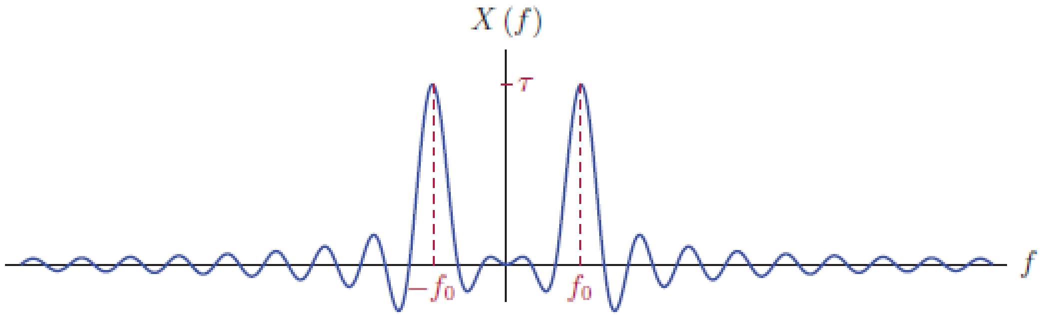
CEDC403: Signals and Systems

Lecture Notes 1: Signal Representation and Modeling: Part A



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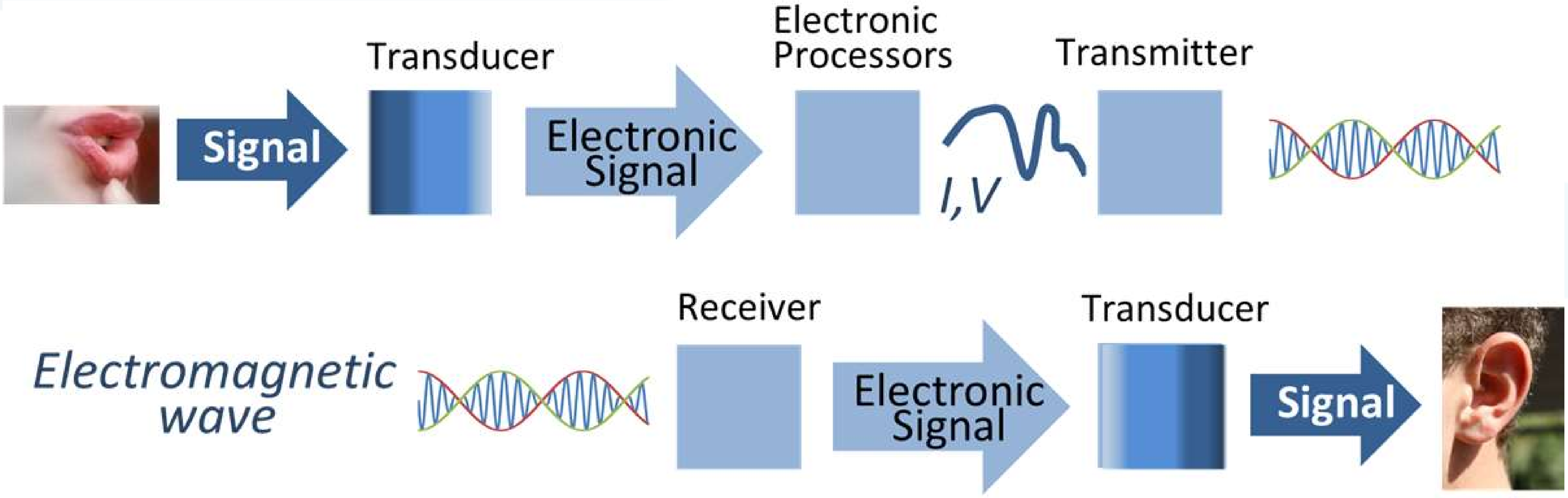
Chapter 1

Signal Representation and Modeling

1. Introduction



- The broadcast example (a commentator in a radio broadcast studio) includes acoustic, electrical and electromagnetic signals.

2. Signals and Systems

- A signal is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- independent variable = time, space, …
- dependent variable = the function value itself.
- Some examples of signals include:
  - a voltage or current in an electronic circuit.
  - the position, velocity, or acceleration of an object.
  - a force or torque in a mechanical system.
  - a flow rate of a liquid or gas in a chemical process.
  - a digital image, digital video, or digital audio.

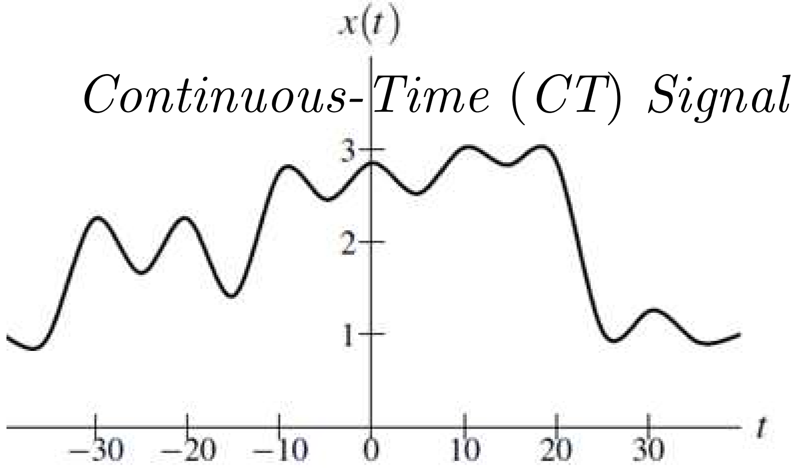
Classification of Signals

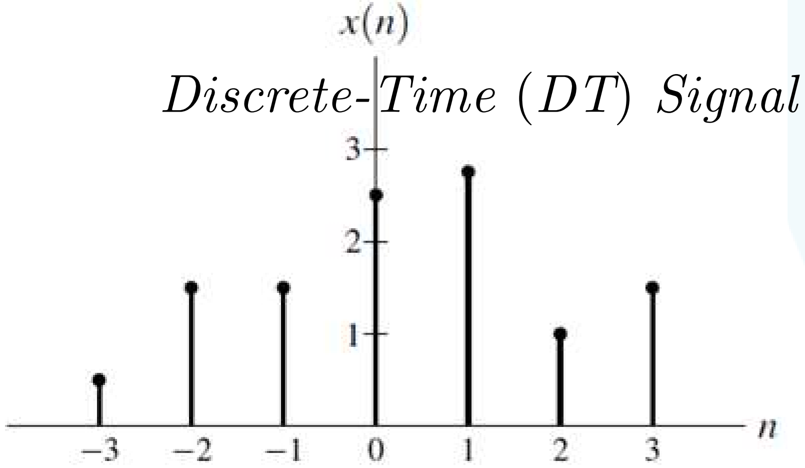
- Number of independent variables (dimensionality):
  - A signal with one independent variable is said to be one dimensional (audio).
  - A signal with more than one independent variable is said to be multi-dimensional (image).
- Continuous or discrete independent variables:
  - A signal with continuous independent variables is said to be continuous time (CT) (voltage waveform). The signal is defined for all values of the independent variable t.
  - A signal with discrete independent variables is said to be discrete time (DT) (stock market index). The signal is defined only at discrete values of time.

- Continuous or discrete dependent variable:
  - A signal with a continuous dependent variable is said to be continuous valued (voltage). A continuous-valued CT signal is said to be analog.
  - A signal with a discrete dependent variable is said to be discrete valued (digital image). A discrete-valued DT signal is said to be digital.
- Deterministic or random signals:
  - A signal whose physical description is known completely, in either a mathematical form or a graphical form, is a deterministic signal.
  - A signal whose values cannot be predicted precisely but are known only in terms of probabilistic description, such as mean value or mean-squared value, is a random signal.

- Periodic and Nonperiodic Signals
  - A periodic signal is one that repeats itself. A CT signal $x(t)$ is said to be periodic with period T if $x(t) = x(t + T)$ for all t (where t is a real number). Likewise, a DT signal $x[n]$ is said to be periodic with period N if $x[n] = x[n + N]$ for all n (where n is an integer).

Graphical Representation of Signals

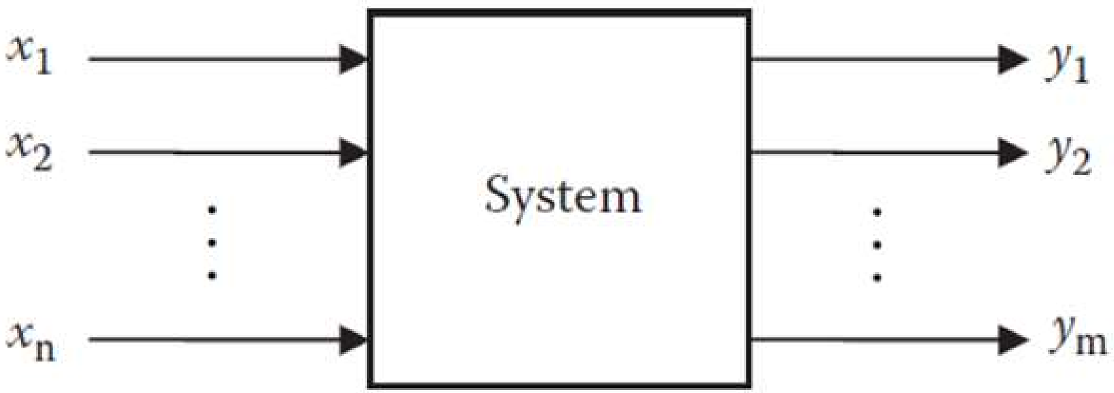




- A system is an entity that processes one or more input signals in order to produce one or more output signals.

system with single-input and single-output

Input Signals



Output Signals

system with many inputs and outputs

Classification of Systems

- Number of Inputs:
  - A system with one input is said to be single input (SI).
  - A system with more than one input is said to be multiple input (MI).
- Number of outputs:
  - A system with one output is said to be single output (SO).
  - A system with more than one output is said to be multiple output (MO).
- Types of signals processed: A system can be classified in terms of the types of signals that it processes:
  - A system that deals with continuous-time signals is called a CT system.
  - A system that deals with discrete-time signals is said to be a DT system.

- A system that handles both continuous- and discrete-time signals, is sometimes referred to as a hybrid system.
- A system that deal with digital signals are referred to as digital.
- A system that handle analog signals are referred to as analog.
- A system interacts with one dimensional signals, the system is referred to as one-dimensional.
- A system handles multi-dimensional signals, the system is said to be multi-dimensional.

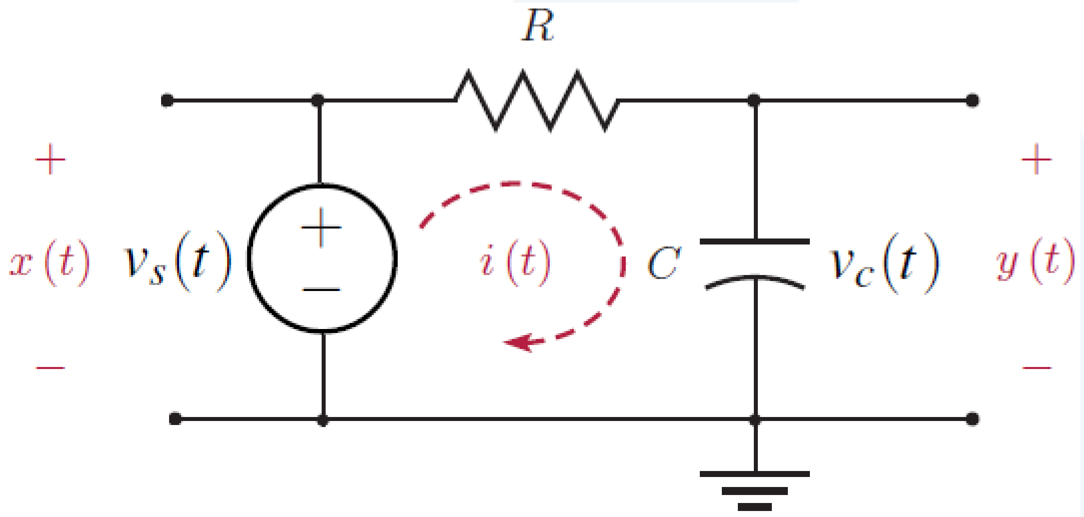
▪ Causal and Noncausal Systems:

- A causal system is one whose present response does not depend on the future values of the input.

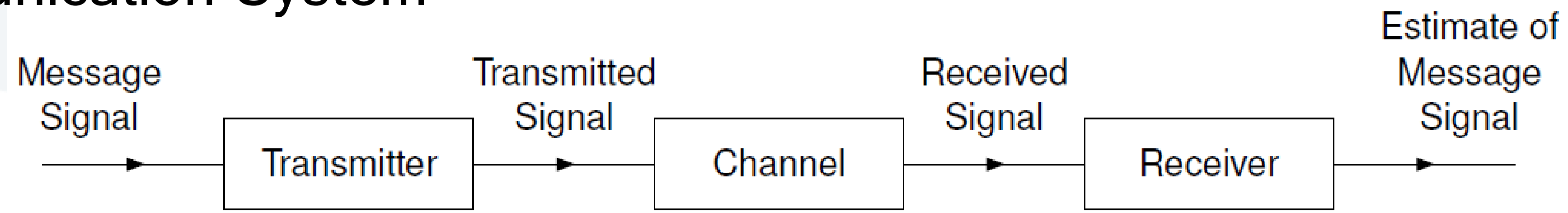
- Linear and Nonlinear Systems.
- Time-Varying and Time-Invariant Systems:
    - A time-varying system is one whose parameters vary with time.
    - In a time-invariant system, a time shift (advance or delay) in the input signal leads to the time shift in the output signal.
- Systems with and without Memory:
    - A memoryless system (static system) is one in which the current output depends only on the current input; it does not depend on the past or future inputs.
    - A system with memory (dynamic system) is one in which the current output depends on the past and/or future input.

