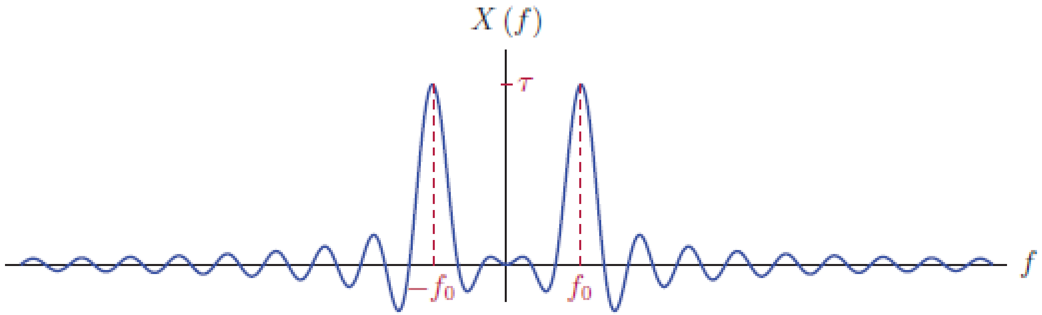
CEDC403: Signals and Systems

Lecture Notes 2: Signal Representation and Modeling: Part B



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Chapter 1

Signal Representation and Modeling

Energy and power definitions

- The energy of a continuous time signal $x(t)$ is given by: $E_x = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt$
- The average power of a continuous time signal $x(t)$ is given by:

  periodic complex signal: $P_x = \dfrac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left| x(t) \right|^2 dt$

  non-periodic complex signal: $P_x = \lim\limits_{T \to \infty} \dfrac{1}{T} \int_{-T/2}^{T/2} \left| x(t) \right|^2 dt$

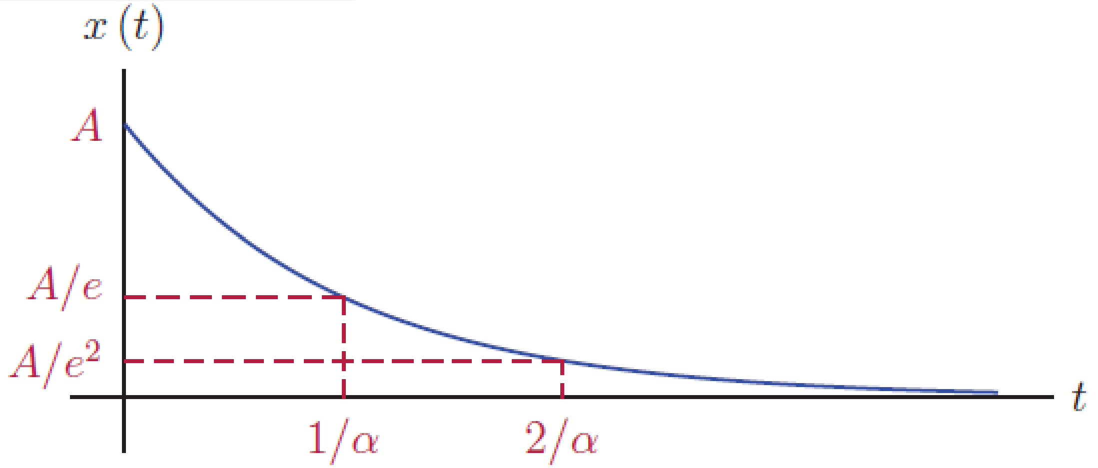
- Energy signals are those that have finite energy and zero power, i.e., $E_x < \infty$, and $P_x = 0$.
- Power signals are those that have finite power and infinite energy, i.e., $E_x \to \infty$, and $P_x < \infty$.

- Example 1: Energy of exponential signal

  Compute the energy of the exponential signal (where $\alpha > 0$).

$$x(t) = \begin{cases} Ae^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E_x = \int_0^{\infty} A^2 e^{-2\alpha t} dt = \frac{A^2}{2\alpha}$$



- Example 2: Power of a sinusoidal signal

$$x(t) = A\sin(2\pi f_0 t + \theta)$$

$$P_x = f_0 \int_{-1/2f_0}^{1/2f_0} A^2 \sin^2(2\pi f_0 t + \theta) dt = \frac{A^2}{2}$$

Symmetry properties

Even and odd symmetry

- A real-valued signal is said to have even symmetry if it has the property: $x(-t) = x(t)$ for all values of t.
- A real-valued signal is said to have odd symmetry if it has the property: $x(-t) = -x(t)$ for all values of t.

Decomposition into even and odd components

- Every real-valued signal $x(t)$ has a unique representation of the form: $x(t) = x_e(t) + x_o(t)$; where the signals x_e and x_o are even and odd, respectively.
- In particular, the signals x_e and x_o are given by:

$$x_e(t) = ½[x(t) + x(-t)] \text{ and } x_o(t) = ½[x(t) - x(-t)]$$

Symmetry properties for complex signals

- A complex-valued signal is said to have conjugate symmetric if it has the property: $x(-t) = x^*(t)$ for all values of t.
- A complex-valued signal is said to have conjugate antisymmetric if it has the property: $x(-t) = -x^*(t)$ for all values of t.

Decomposition of complex signals

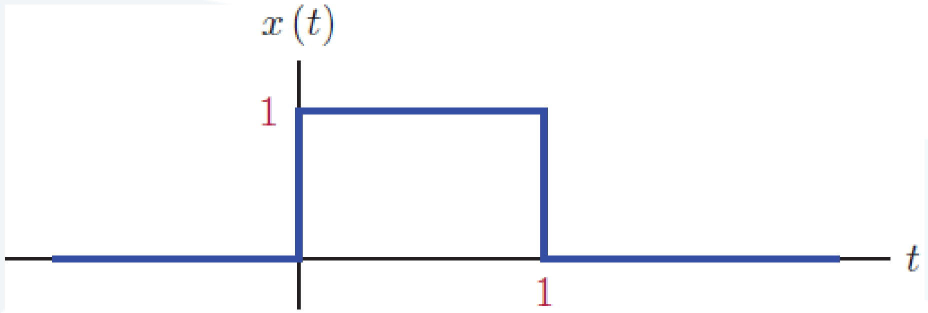
- Every complex-valued signal $x(t)$ has a unique representation of the form: $x(t) = x_E(t) + x_O(t)$; where the signals x_E and x_O are conjugate symmetric and conjugate antisymmetric, respectively.
- In particular, the signals x_E and x_O are given by:

$$x_E(t) = ½[x(t) + x^*(-t)] \text{ and } x_O(t) = ½[x(t) - x^*(-t)]$$

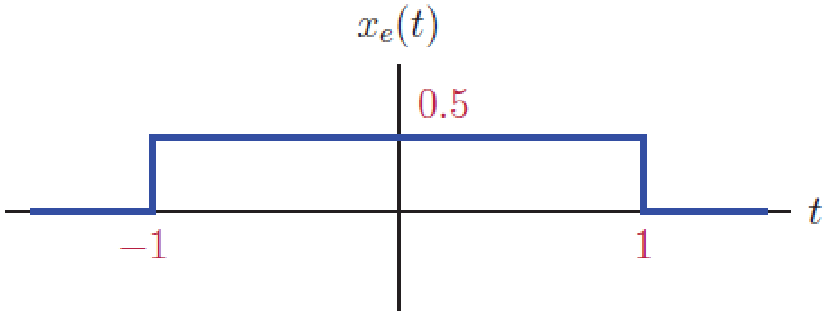
- Example 3: Even and odd components of a rectangular pulse

