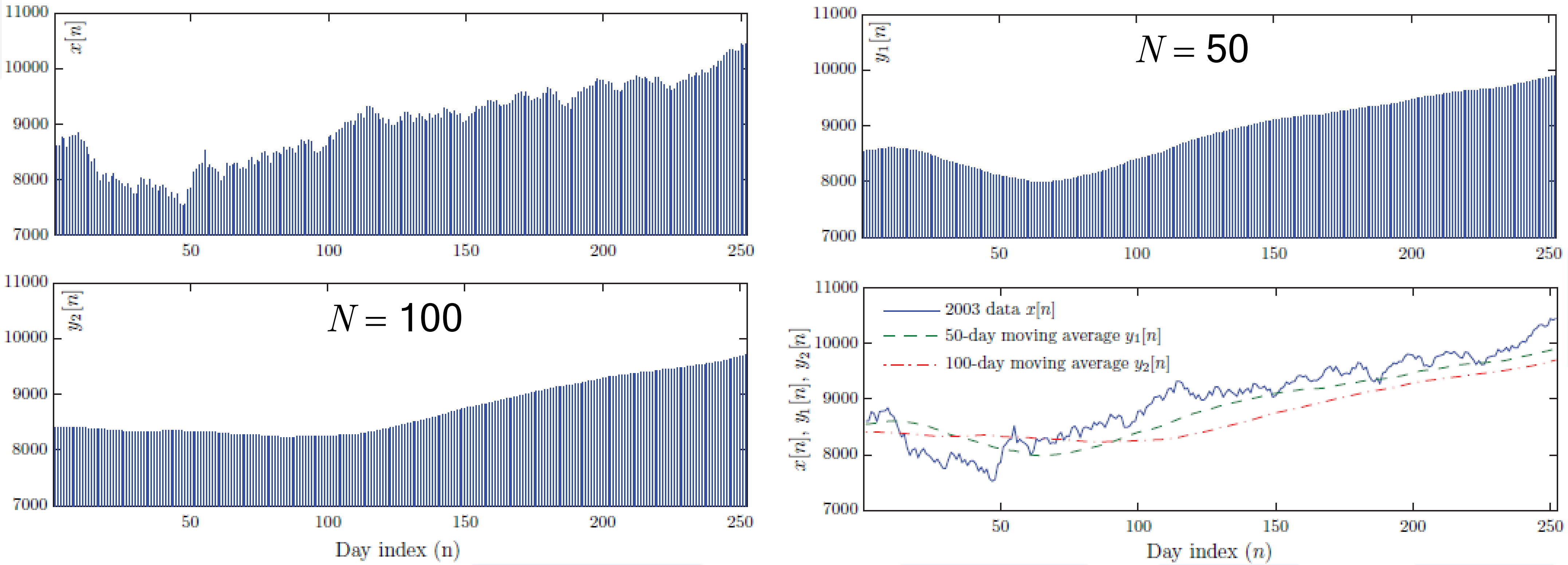
$$y[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1] \qquad y[n] = \tfrac{1}{4}x[n] + \tfrac{1}{4}x[n-1] + \tfrac{1}{4}x[n-2] + \tfrac{1}{4}x[n-3]$$



Length-2 moving-average filter



Length-4 moving-average filter



- Example 5: Exponential smoother
- An exponential smoother which employs a difference equation with feedback.

- The current output sample is computed as a mix of the current input sample and the previous output sample through the equation.