Examples of Systems:

- One very basic system is the resistor-capacitor (RC) network. Here, the input would be the source voltage v_s and the output would be the capacitor voltage v_c.

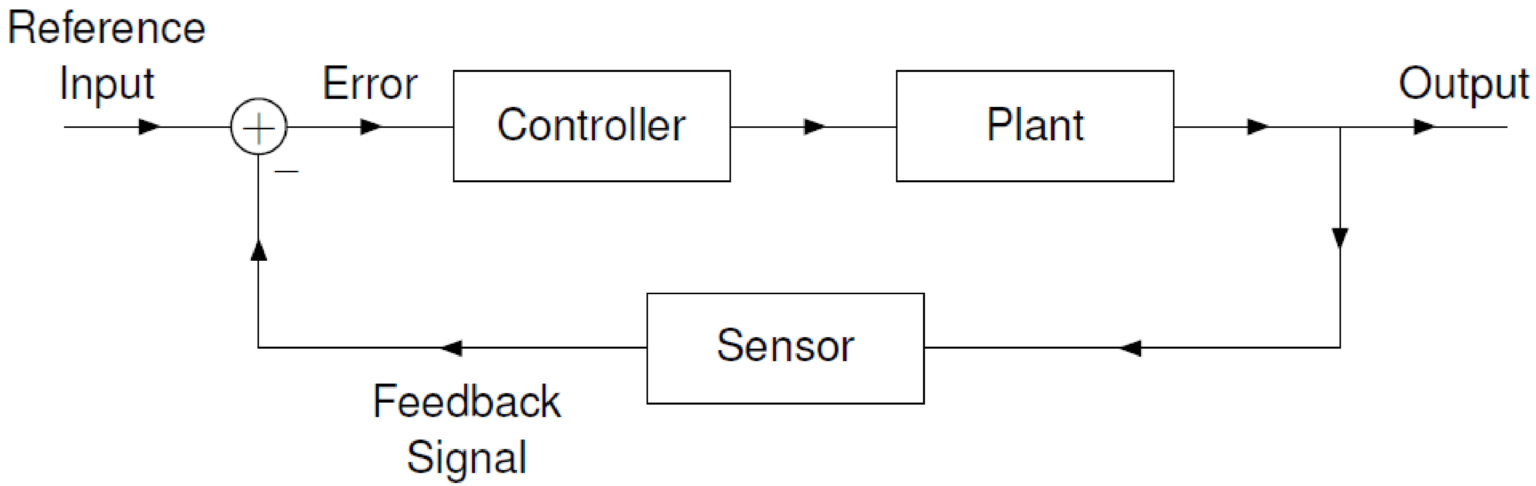


- Communication System



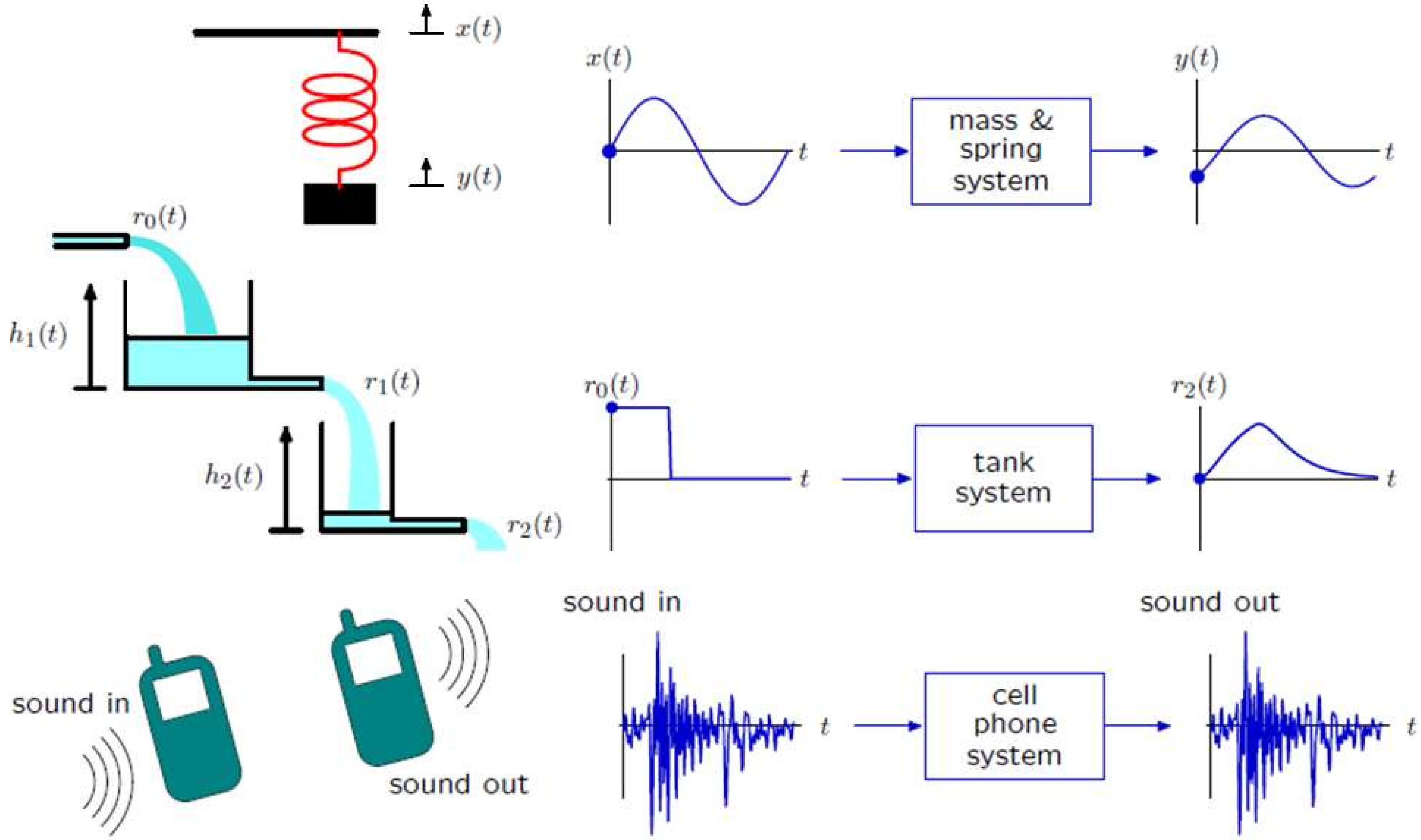
General Structure of a Communication System

- Feedback Control System



General Structure of a Feedback Control System

- The Signals and Systems approach has broad application: electrical, mechanical, optical, acoustic, biological, financial, ...

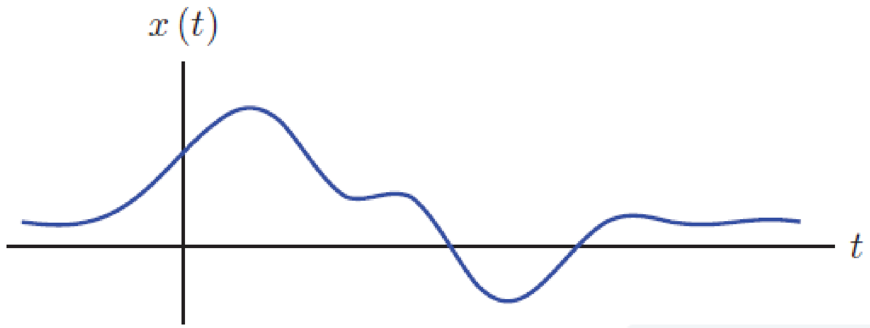


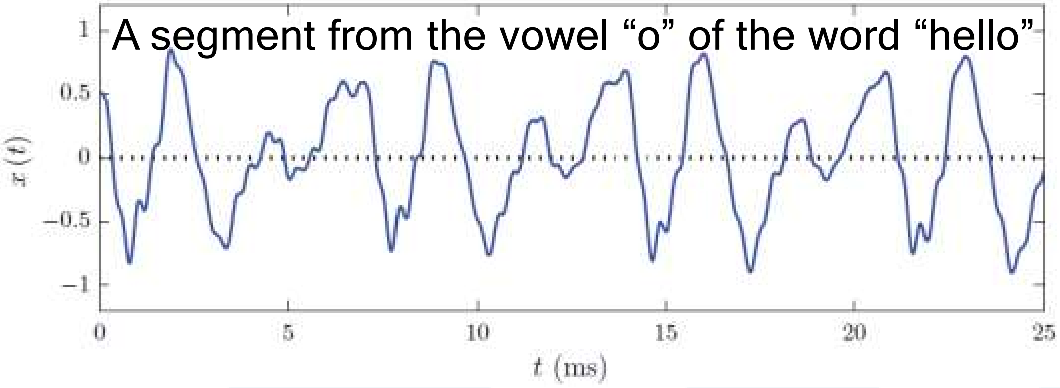
3. Mathematical Modeling of Signals

- Understand the characteristics of the signal in terms of its behavior in time and in terms of the frequencies it contains (signal analysis).
- Develop methods of creating signals with desired characteristics (signal synthesis).
- Understand how a system responds to a signal and why (system analysis).
- Develop methods of constructing a system that responds to a signal in some prescribed way (system synthesis).
- The mathematical model for a signal is in the form of a formula, function, algorithm or a graph that approximately describes the time variations of the physical signal.

4. Continuous-Time Signals

- Consider $x(t)$, a mathematical function of time chosen to approximate the strength of the physical quantity at the time instant t.
- The signal $x(t)$, is referred to as a continuous-time signal or an analog signal. t is the independent variable, and x is the dependent variable.



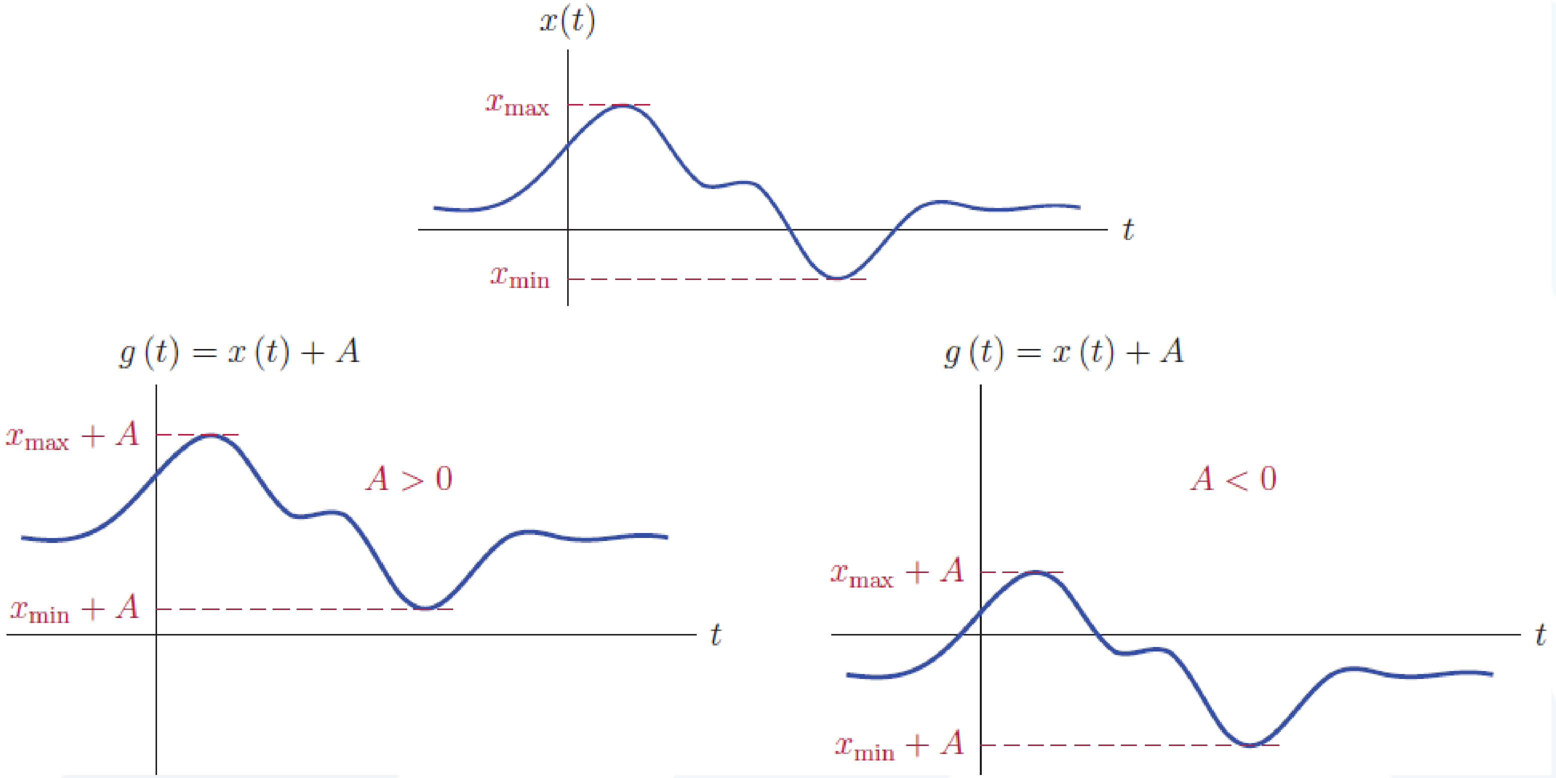


- Some signals can be described analytically. For ex., the function $x(t) = 5\sin(12t)$, or by segments as:

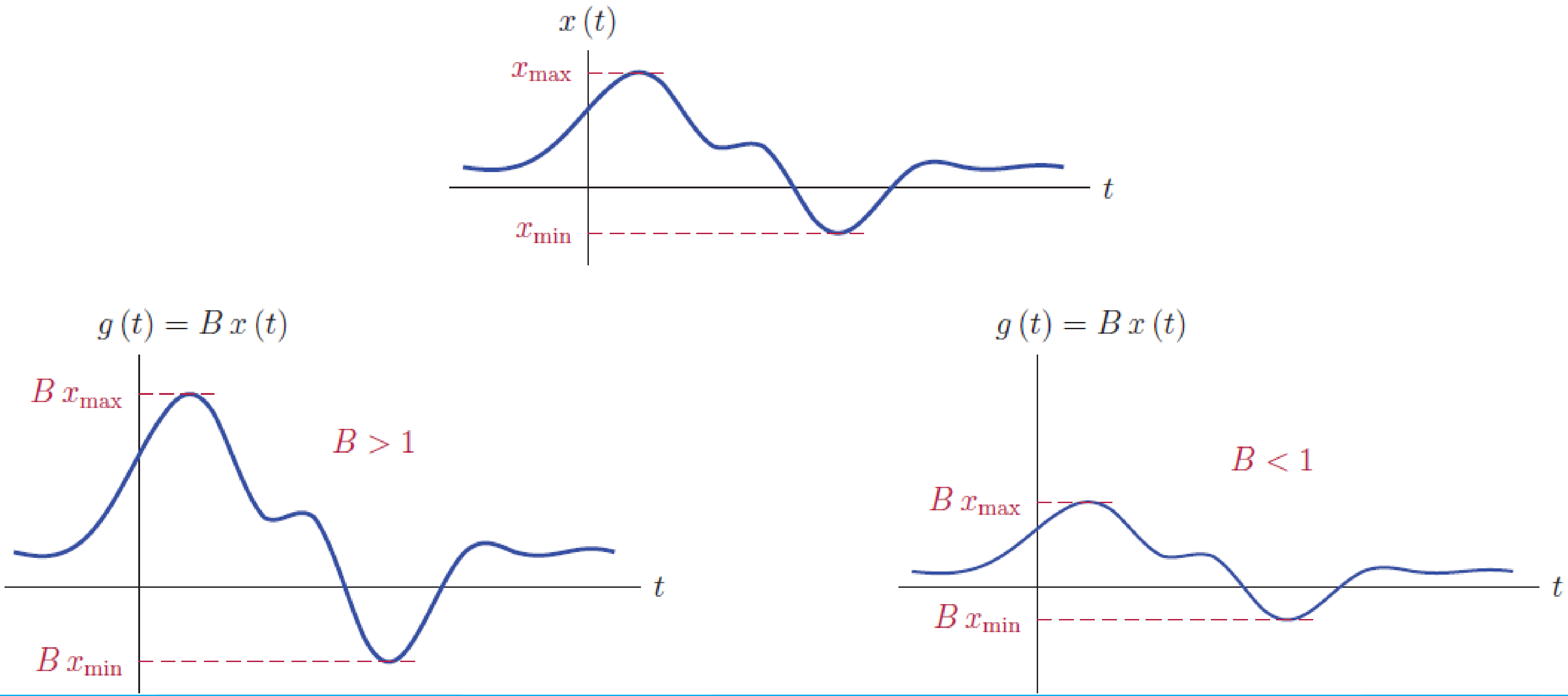
$$x(t) = \begin{cases} e^{-3t} - e^{-5t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Signal operations

- Amplitude shifting maps the input signal x to the output signal g as given by $g(t) = x(t) + A$, where A is a real number.

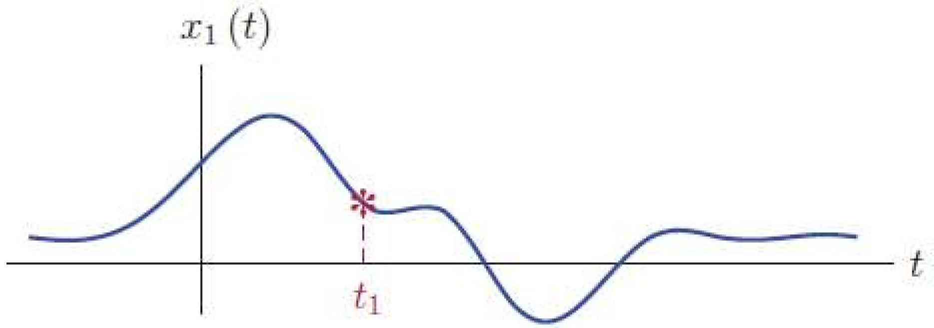


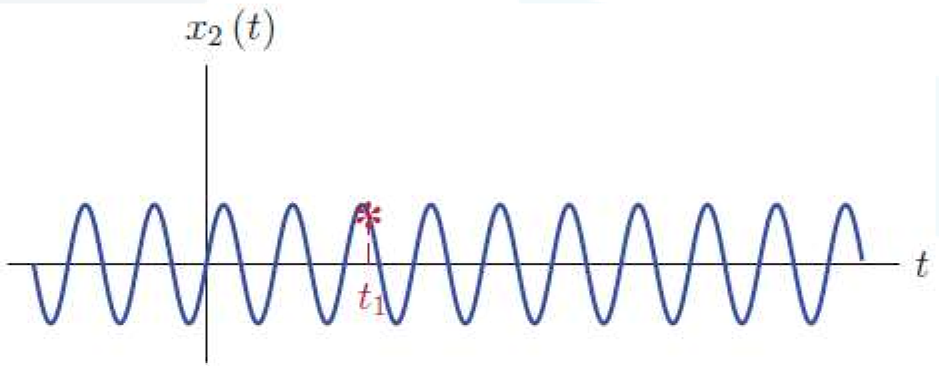
- Amplitude scaling maps the input signal x to the output signal g as given by $g(t) = Bx(t)$, where B is a real number.
- Geometrically, the output signal g is expanded/compressed in amplitude.

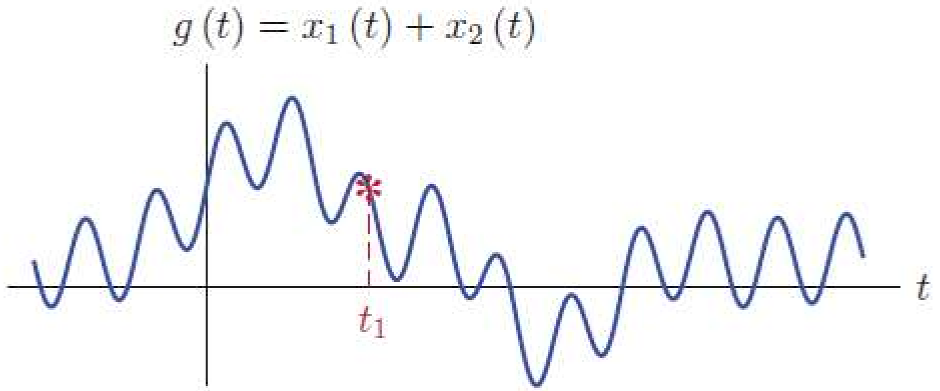


- Addition and Multiplication of two signals

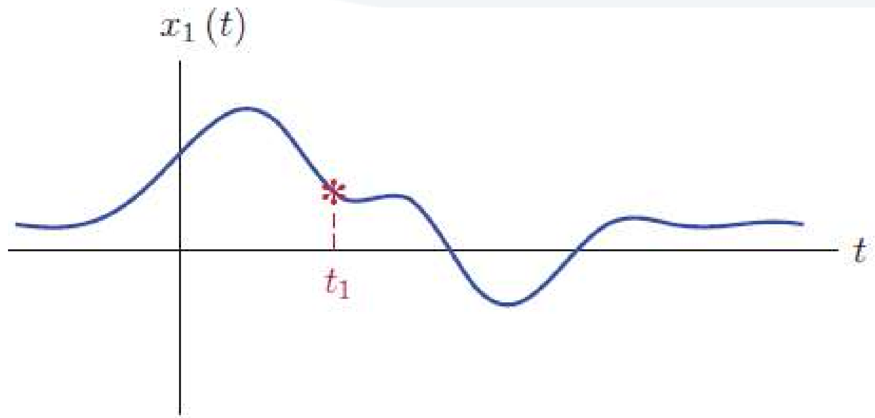
  Addition of two signals is accomplished by adding the amplitudes of the two signals at each time instant. $g(t) = x_1(t) + x_2(t)$.

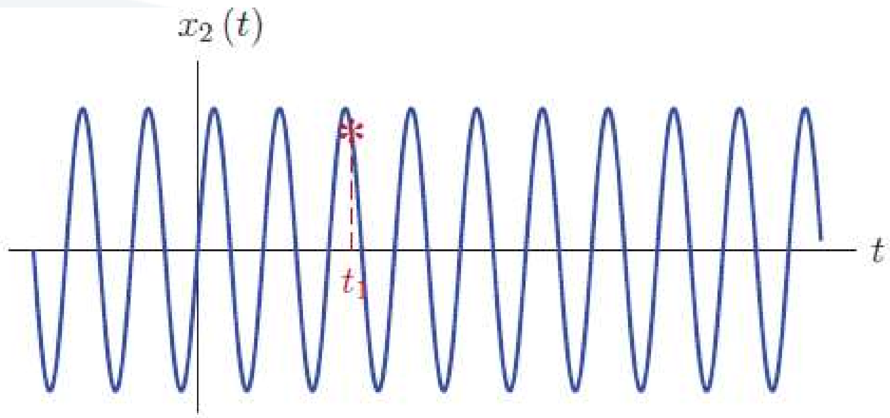


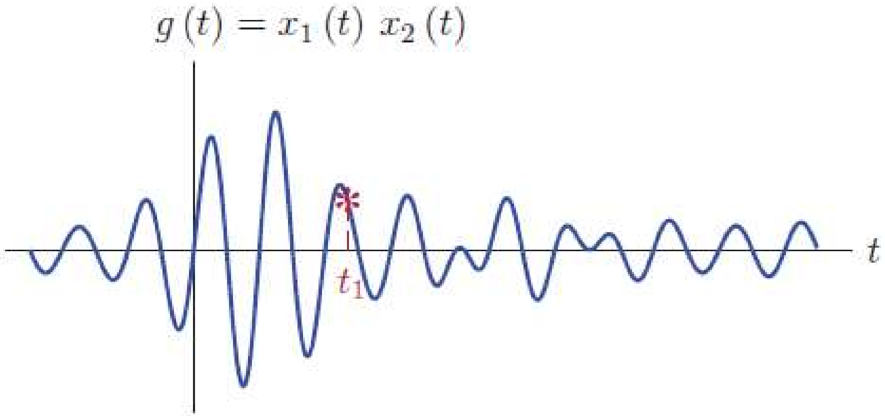




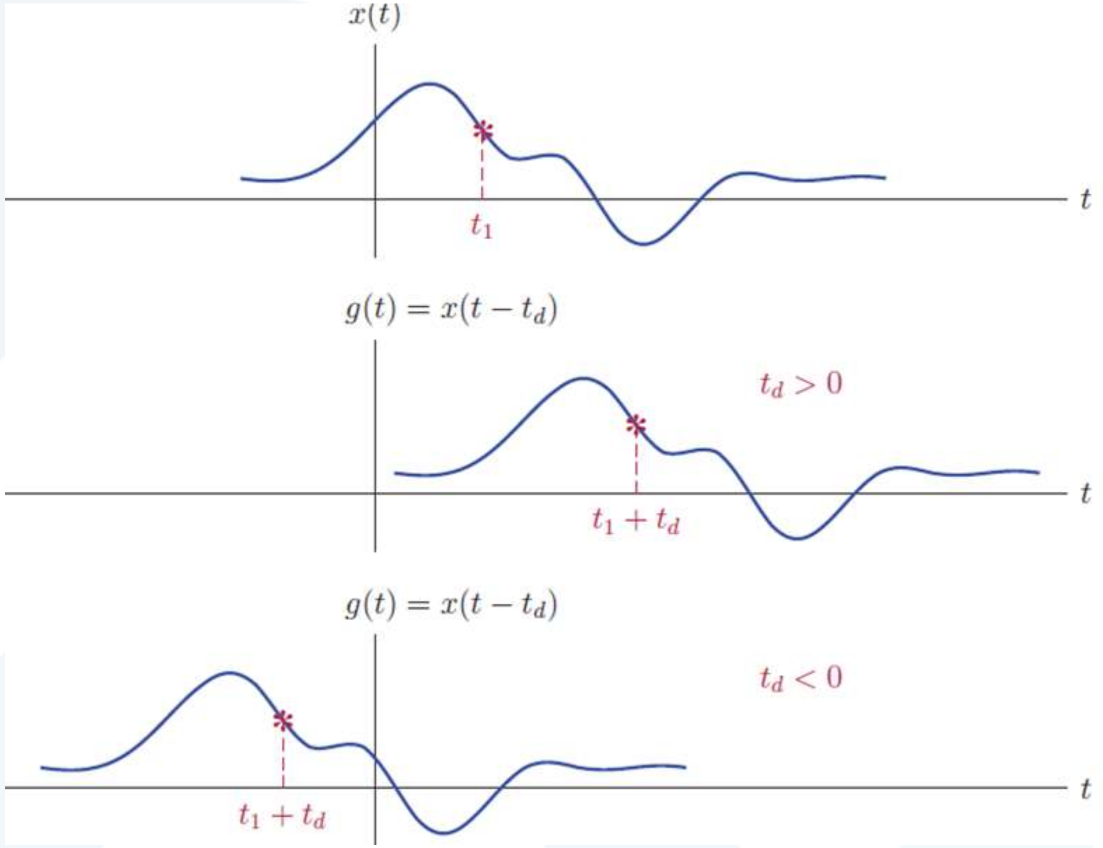
Multiplication of two signals is accomplished by multiplying the amplitudes of the two signals at each time instant. $g(t) = x_1(t).x_2(t)$.