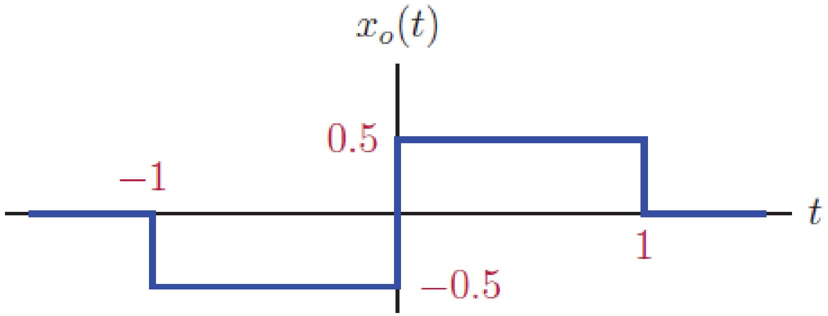
Determine the even and the odd components of the rectangular pulse signal.

$$\Pi\left(t-\tfrac{1}{2}\right)=\begin{cases}1 & \text{if } 0<t<1\\ 0 & \text{otherwise}\end{cases}$$



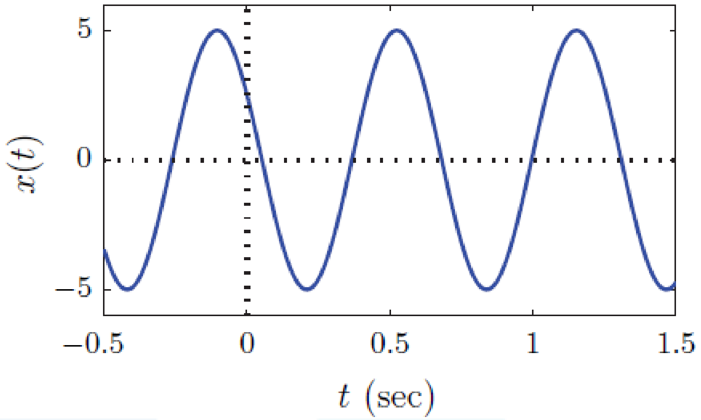
$$x_e(t)=\frac{\Pi\left(t-\frac{1}{2}\right)+\Pi\left(-t-\frac{1}{2}\right)}{2}=\frac{\Pi(t/2)}{2},\qquad x_o(t)=\frac{\Pi\left(t-\frac{1}{2}\right)-\Pi\left(-t-\frac{1}{2}\right)}{2}$$





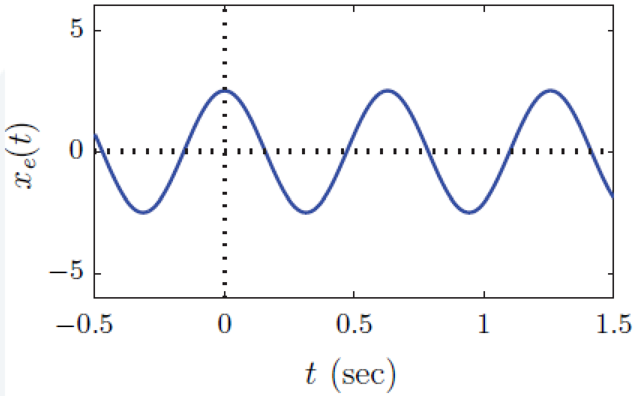
- Example 4: Even and odd components of a sinusoidal signal

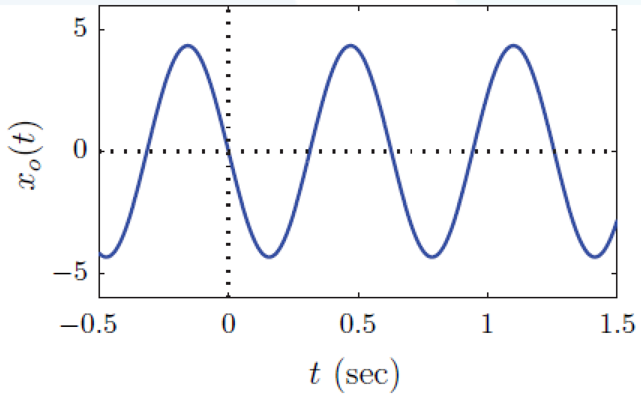
Determine the even and the odd components of the sinusoidal signal $x(t) = 5\cos(10t + \pi/3)$.



$$x_e(t) = \frac{5}{2}\cos(10t + \pi/3) + \frac{5}{2}\cos(-10t + \pi/3)$$
$$= 2.5\cos(10t)$$

$$x_o(t) = \frac{5}{2}\cos(10t + \pi/3) - \frac{5}{2}\cos(-10t + \pi/3)$$
$$= -\frac{5\sqrt{3}}{2}\sin(10t)$$





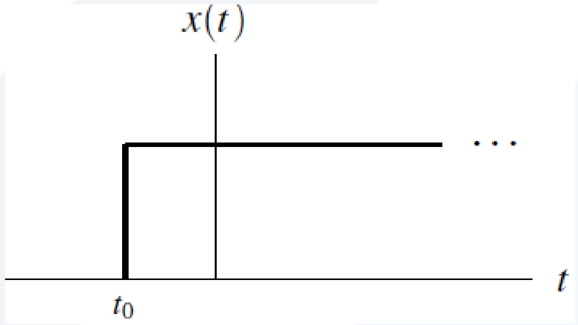
- Example 5: Symmetry of a complex exponential signal

  Consider the complex exponential signal $x(t) = Ae^{j\omega t}$, A: real

  $$x(-t) = Ae^{-j\omega t}) = (Ae^{-j\omega t})^* = x^*(t) \Rightarrow \text{the signal } x(t) \text{ is conjugate symmetric.}$$

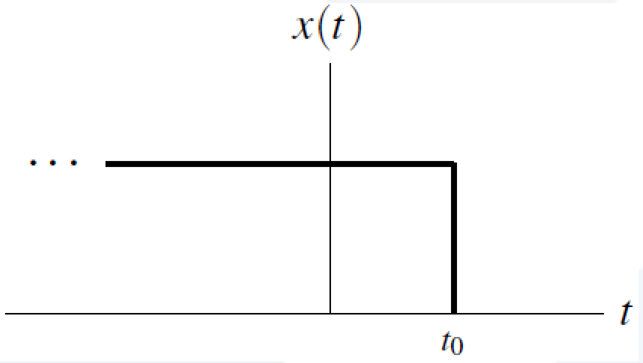
Right and Left-Sided Signals

- A signal x is said to be right sided if, for some (finite) real constant t_0, the following condition holds: $x(t) = 0$ for all $t < t_0$ (i.e., x is only potentially nonzero to the right of t_0).