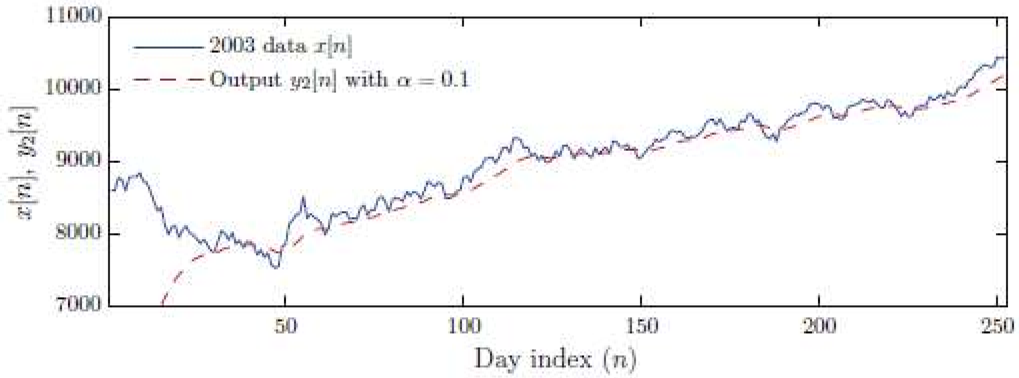
$$y[n] = (1 - \alpha)\ y[n-1] + \alpha x[n]$$

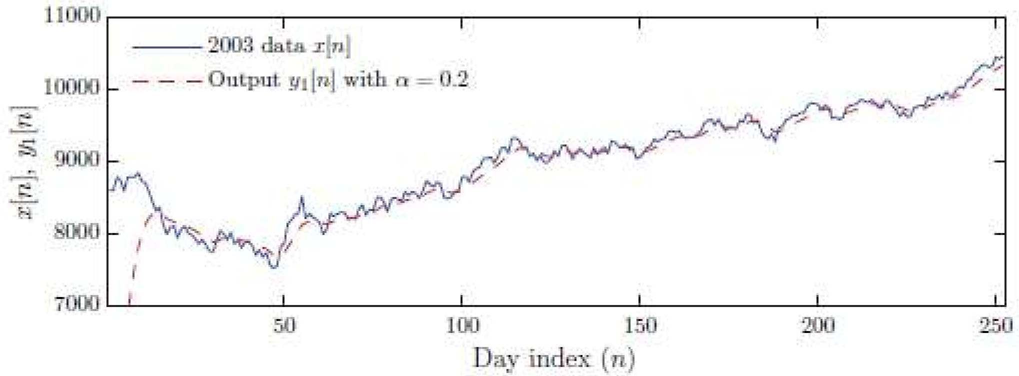
- The parameter $0 < \alpha < 1$ is a constant, it controls the degree of smoothing.

$$y[0] = (1 - \alpha)\ y[-1] + \alpha x[0]$$

$$y[1] = (1 - \alpha)\ y[0] + \alpha x[1]$$

$$y[2] = (1 - \alpha)\ y[1] + \alpha x[2]$$





4. Constant-Coefficient Linear Difference Equations

- In general, DTLTI systems can be modeled with linear difference equations that have constant coefficients in the form:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] =$$
$$b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

or it can be expressed in the form $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

- The order of the DE ($=$ the order of the system it represents) $=$ max(N, M).
- The orders of the length-N moving average filter is $N-1$.
- A constant-coefficient linear DE has a family of solutions. To find a unique solution for $n \geq n_0$, the initial values $y[n_0-1]$, ... , $y[n_0-N]$ are needed.

- The linear difference equation $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

  represents a linear system provided that all initial conditions are equal to zero: $y[n_0 - k] = 0$ for $k = 1,\dots, N$. And represents a time invariance system.

Solving Linear Difference Equations

Solution of the general linear difference equation

- 2 separate components of the output signal $y[n]$ as follows: $y[n] = y_h[n] + y_p[n]$.
- $y_h[n]$, is the solution of the homogeneous linear difference equation found by setting $x[n] = 0$ for all values of n.

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

- $y_h[n]$ is called the natural response of the system.

- $y_h[n]$ depends on the structure of the system as well as the initial state of the system $y[n_0-1]$, $y[n_0-2]$,... , $y[n_0-N]$. It does not depend, on the input signal.
- For a stable system, $y_h[n]$ tends to gradually disappear in time.
- $y_p[n]$ is due to the input signal $x[n]$ being applied to the system. It is referred to as the particular solution of the difference equation.
- $y_p[n]$ depends on the input signal $x[n]$ and the internal structure of the system, but it does not depend on the initial state of the system.
- The particular solution $y_p[n]$ represents any solution of the DE for the given input. It is also called Forced response $y_\phi[n]$.