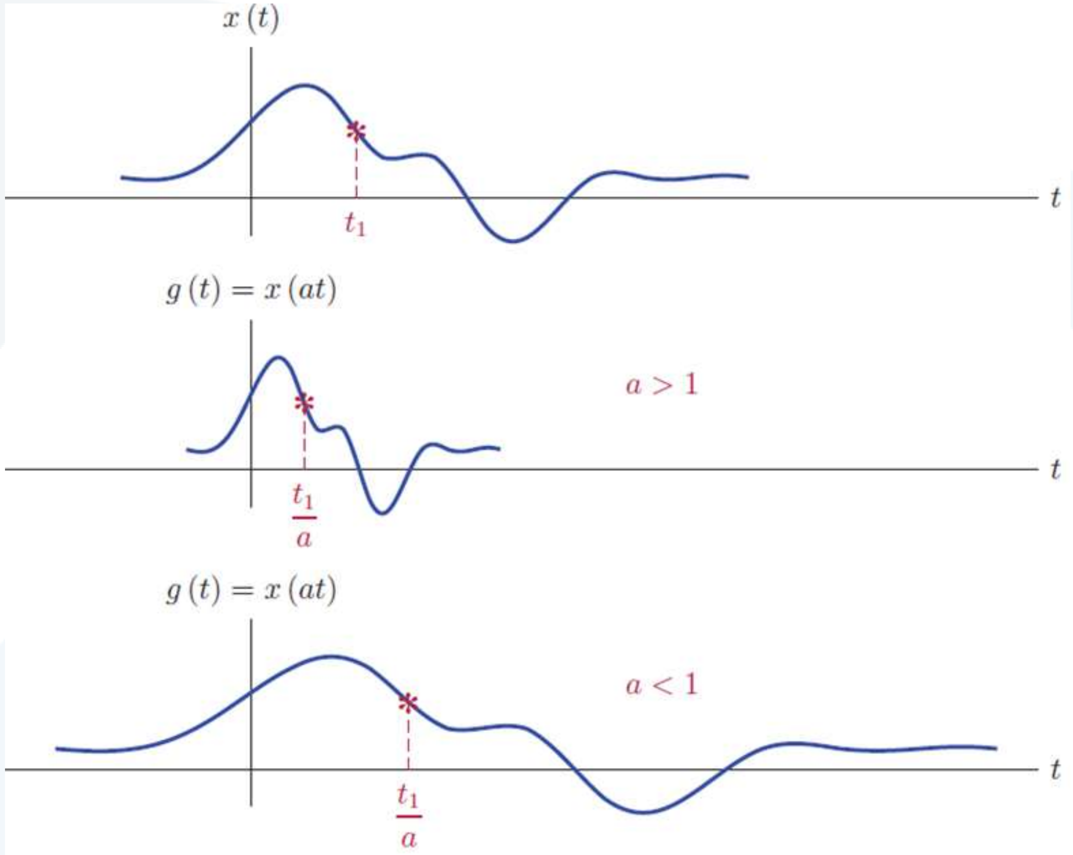




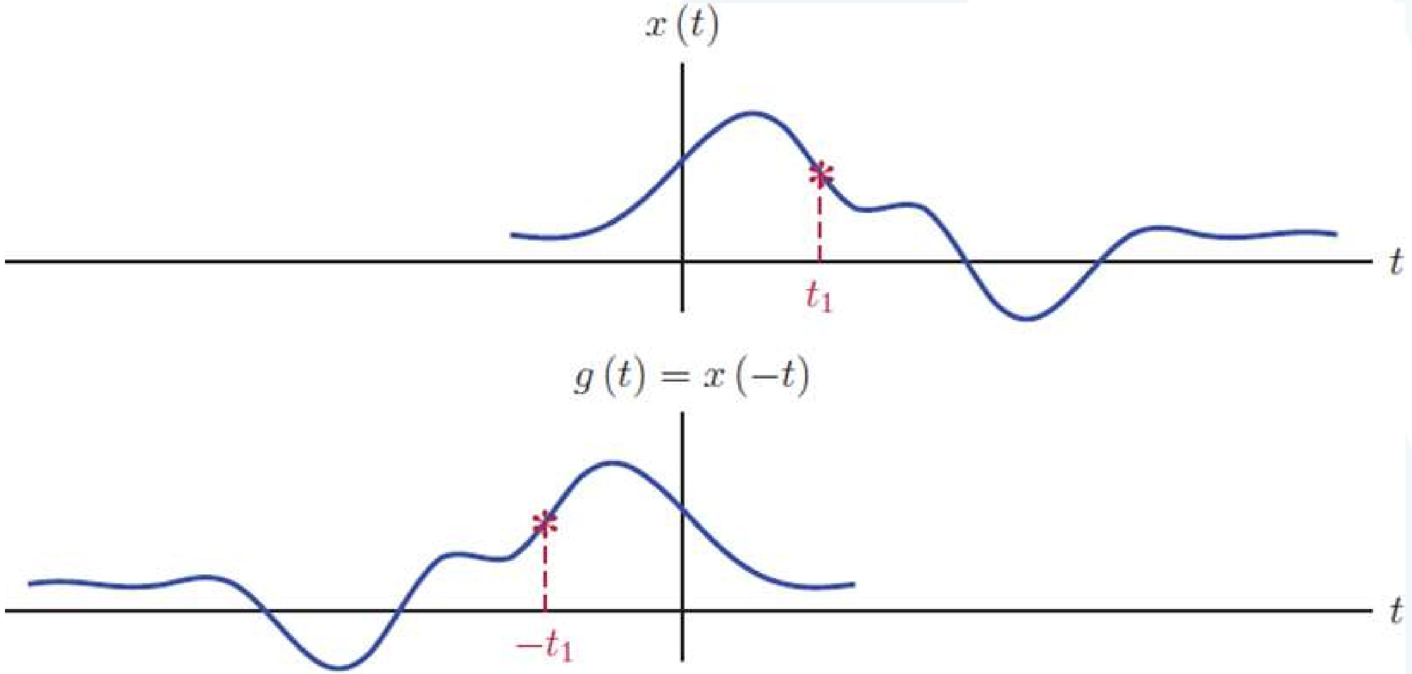
- Time shifting (also called translation) maps the input signal x to the output signal g as given by: $g(t) = x(t - t_d)$; where t_d is a real number.
- Such a transformation shifts the signal (to the left or right) along the time axis.
- If $t_d > 0$, g is shifted to the right by $|t_d|$, relative to x (i.e., delayed in time).
- If $t_d < 0$, g is shifted to the left by $|t_d|$, relative to x (i.e., advanced in time).



- Time scaling (also called dilation) maps the input signal x to the output signal g as given by: $g(t) = x(at)$; where a is a strictly positive real number.
- Such a transformation is associated with a compression/expansion along the time axis.
- If $a > 1$, g is compressed along the horizontal axis by a factor of a, relative to x.
- If $a < 1$, g is expanded (stretched) along the horizontal axis by a factor of $1/a$, relative to x.



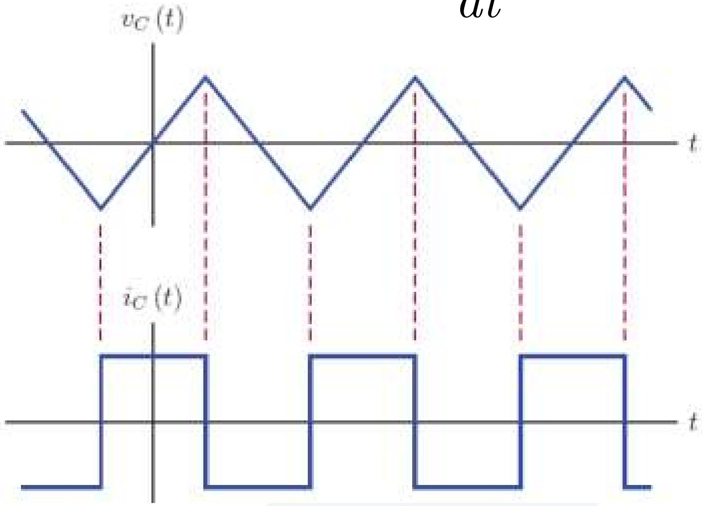
- Time reversal (also known as reflection) maps the input signal x to the output signal g as given by $g(t) = x(-t)$.
- Geometrically, the output signal g is a reflection of the input signal x about the (vertical) line $t = 0$.



- Integration and differentiation

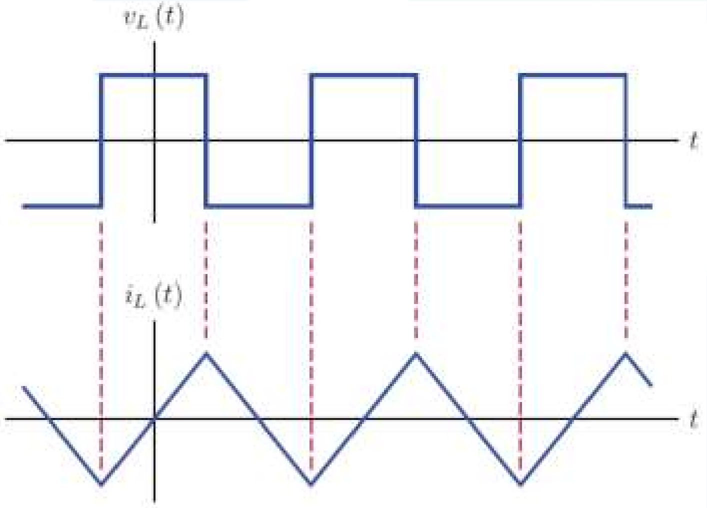
Given a continuous-time signal $x(t)$, a new signal $g(t)$ may be defined as its time **derivative** in the form: $g(t) = dx(t)/dt$. Similarly, a signal can be defined as the **integral** of another signal in the form: $g(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$$i_C(t) = C\frac{dv_C(t)}{dt}$$



ideal capacitor

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau$$



ideal inductor

- Sum of periodic signals

  For two periodic signals x_1 and x_2 with fundamental periods T_1 and T_2, respectively, and the sum $y = x_1 + x_2$:

  - The sum y is periodic if and only if the ratio T_1/T_2 is a rational number (i.e., the quotient of two integers).
  - If y is periodic, its fundamental period is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$), where $T_1/T_2 = q/r$ and q and r are integers and coprime (i.e., have no common factors). (Note that rT_1 is simply the least common multiple of T_1 and T_2).