- A signal x is said to be causal if $x(t) = 0$ for all $t < 0$.

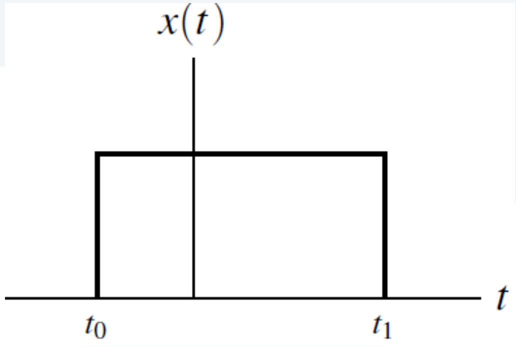
- A causal signal is a special case of a right-sided signal.
- A causal signal is not to be confused with a causal system.
- A signal x is said to be left sided if, for some (finite) real constant t_0, the following condition holds: $x(t) = 0$ for all $t > t_0$ (i.e., x is only potentially nonzero to the left of t_0).



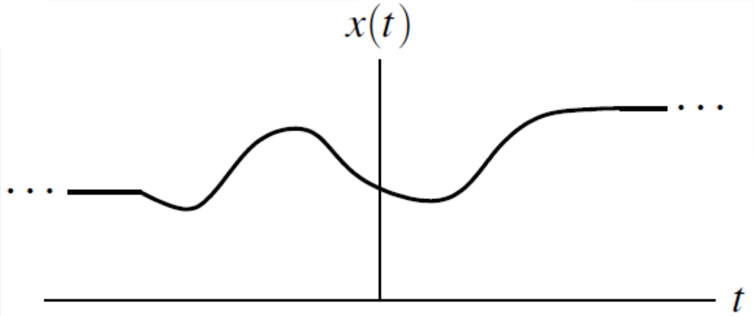
- A signal x is said to be anticausal if $x(t) = 0$ for all $t > 0$.
- An anticausal signal is a special case of a left-sided function.
- An anticausal signal is not to be confused with a anticausal system.

Finite-Duration and Two-Sided Signals

- A signal that is both left sided and right sided is said to be finite duration (or finite support).



- A signal that is neither left sided nor right sided is said to be two sided.



Bounded Signals

- A signal x is said to be bounded if there exists some (finite) positive real constant A such that $|x(t)| \leq A$ for all t (i.e., $x(t)$ is finite for all t).
- For example, the sine and cosine functions are bounded, since

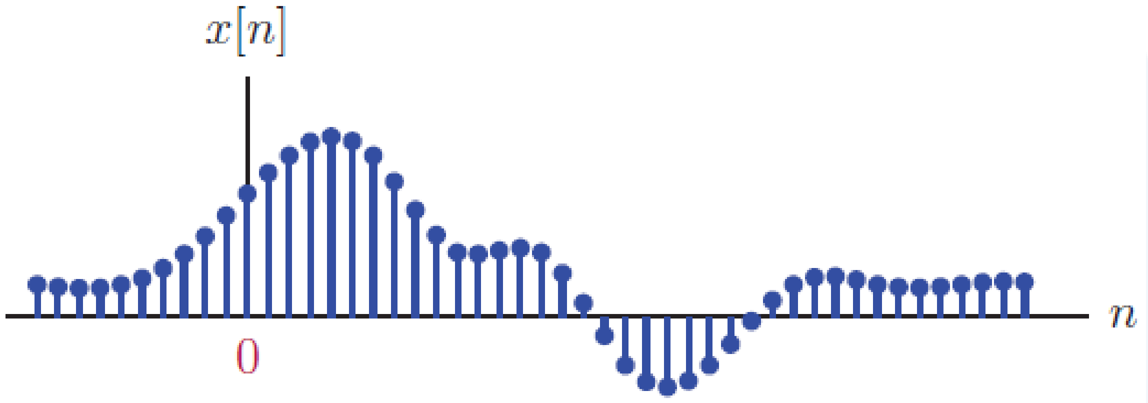
$$|\sin t| \leq 1 \text{ for all } t \text{ and } |\cos t| \leq 1 \text{ for all } t$$

- In contrast, the tangent signal and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t \to \frac{\pi}{2}} \left|\tan t\right| = \infty \quad \text{and} \quad \lim_{t \to \infty} \left|p(t)\right| = \infty$$

5. Discrete-Time Signals

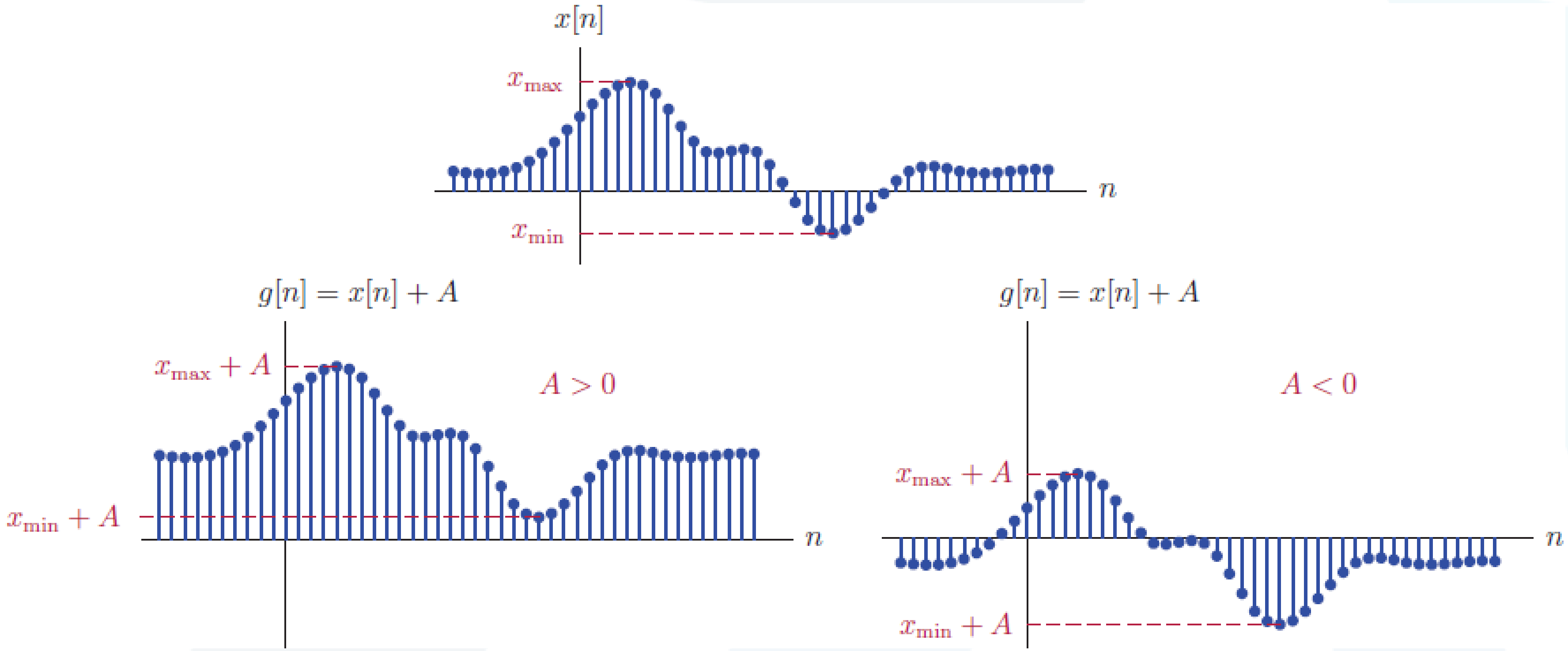
- Discrete-time signals are not defined at all time instants. they are defined only at time instants that are integer multiples of a fixed time increment T, that is, at $t = nT$.
- Consequently, the mathematical model for a discrete-time signal is a function $x[n]$ in which independent variable n is an integer, and is referred to as the sample index.