Finding the natural response of a discrete-time system

General homogeneous difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

- The characteristic equation: $\sum_{k=0}^{N} a_k z^{-k} = 0$
- To obtain the characteristic equation, substitute: $y[n-k] \rightarrow z^{-k}$
- Characteristic polynomial of the DTLTI system:

$$a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z^1 + a_N = a_0 (z - z_1)(z - z_2) \cdots (z - z_N) = 0$$

z_1, z_2, ..., z_N are the roots of the characteristic polynomial be:

$$y_h[n] = c_1 z_1^n + c_2 z_2^n + \cdots + c_N z_N^n = \sum_{k=1}^{N} c_k z_k^n$$

- The coefficients c_1, c_2, ..., c_N are determined from the initial conditions.
- Example 6: Natural response of second-order system

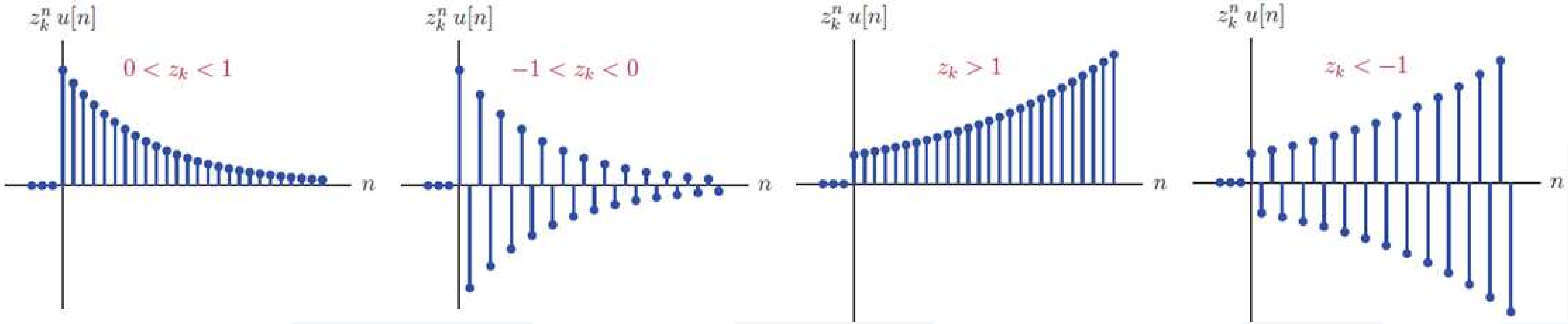
$$y[n] - \tfrac{5}{6} y[n-1] + \tfrac{1}{6} y[n-2] = 0 \qquad n \geq 0,\ y[-1] = 19 \text{ and } y[-2] = 53$$

$$z^2 - \frac{5}{6}z + \frac{1}{6} = (z - \frac{1}{2})(z - \frac{1}{3}) = 0 \Rightarrow y_h[n] = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n, \text{ for } n \geq 0$$

$y_h[-1] = 19$, and $y_h[-2] = 53 \Rightarrow c_1 = 2,\ c_2 = 5$ $\quad y_h[n] = 2\left(\frac{1}{2}\right)^n u[n] + 5\left(\frac{1}{3}\right)^n u[n]$