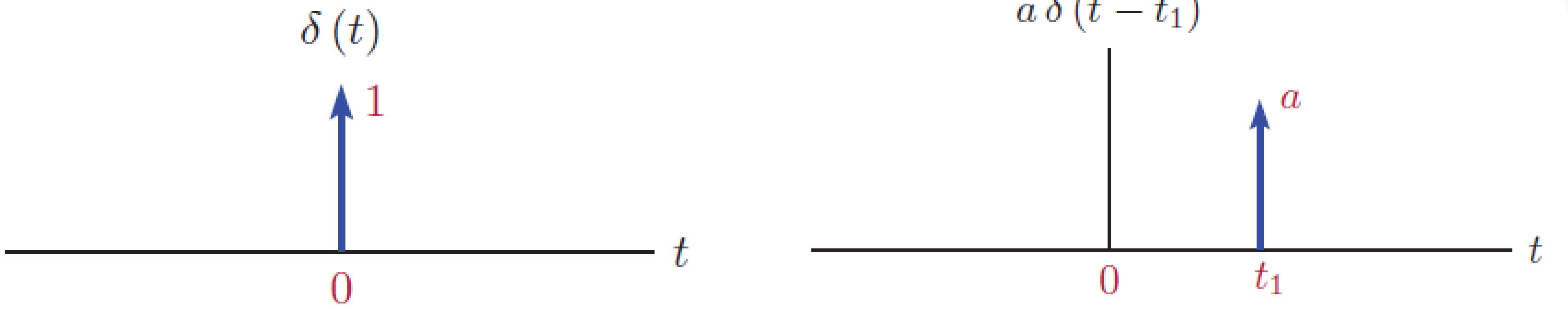
Basic building blocks for continuous-time signals

Unit-impulse function

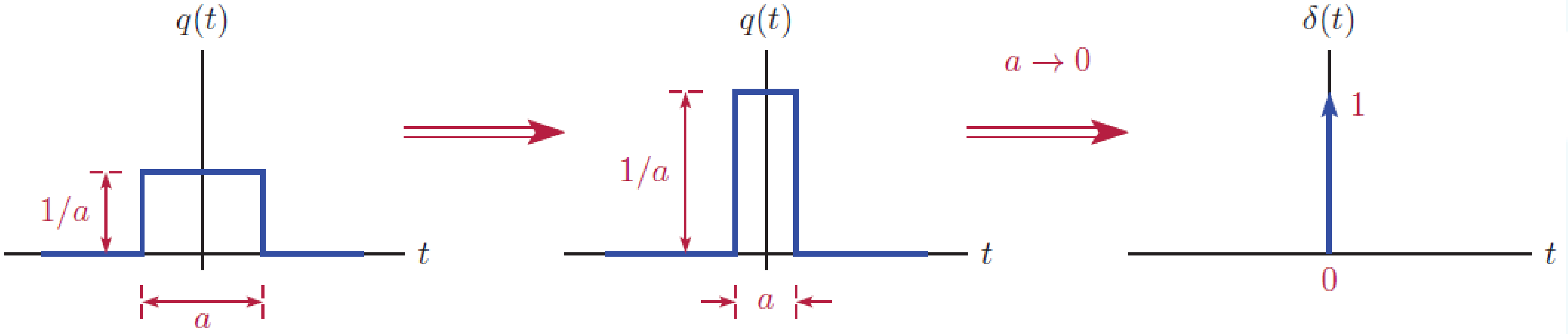
- The unit-impulse function (Dirac delta function or delta function), denoted δ, is defined by:

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \text{undefined}, & \text{if } t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a generalized function.



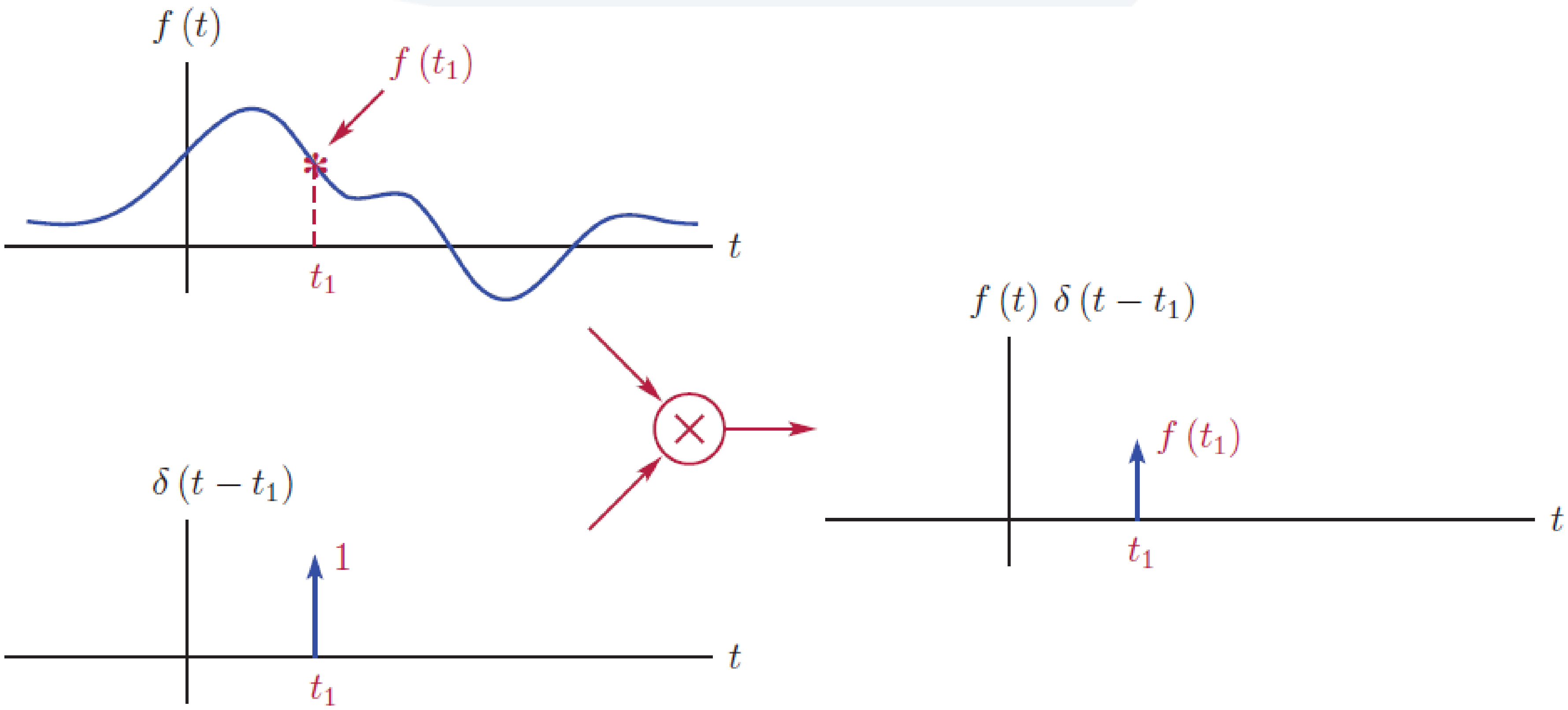
- Define $q(t) = \begin{cases} 1/a, & |t| < a/2 \\ 0, & |t| > a/2 \end{cases}$
- Clearly, for any choice of a, $\int_{-\infty}^{\infty} q(t)dt = 1$
- The function δ can be obtained as the following limit: $\delta(t) = \lim_{a \to 0} q(t)$



- Sampling property. For any continuous function f and any real constant t_1, $f(t)\delta(t - t_1) = f(t_1)\delta(t - t_1)$.

- Sifting property. For any continuous function f and any real constant t_1:

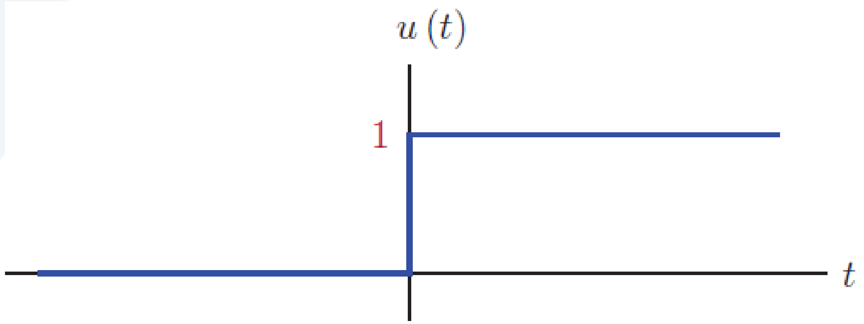
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_1)dt = f(t_1)$$



Unit-Step Function

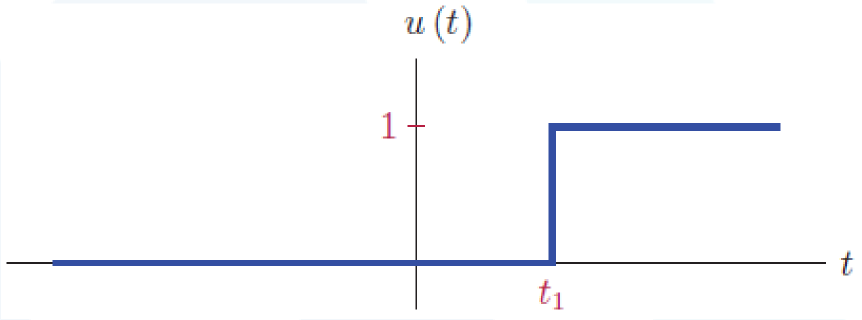
- The unit-step function (also known as the Heaviside function), denoted u, is defined as:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



- A time shifted version of the unit-step function:

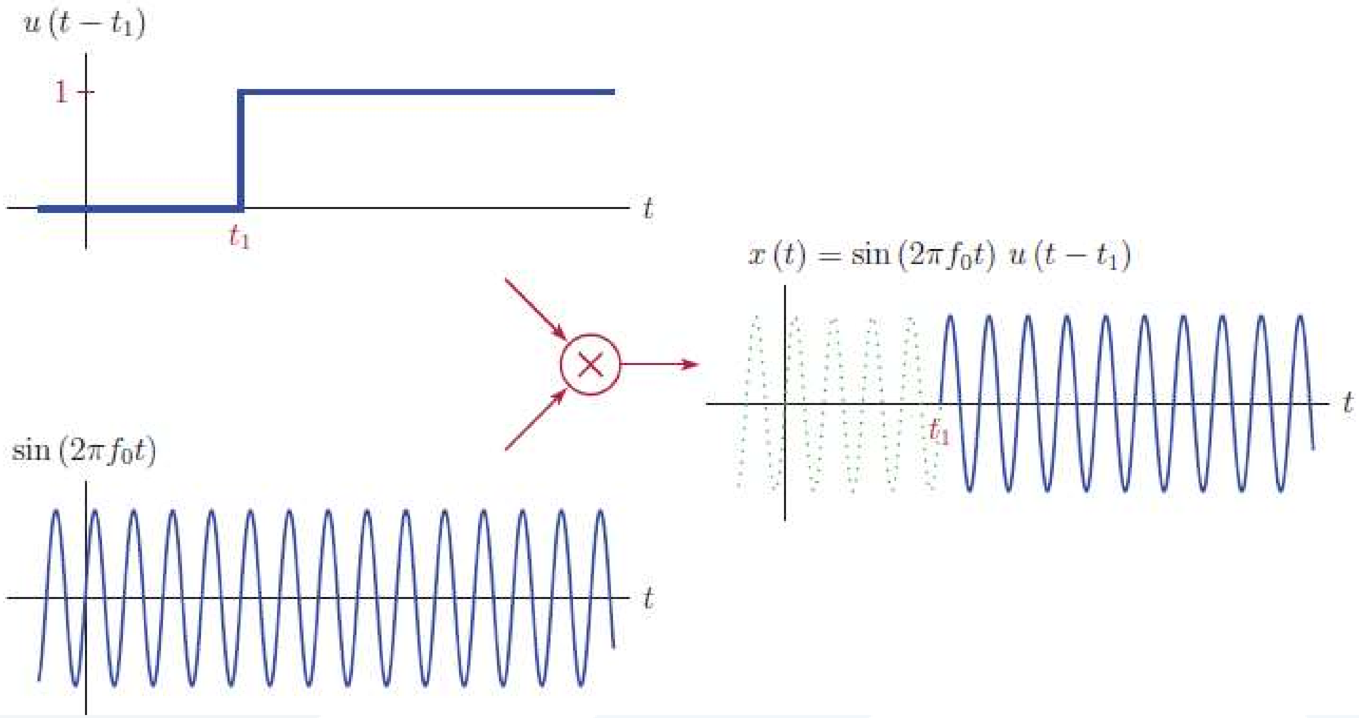
$$u(t - t_1) = \begin{cases} 1, & t \geq t_1 \\ 0, & t < t_1 \end{cases}$$



- Signals begin at $t = 0$ (causal signals) can be described in terms of $u(t)$.