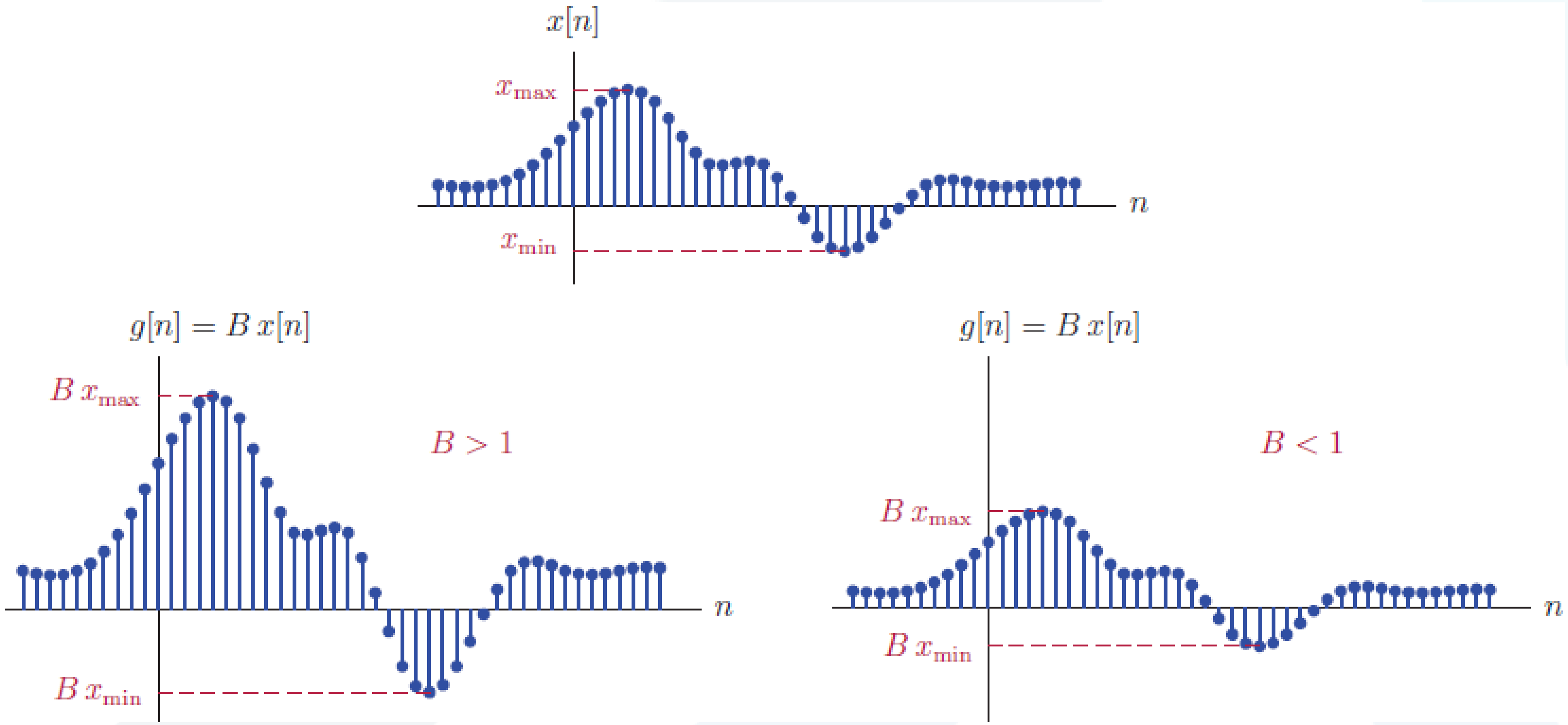
- Sometimes discrete-time signals are also modeled using mathematical functions: $x[n] = 3\sin[0.2n]$.
- In a discrete-time signal the time variable is discrete, yet the amplitude of each sample is continuous.
- If, in addition to limiting the time variable to the set of integers, we also limit the amplitude values to a discrete set, the resulting signal is called a digital signal.
- In the simplest case there are only two possible values for the amplitude of each sample, typically indicated by "0" and "1". The corresponding signal is called a binary signal.

Signal operations

- Amplitude shifting maps the input signal $x[n]$ to the output signal g as given by $g[n] = x[n] + A$, where A is a real number.