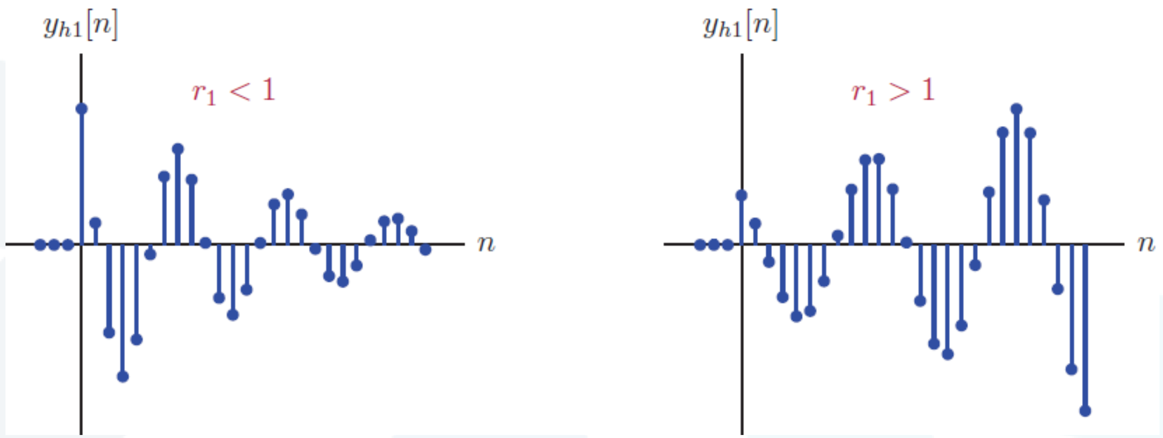
Case 1: All roots are distinct and real-valued $\quad y[n] = \sum_{k=1}^{N} c_k z_k^n, \text{ for } n \geq n_0$

- If $|z_k| < 1$ then z_k^n decays exponentially over time.
- Conversely, $|z_k| > 1$ leads to a term z_k^n that grows exponentially.



Case 2: Characteristic polynomial has complex-valued roots

- Any complex roots of the characteristic polynomial must appear in conjugate pairs.



$$z_{1a} = r_1 e^{j\Omega_1}, \qquad z_{1b} = r_1 e^{-j\Omega_1}$$

$$y_{h1}[n] = d_1 r_1^n \cos(\Omega_1 n) + d_2 r_1^n \sin(\Omega_1 n)$$

Case 3: Characteristic polynomial has some multiple roots

$$a_0(z - z_1)(z - z_2)\cdots(z - z_N) = 0 \qquad z_1 = z_2$$

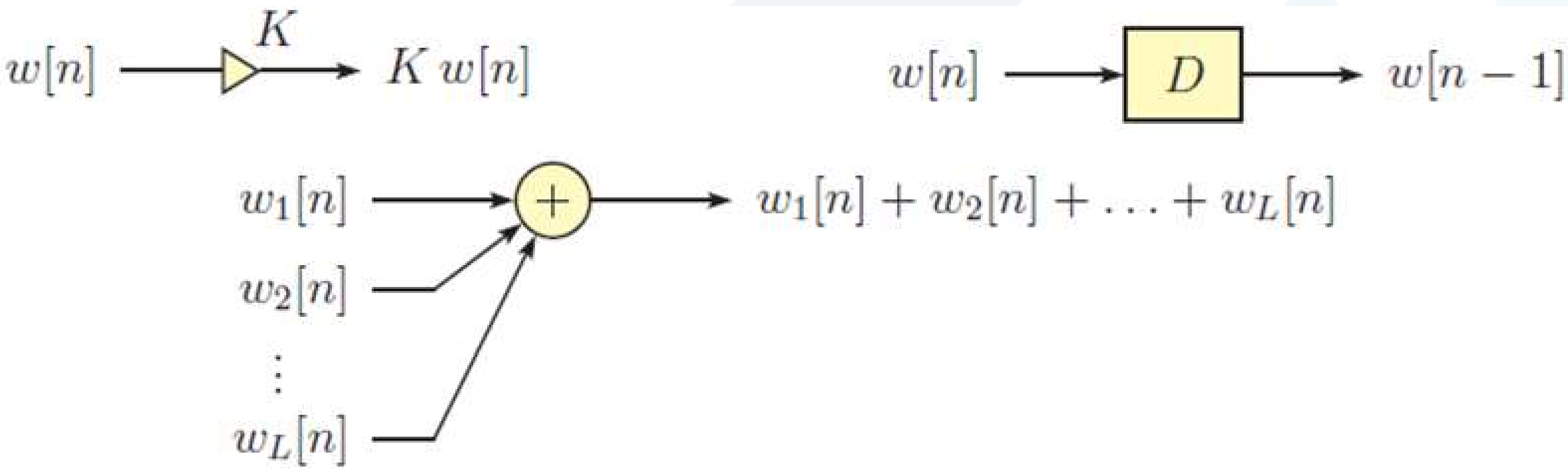
$y_h[n] = c_{11} z_1^n + c_{12} n z_1^n$ + other terms