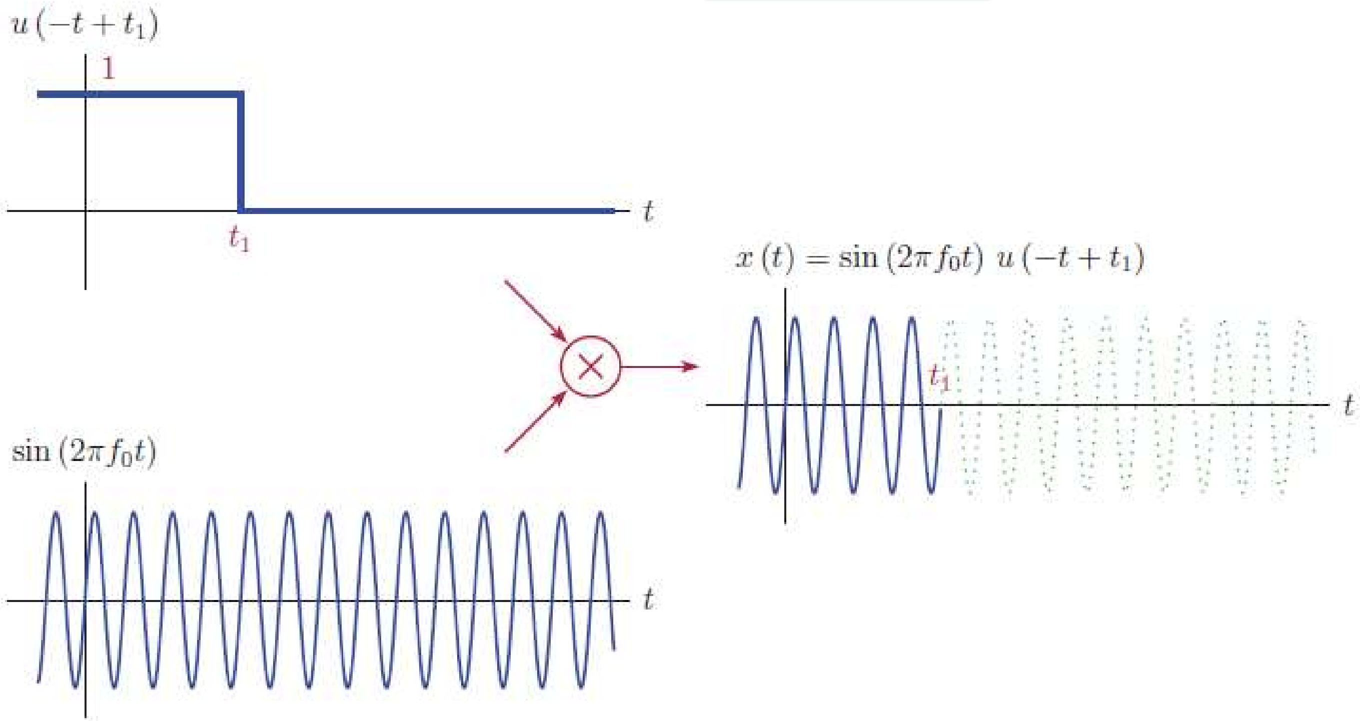
- Using the unit-step function to turn a signal on at a specified time instant:

$$x(t)u(t-t_1) = \begin{cases} \sin(2\pi f_0 t), & t \geq t_1 \\ 0, & t < t_1 \end{cases}$$

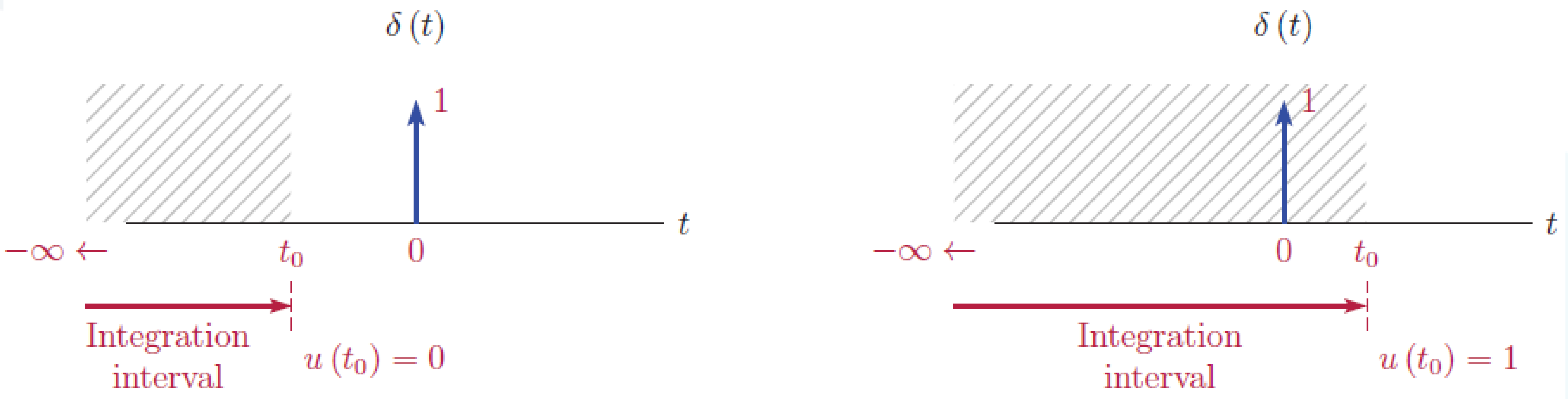


- Using the unit-step function to turn a signal off at a specified time instant:

$$x(t)u(-t+t_1) = \begin{cases} \sin(2\pi f_0 t), & t \le t_1 \\ 0, & t > t_1 \end{cases}$$

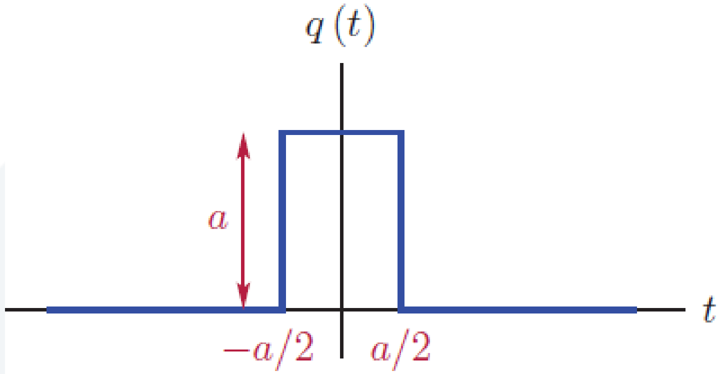


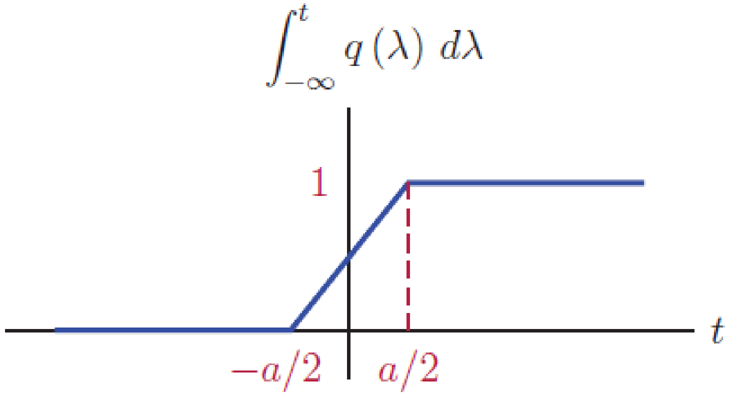
- The Relationship between the unit-step function and the unit-impulse function:



$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$\delta(t) = \frac{du(t)}{dt}$$



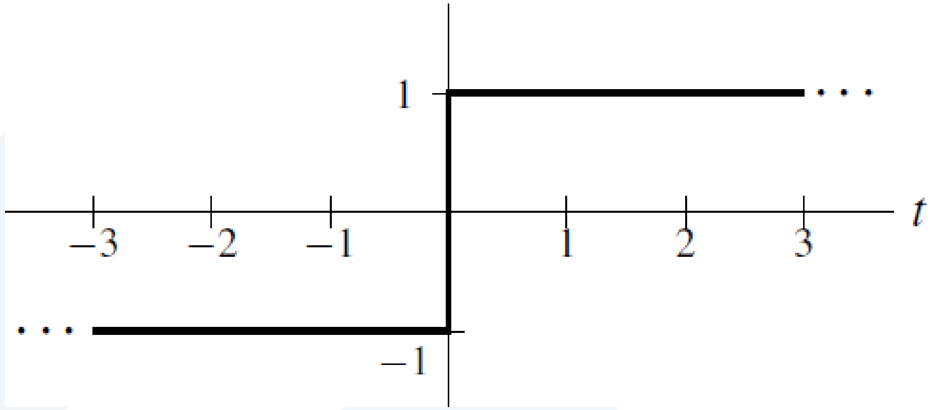


Signum Function

- The signum function, denoted sgn, is defined as:

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

- From its definition, one can see that the signum function simply computes the sign of a number.

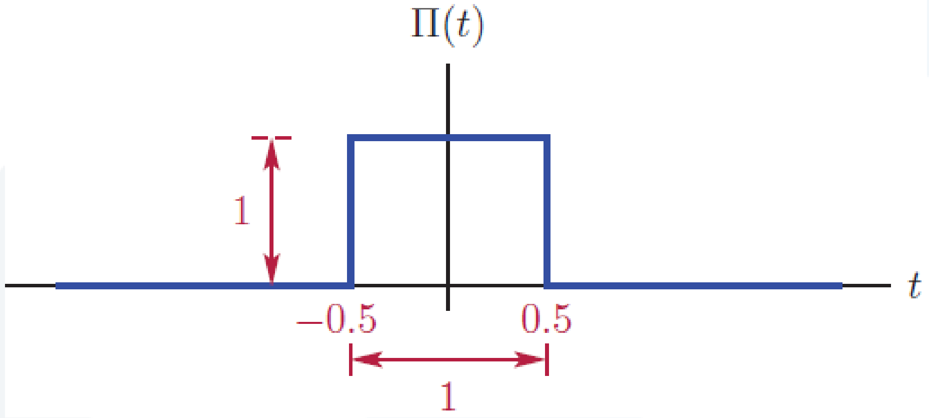


Unit-pulse function

- The unit-pulse function (also called the unit-rectangular pulse function), denoted rect, is given by:

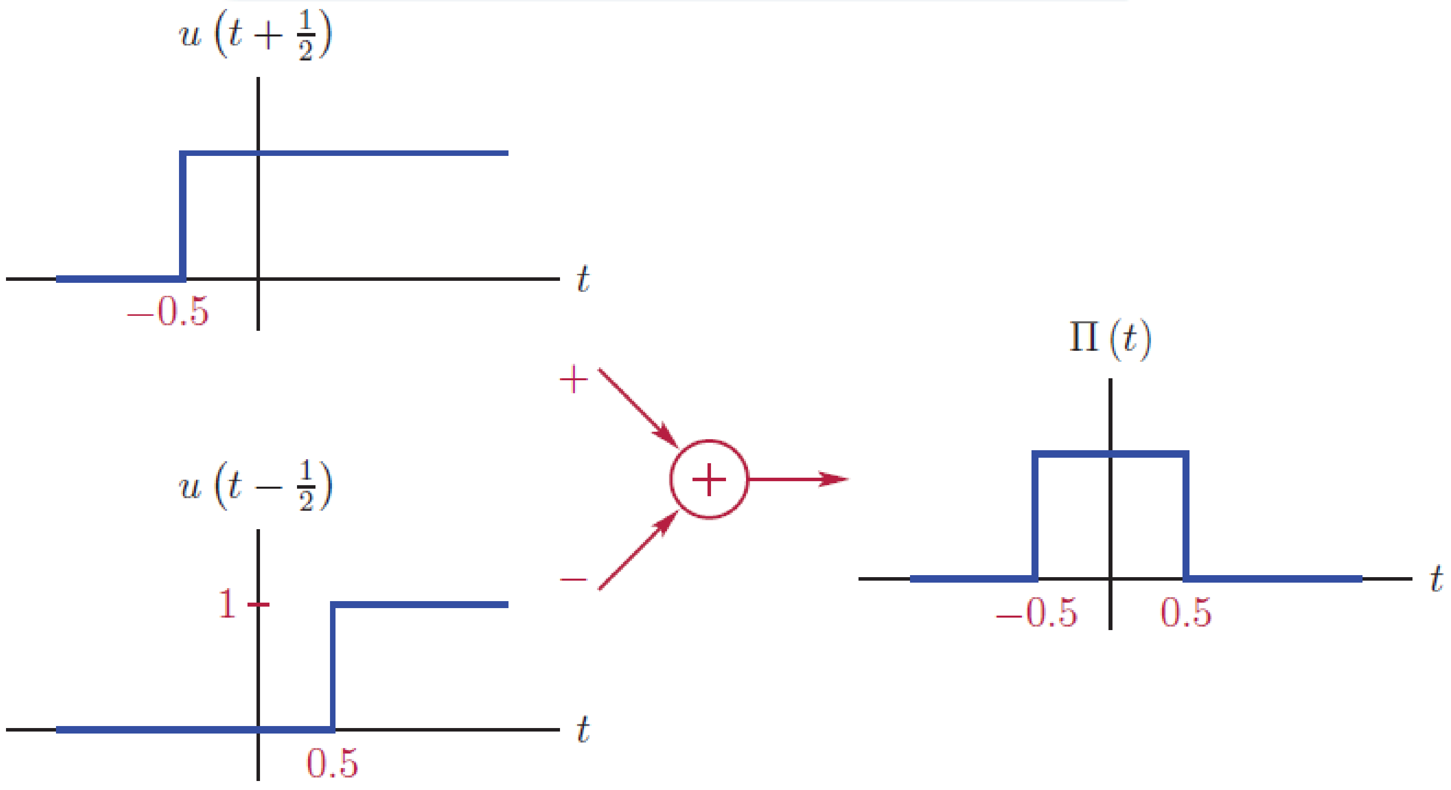
$$\operatorname{rect} t = \prod(t) = \begin{cases} 1, & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual value of $\operatorname{rect} t$ at $t = \pm½$ is unimportant. Sometimes ≠ values are used.