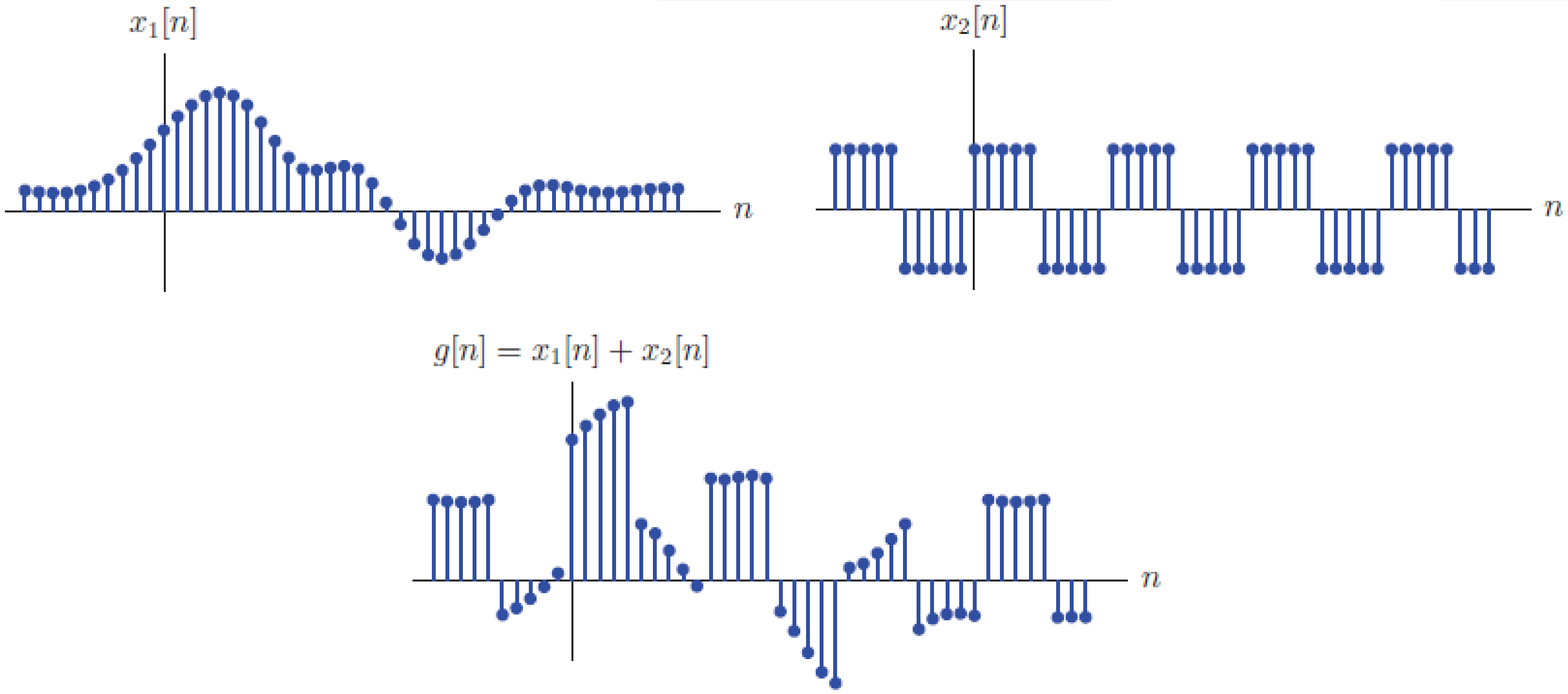


- Amplitude scaling maps the input signal x to the output signal g as given by $g[n] = Bx[n]$, where B is a real number.
- Geometrically, the output signal g is expanded/compressed in amplitude.

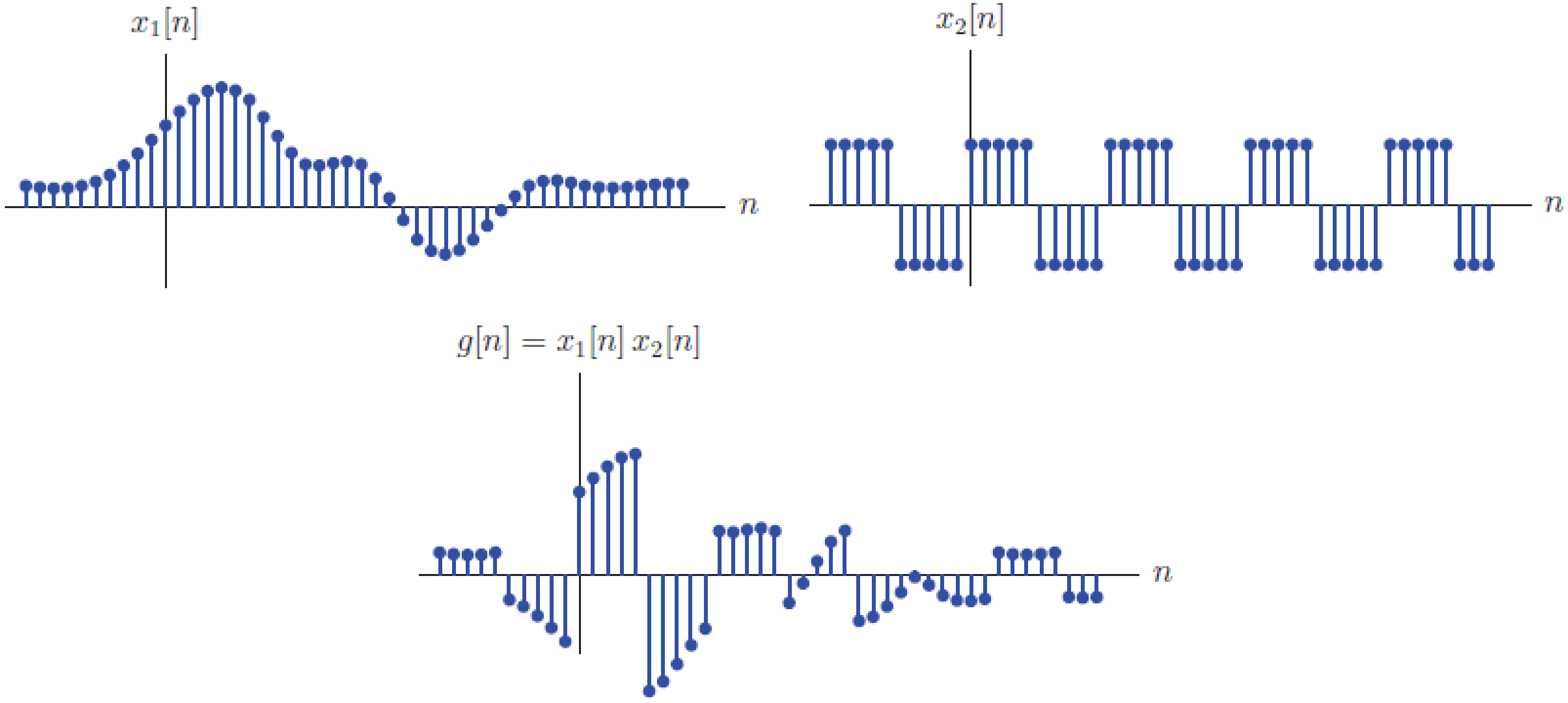


- Addition and Multiplication of two signals

Addition of two signals is accomplished by adding the amplitudes of the two signals at each time instant. $g[n] = x_1[n] + x_2[n]$.