- In general, a root of multiplicity r requires r terms in the homogeneous solution.

$$y_h[n] = c_{11} z_1^n + c_{12} n z_1^n + \cdots + c_{1r} n^{r-1} z_1^n + \text{other terms}$$

5. Block Diagram Representation of Discrete-Time Systems

- Block diagrams for discrete-time systems are constructed using three types of components, namely multiplication of a signal by a constant gain factor, addition of two signals, and time shift of a signal.



- The technique for finding a block diagram from a difference equation is best explained with an example.

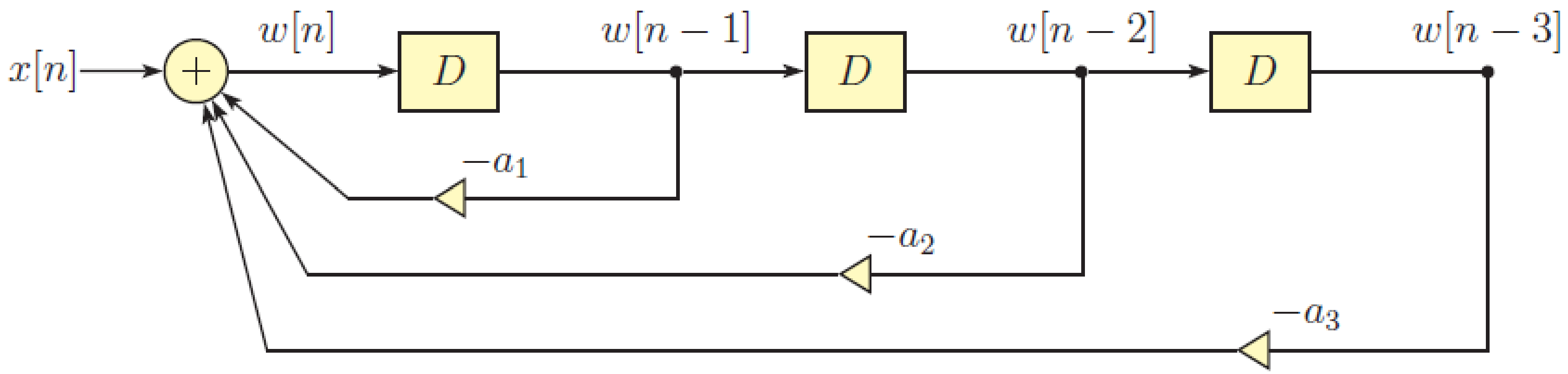
$$y[n] + a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