- Constructing a unit-pulse function from unit-step functions:

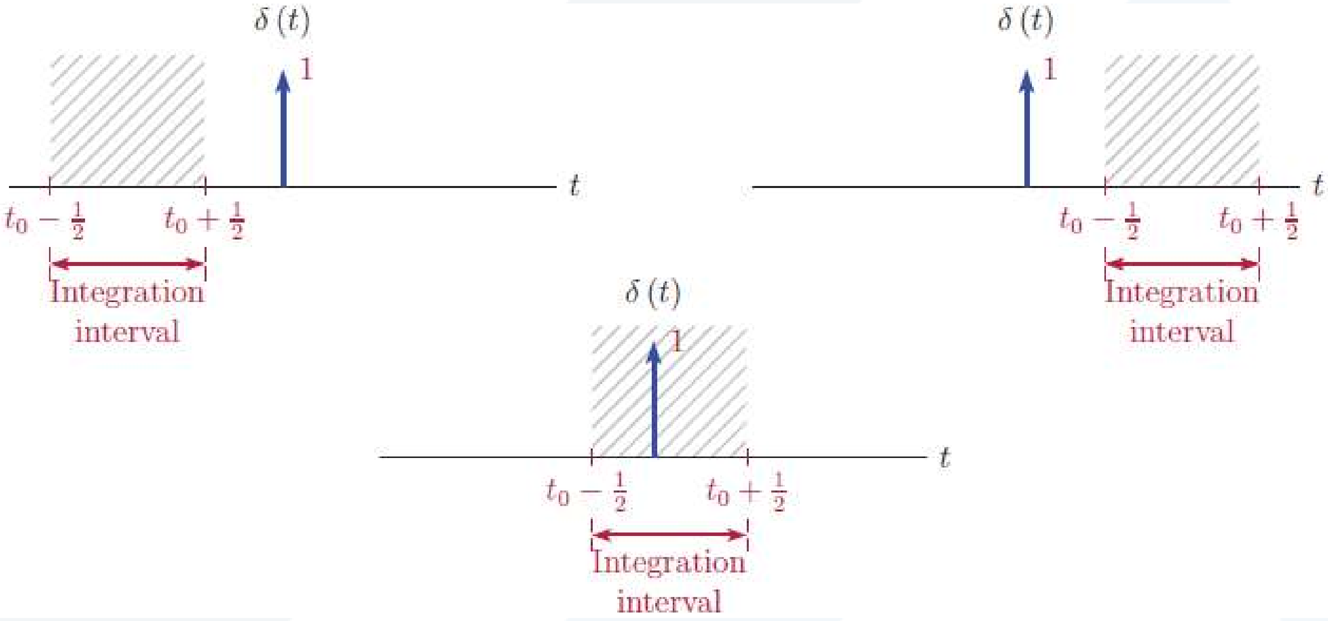
$$\prod(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2})$$



- Constructing a unit-pulse function from unit- impulse functions:

$$\prod(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2}) = \int_{-\infty}^{t+1/2} \delta(\tau)d\tau - \int_{-\infty}^{t-1/2} \delta(\tau)d\tau = \int_{t-1/2}^{t+1/2} \delta(\tau)d\tau$$

$$\int_{t-1/2}^{t+1/2} \delta(\tau)d\tau = \begin{cases} 1, & t-\frac{1}{2} < 0 \text{ and } t+\frac{1}{2} > 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

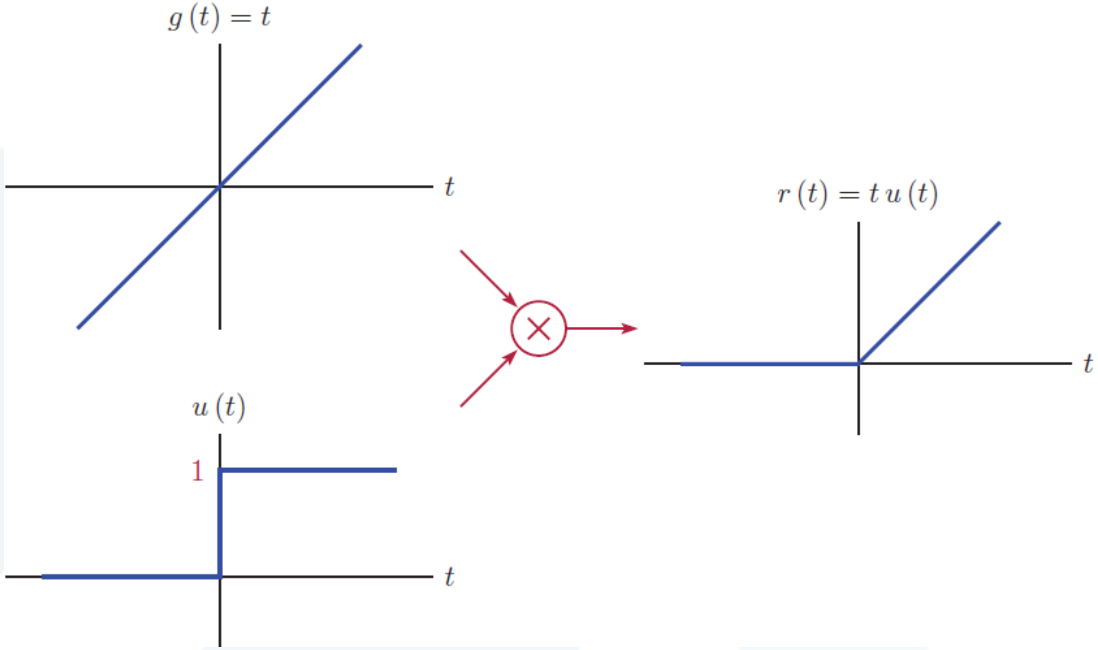


Unit-Ramp Function

- The unit-ramp function, denoted r, is defined as:

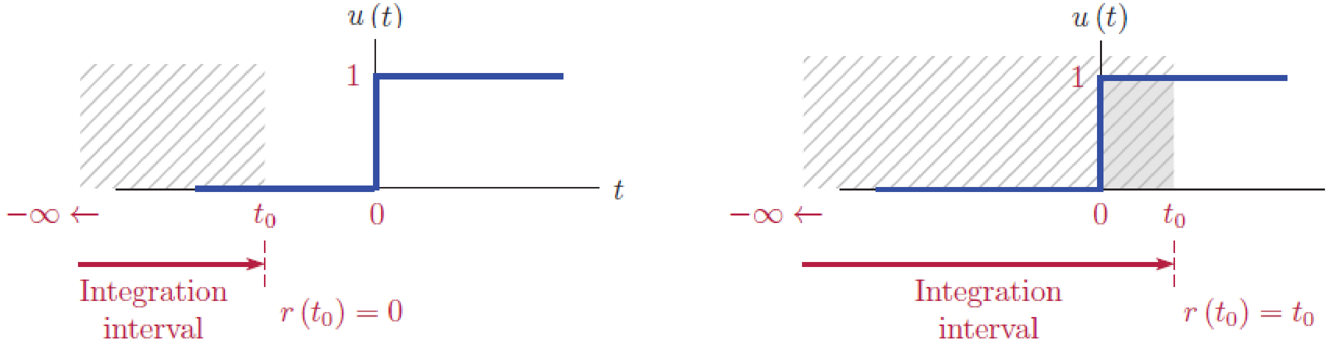
$$r(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

or, equivalently: $r(t) = tu(t)$.



- Constructing a unit-ramp function from a unit-step:

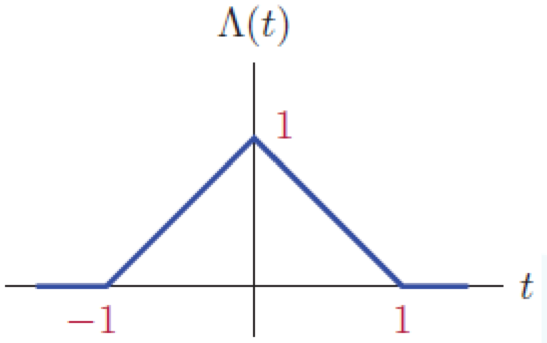
$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$



Unit Triangular Function

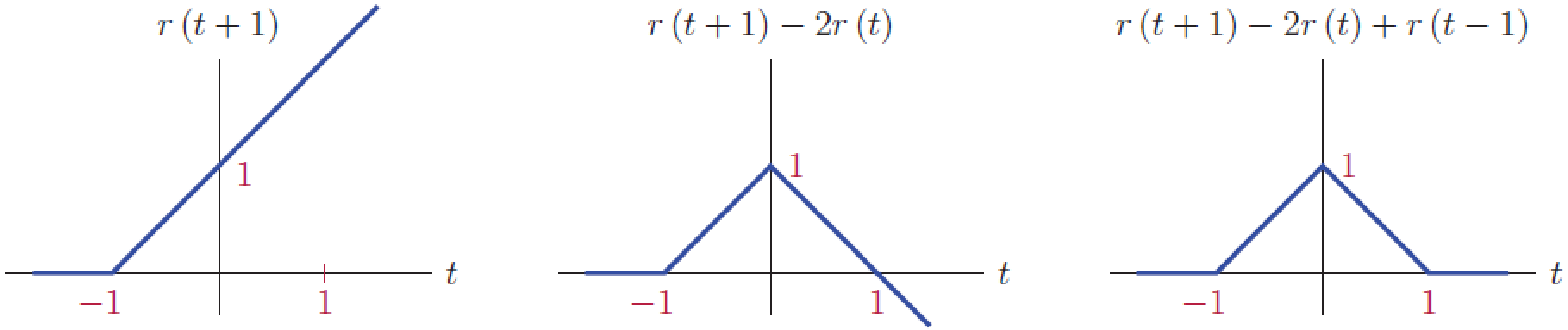
- The unit triangular function (unit-triangular pulse function), denoted tri, is defined as:

$$\operatorname{tri} t = \Lambda(t) = \begin{cases} 1 - |t|, & \text{if } |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$$



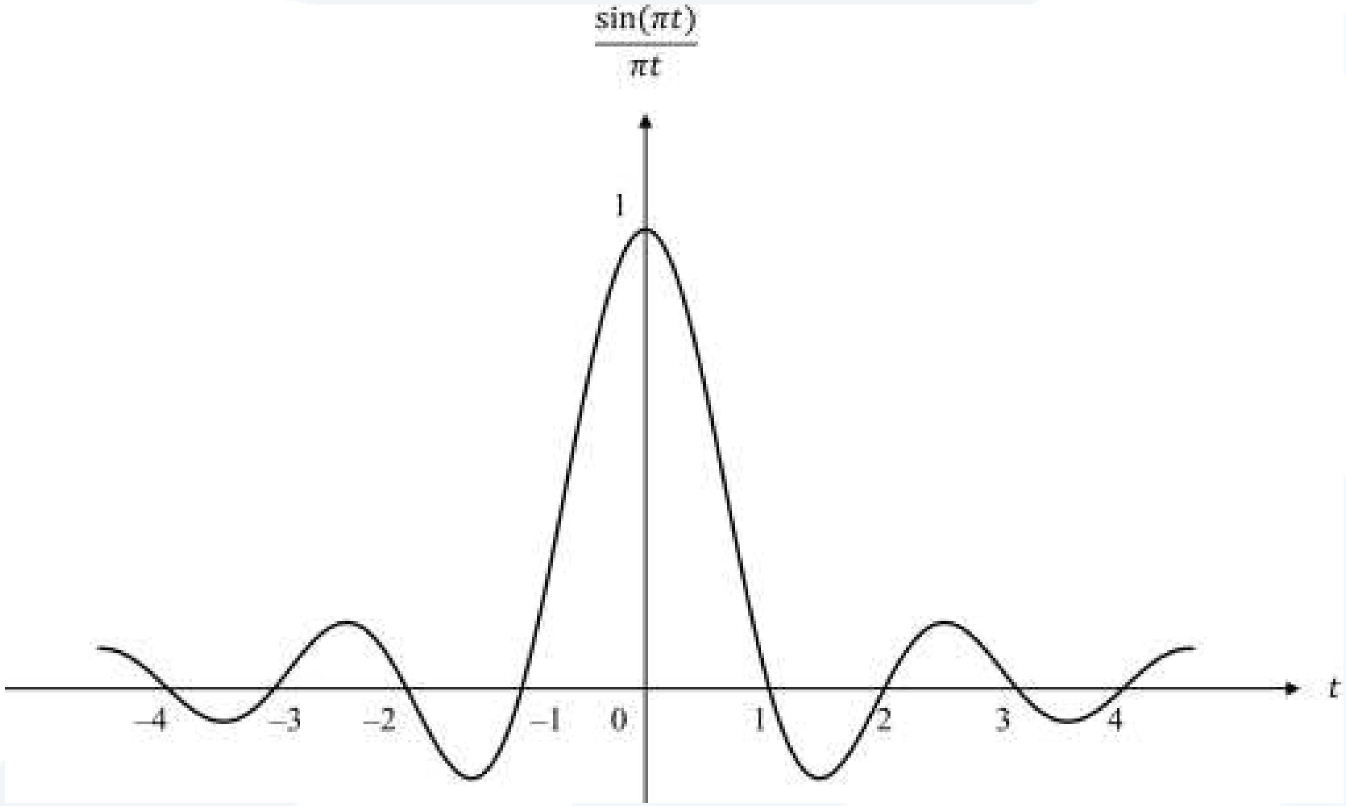
- Constructing a unit-triangle using unit-ramp functions:

$$\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$$



Cardinal Sine Function

- The cardinal sine function, denoted sinc, is given by $\mathrm{sinc}\,t = \dfrac{\sin(\pi t)}{\pi t}$



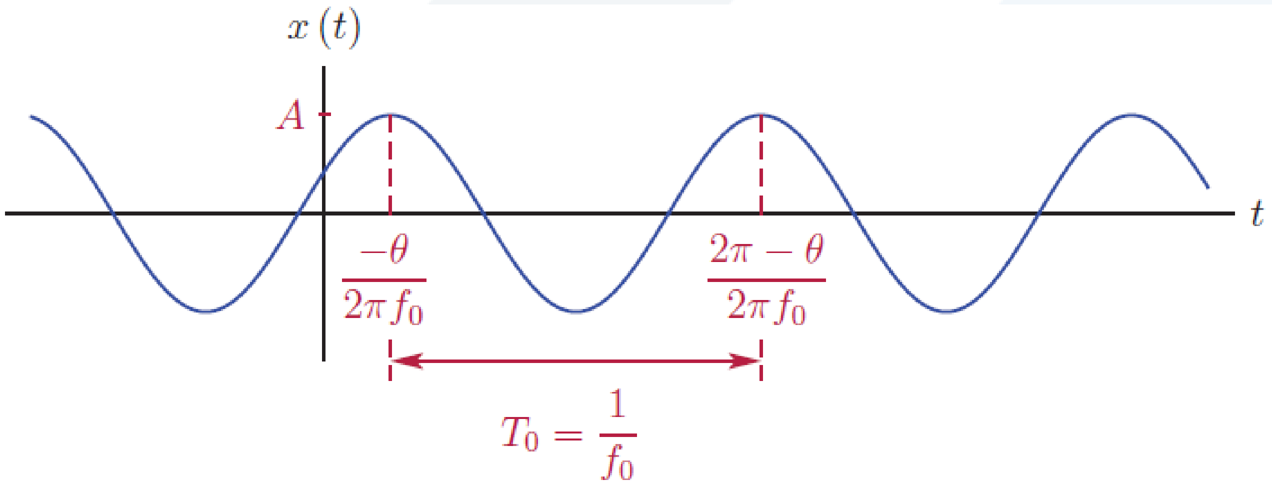
Sinusoidal Signal

- A real sinusoidal function is a function of the form:

$$x(t) = A\cos(\omega_0 t + \theta)$$

where A is the amplitude of the signal, ω_0 is the radian frequency (rad/s), and θ is the initial phase angle (rad), all are real constants.