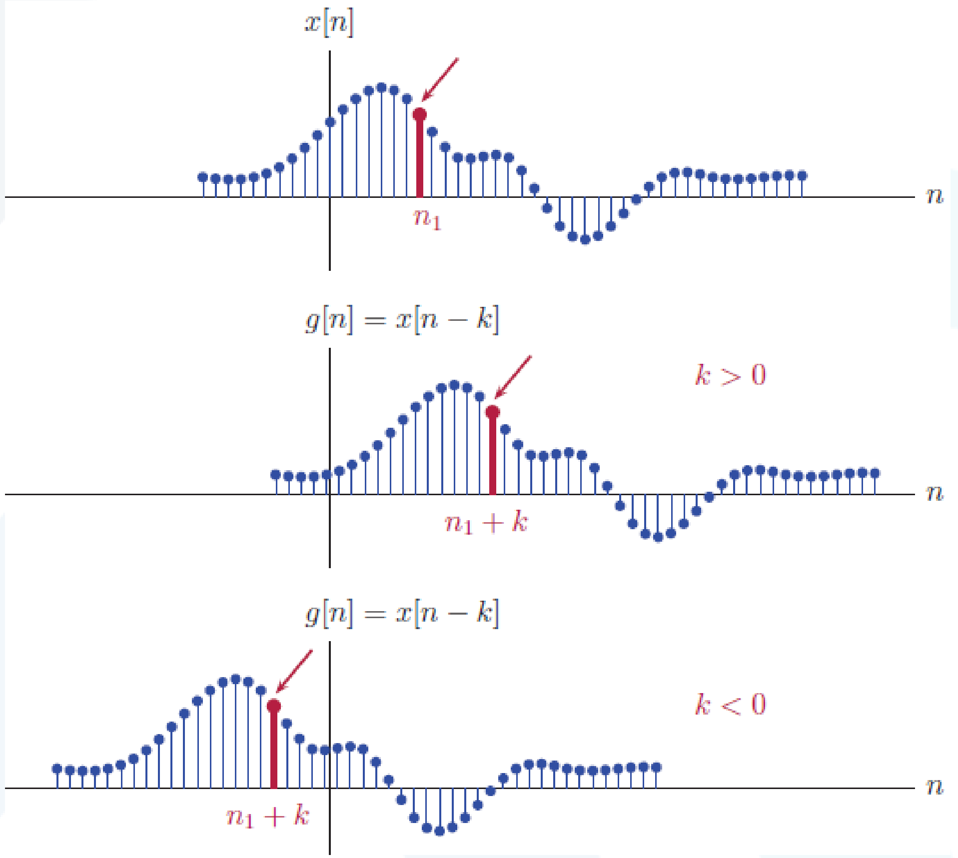


Multiplication of two signals is accomplished by multiplying the amplitudes of the two signals at each time instant. $g[n] = x_1[n]\ x_2[n]$.



- Time shifting (also called translation) maps the input signal x to the output signal g as given by: $g[n] = x[n-k]$; where k is an integer.

- Such a transformation shifts the signal (to the left or right) along the time axis.
- If $k > 0$, g is shifted to the right by $|k|$, relative to x (i.e., delayed in time).
- If $k < 0$, g is shifted to the left by $|k|$, relative to x (i.e., advanced in time).



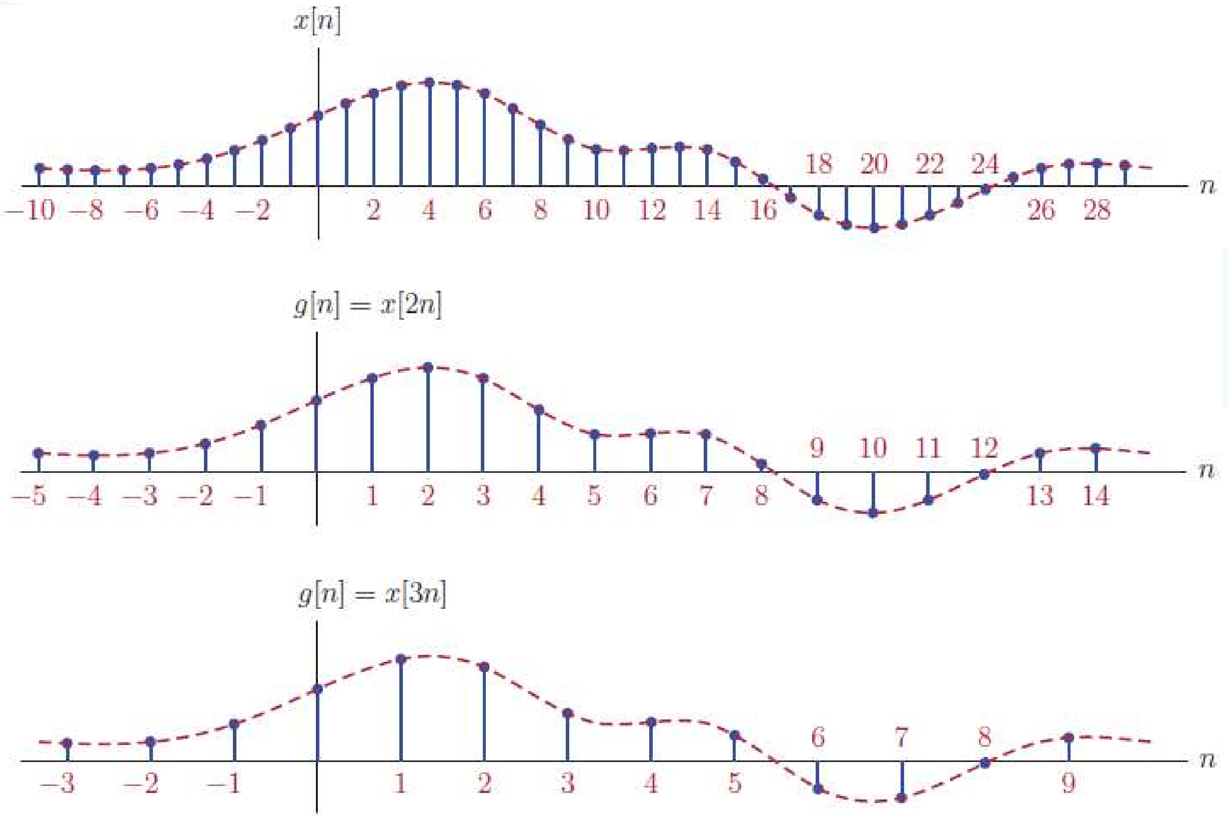
- Time scaling maps the input signal x to the output signal g as given by:

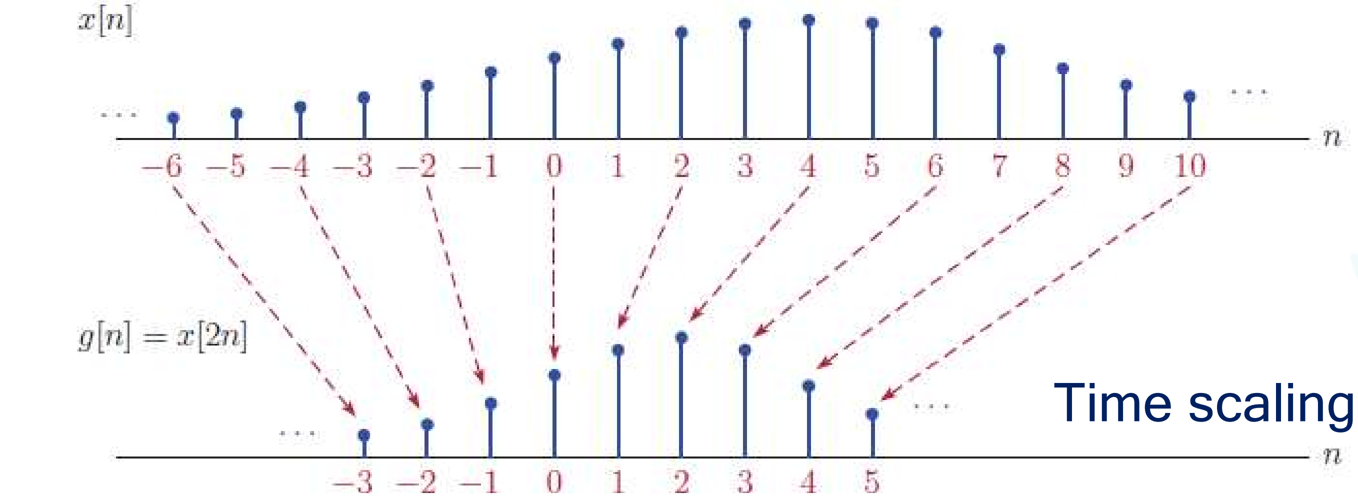
$$g[n] = x[kn]; \quad \text{downsampling}$$