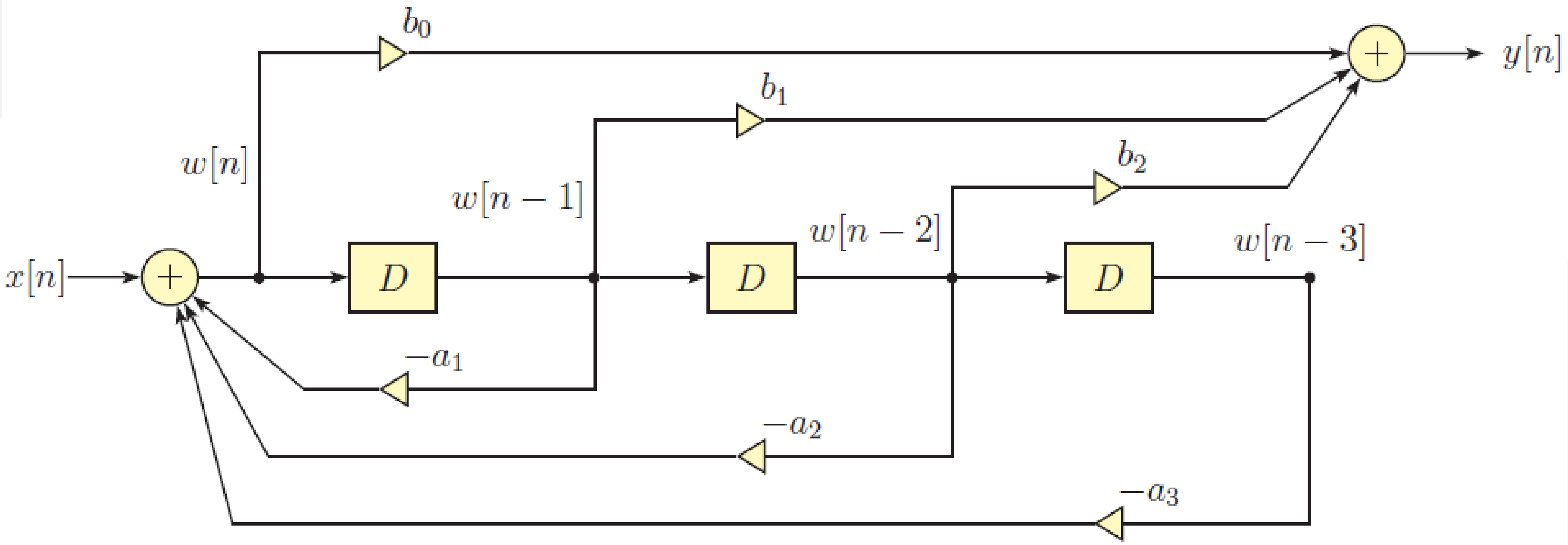
We will introduce an intermediate variable $w[n]$:

$$w[n] + a_1 w[n-1] + a_2 w[n-2] + a_3 w[n-3] = x[n]$$

$$y[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2]$$

One possible block diagram implementation of the difference equation $w[n]$ is:





The completed block diagram

Imposing initial conditions

- Initial values of $y[-1]$, $y[-2]$, and $y[-3]$, need to be converted to corresponding initial values of $w[-1]$, $w[-2]$, and $w[-3]$.

6. Impulse Response and Convolution

Convolution operation for DTLTI systems

- The (DT) convolution of x and h, denoted $x * h$, is defined as the function:

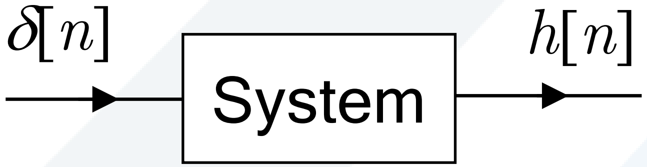
$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Properties of Convolution

- Commutative. That is, for any two functions x and h, $x * h = h * x$.
- Associative. That is, for any functions x, h_1, and h_2, $(x * h_1) * h_2 = x * (h_1 * h_2)$.
- Distributive. That is, for any functions x, h_1, and h_2, $x * (h_1 + h_2) = x * h_1 + x * h_2$.
- For any function x, $x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$
- Moreover, δ is the convolutional identity. That is, for any function x, $x * \delta = x$.

Finding impulse response of a DTLTI system

- The response h of a system T to the input δ is called the impulse response of the system.
- For any LTI system with input x, output y, and impulse response h: $y = x * h$.
- Furthermore, a LTI system is completely characterized by its impulse response.



Step Response of a DTLTI system

- The response s of a system T to the input u is called the step response of the system.

$$s[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=0}^{\infty} u[k]h[n-k]$$

- The impulse response h and step response s of a LTI system are related as

$$h[n] = s[n] - s[n-1]$$

- Example 7: Impulse response of moving average filters

Length-2 moving average filter: $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

$$h_2[n] = T\{\delta[n]\} = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1] = \{\underset{\uparrow}{\tfrac{1}{2}}, \tfrac{1}{2}\}$$

Length-4 moving average filter: $y[n] = \frac{1}{4}x[n] + \frac{1}{4}x[n-1] + \frac{1}{4}x[n-2] + \frac{1}{4}x[n-3]$

$$h_4[n] = T\{\delta[n]\} = \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3] = \{\underset{\uparrow}{\tfrac{1}{4}}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\}$$