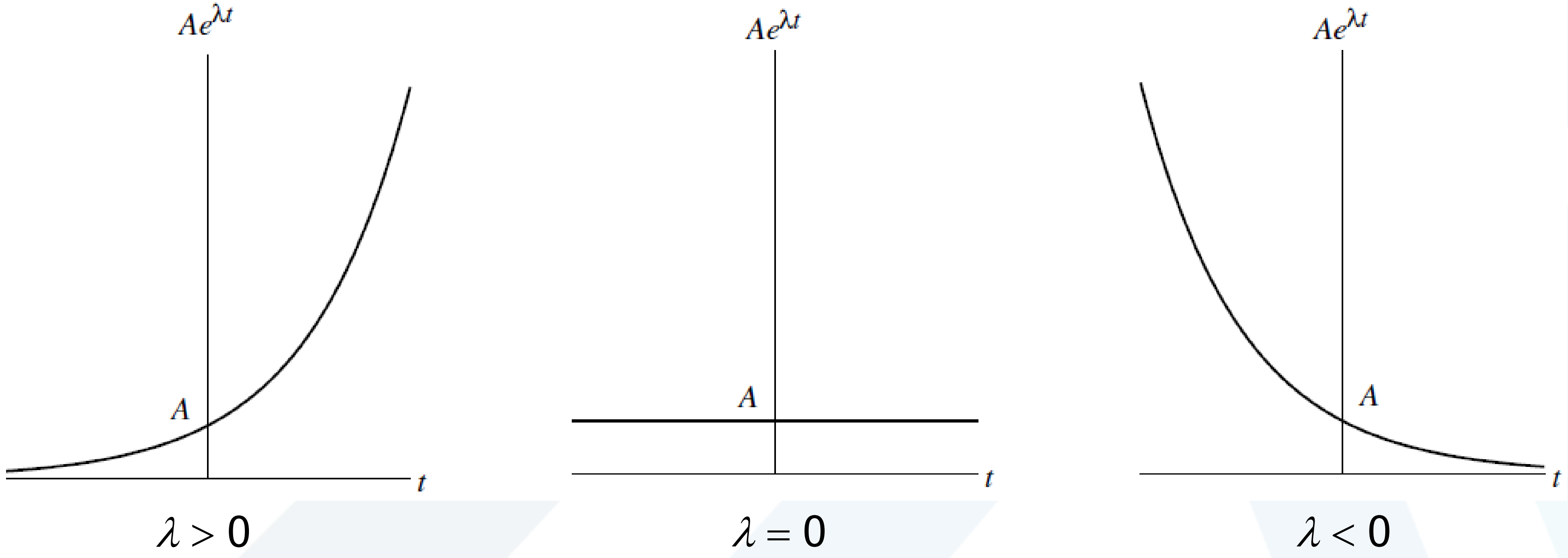
$\omega_0 = 2\pi f_0$ where f_0 is the frequency (Hz), $T_0 = 1/f_0$ is the period (s).



Complex Exponential Function

- A complex exponential function is a function of the form $x(t) = Ae^{\lambda t}$, where A and λ are complex constants.
- A complex exponential can exhibit one of a number of distinct modes of behavior, depending on the values of A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.
- A real exponential function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be real numbers.
- A real exponential can exhibit one of three distinct modes of behavior, depending on the value of λ, as illustrated below.

- If $\lambda > 0$, $x(t)$ increases exponentially as t increases (growing exponential).
- If $\lambda < 0$, $x(t)$ decreases exponentially as t increases (decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the constant A.



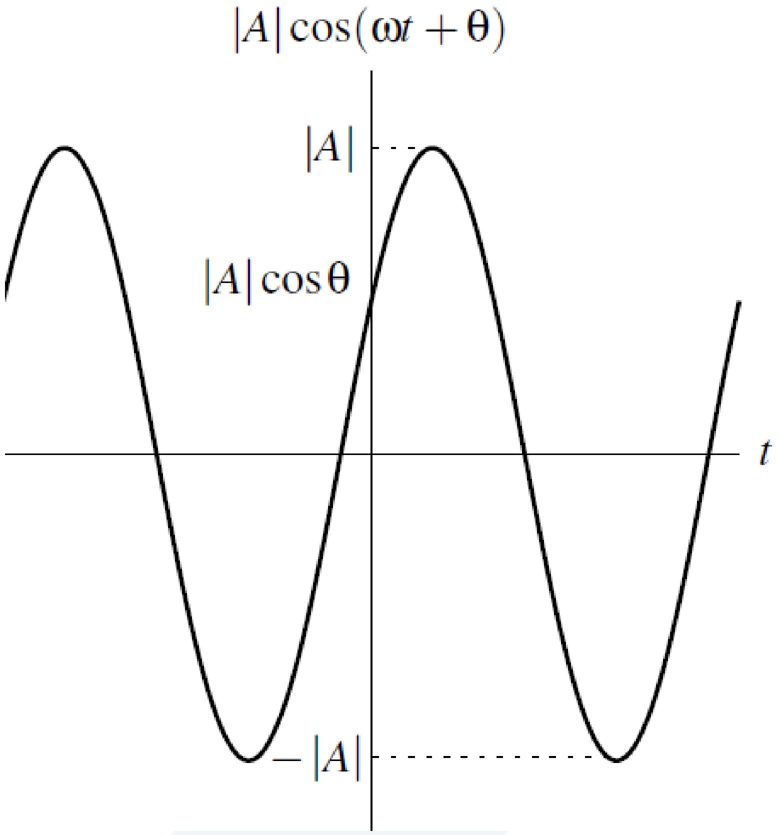
Complex Sinusoidal Function

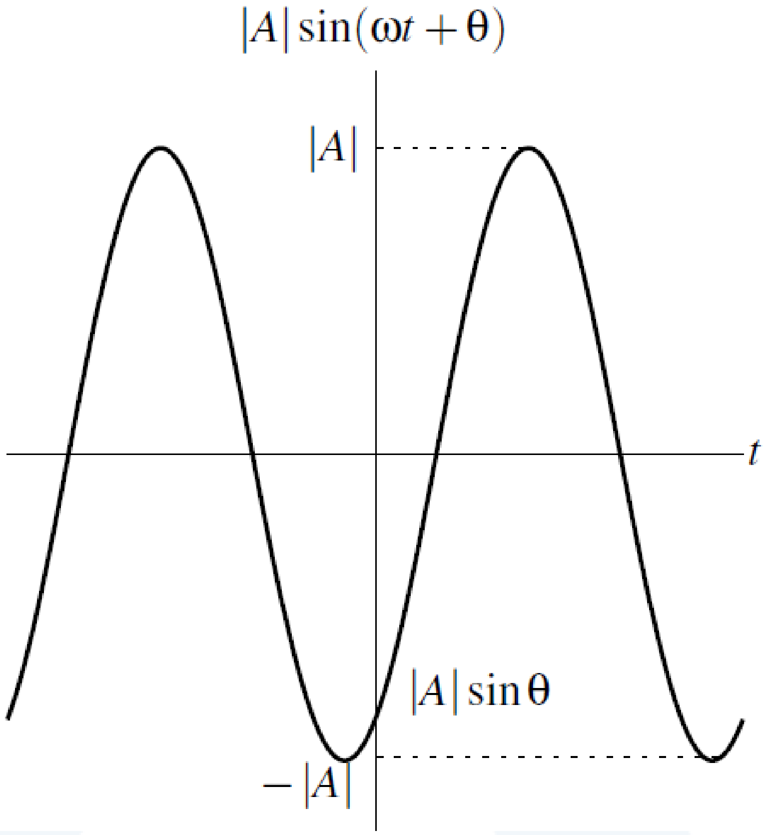
- A complex sinusoidal function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is complex and λ is purely imaginary (i.e., $\mathrm{Re}\{\lambda\} = 0$).
- That is, a complex sinusoidal function is a function of the form $x(t) = Ae^{j\omega t}$, where A is complex and ω is real.
- By expressing A in polar form as $A = |A|e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as:

$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the same except for a time shift.

- Also, x is periodic with fundamental period $T = 2\pi/|\omega|$ and fundamental frequency $|\omega|$.



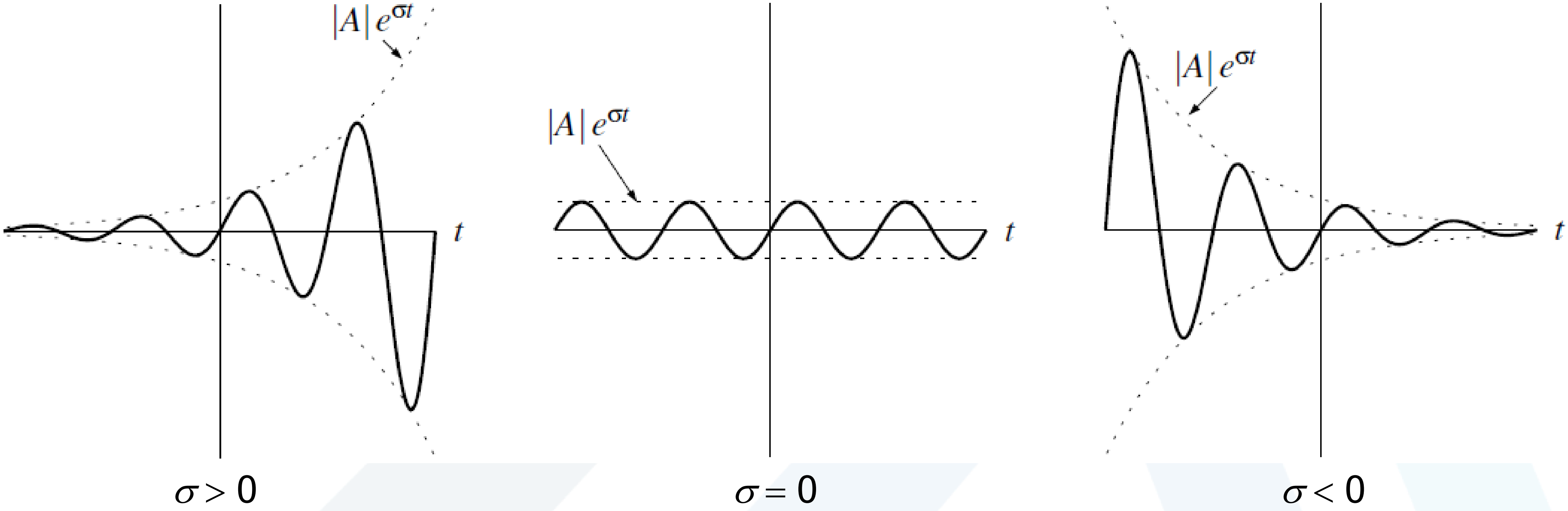


- In the most general case of a complex exponential function $x(t) = Ae^{\lambda t}$, A and λ are both complex.
- Letting $A = |A|e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as:

$$x(t) = \underbrace{|A|e^{\sigma t}\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|e^{\sigma t}\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}$$

- One of three distinct modes of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, Re$\{x\}$ and Im$\{x\}$ are real sinusoids.
- If $\sigma > 0$, Re$\{x\}$ and Im$\{x\}$ are each the product of a real sinusoid and a growing real exponential.

- If $\sigma < 0$, Re$\{x\}$ and Im$\{x\}$ are each the product of a real sinusoid and a decaying real exponential.



- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as:

$$A e^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t)$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities:

$$A\cos(\omega t+\theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$

$$A\sin(\omega t+\theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right]$$

Impulse decomposition for continuous-time signals

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta)\prod\left(\frac{t-n\Delta}{\Delta}\right)$$