and

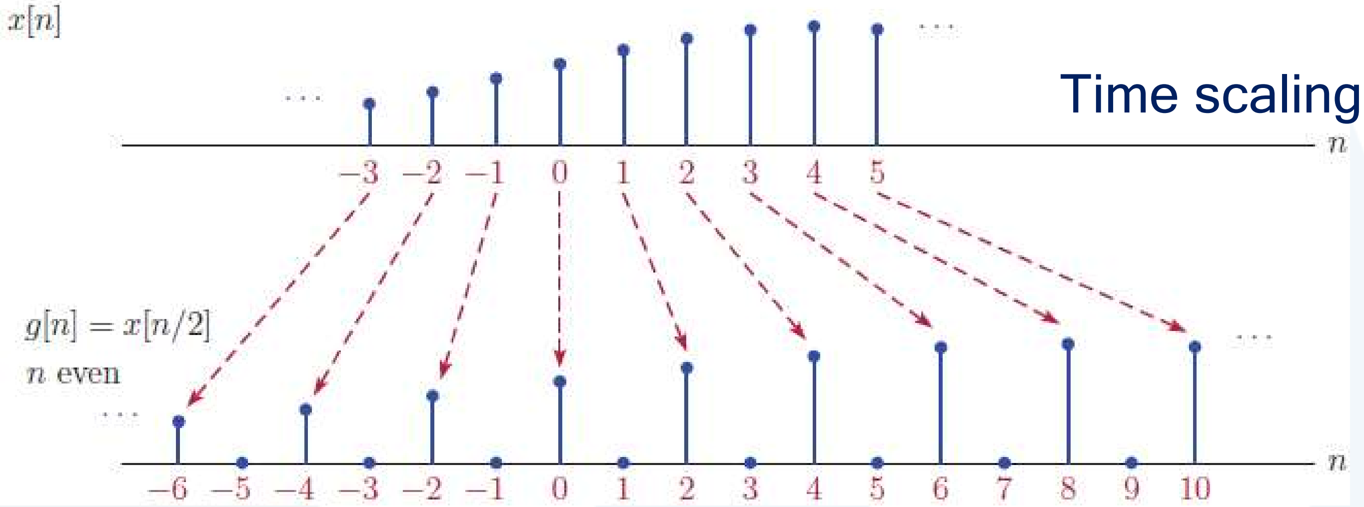
$$g[n] = x[n/k]; \text{ upsampling}$$

where k is a strictly positive integer.



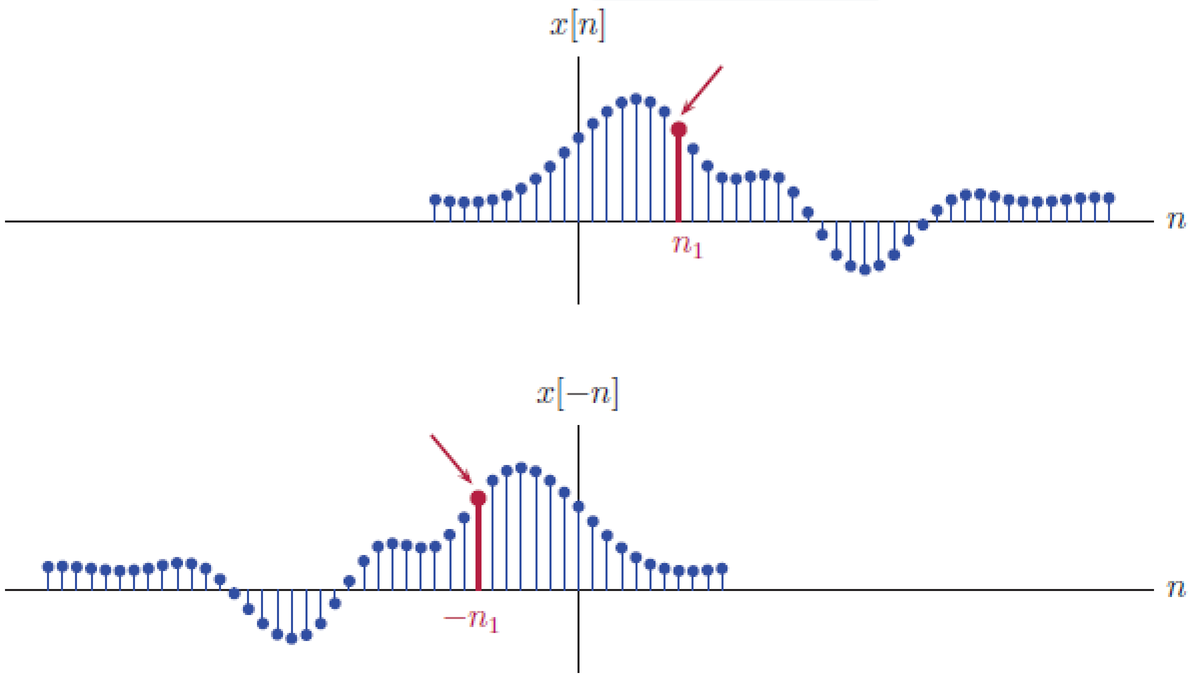


Time scaling (downsampling)



Time scaling (upsampling)

- Time reversal (also known as reflection) maps the input signal x to the output signal g as given by $g[n] = x[-n]$.
- Geometrically, the output signal g is a reflection of the input signal x about the (vertical) line $n = 0$.