Length-N moving average filter $y[n] = \dfrac{1}{N}\sum_{k=0}^{N-1} x[n-k]$

$$h_N[n] = T\{\delta[n]\} = \frac{1}{N}\sum_{k=0}^{N-1}\delta[n-k]$$

$$h_N[n] = \begin{cases} \frac{1}{N}, & n = 0, \cdots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$h_N[n] = \frac{1}{N}\left(u[n] - u[n-N]\right)$$

- Example 8: Impulse response of exponential smoother

$$y[-1] = 0$$

$$y_h[n] = c(1-\alpha)^n \qquad y_p[n] = k \Rightarrow k = 1$$

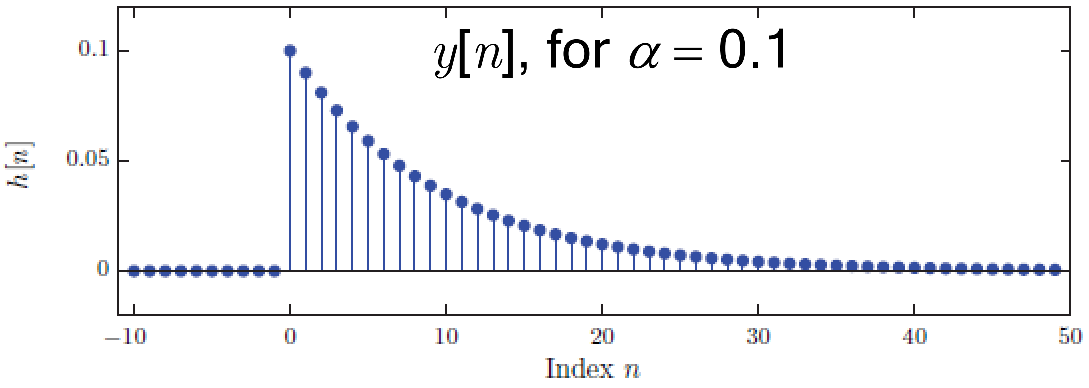
$$y[n] = y_h[n] + y_p[n] = c(1-\alpha)^n + 1$$

$$y[-1] = 0 \Rightarrow c = -(1-\alpha)$$

$$s[n] = 1 - (1-\alpha)^{n+1}, \text{ for } n \geq 0$$

$$s[n] = [1 - (1-\alpha)^{n+1}]u[n]$$

$$h[n] = s[n] - s[n-1] = \alpha(1-\alpha)^n u[n]$$



7. Causality and Stability in Discrete-Time Systems

- A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.

$$y[n] = y[n-1] + x[n] - 3x[n-1] \quad \text{is causal}$$

$$y[n] = y[n-1] + x[n] - 3x[n+1] \quad \text{is non causal}$$

- Causal systems can be implemented in real-time processing mode.

- For DTLTI systems the causality property can be related to the impulse response of the system $h[n] = 0$ for all $n < 0$:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

9. Stability in Continuous-Time Systems

- A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal is guaranteed to produce a bounded output signal.

- An input signal $x[n]$ is said to be bounded if an upper bound B_x exists such that $x[n] < B_x < \infty$ for all values of the integer index n.

- For stability of a discrete-time system: $x[n] < B_x < \infty \Rightarrow y[n] < B_y < \infty$.

- For a DTLTI system to be stable, its impulse response must be absolute summable:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- Example 9: Stability of a length-2 moving-average filter

  Comment on the stability of the length-2 moving-average filter described by the difference equation $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

$$|y[n]| = \left|\tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1]\right| \le \tfrac{1}{2}|x[n]| + \tfrac{1}{2}|x[n-1]|$$

  Since we assume $|x[n]| < B_x$ for all n, $|y[n]| \le \frac{1}{2}B_x + \frac{1}{2}B_x = B_x$

- For a causal DTLTI system to be stable, the magnitudes of all roots of the characteristic polynomial must be less than unity.
- If a circle is drawn on the complex plane with its center at the origin and its radius equal to unity, all roots of the characteristic polynomial must lie inside the circle for the corresponding causal DTLTI system to be stable.