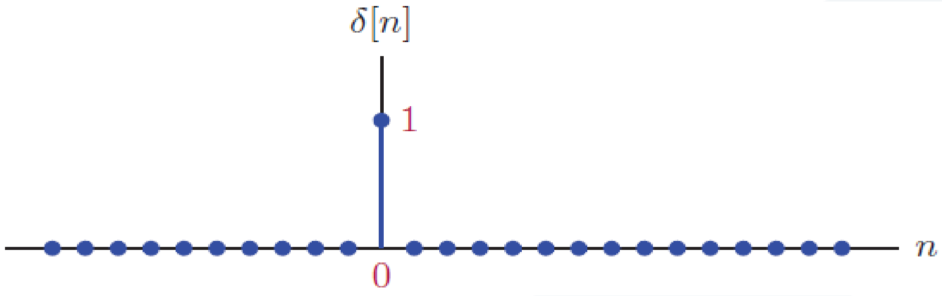


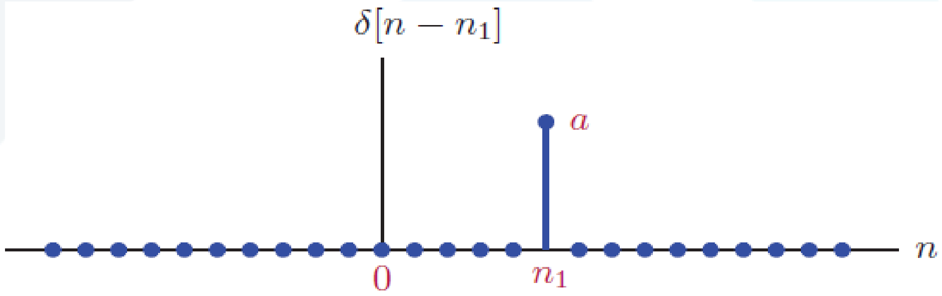
Basic building blocks for discrete-time signals

Unit-impulse Signal

- The unit-impulse signal, denoted δ, is defined by:

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \qquad a\delta[n - n_1] = \begin{cases} a, & \text{if } n = n_1 \\ 0, & \text{if } n \neq n_1 \end{cases}$$



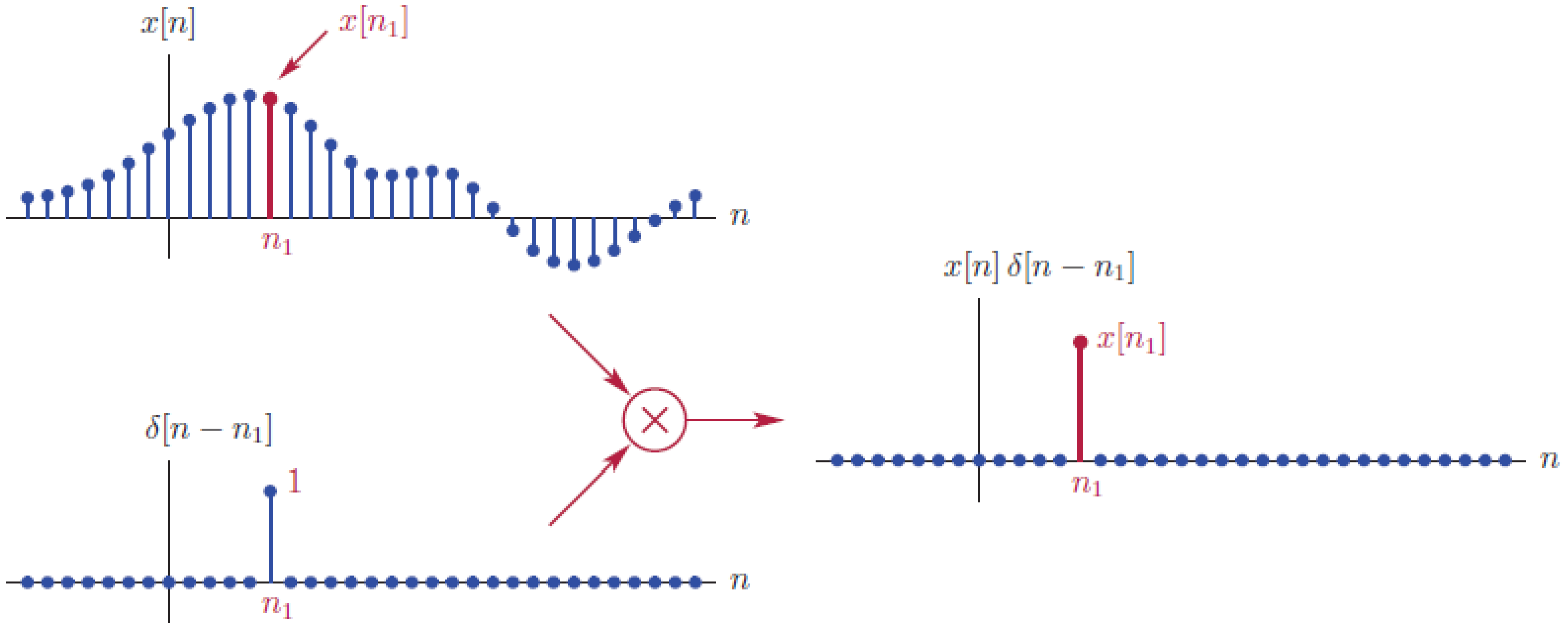


- Sampling property of the unit-impulse signal:

$$x[n]\delta[n - n_1] = x[n_1]\delta[n - n_1] = \begin{cases} x[n_1], & n = n_1 \\ 0, & n \neq n_1 \end{cases}$$

- Sifting property of the unit-impulse signal

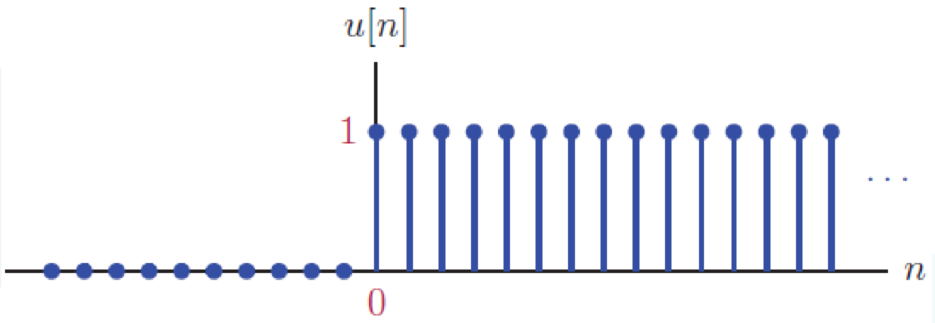
$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_1] = x[n_1]$$



Unit-Step Signal

- The unit-step signal, denoted u, is defined as:

$$u[n] = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

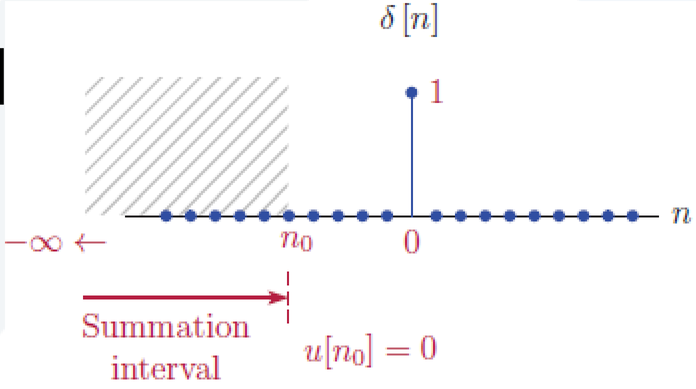


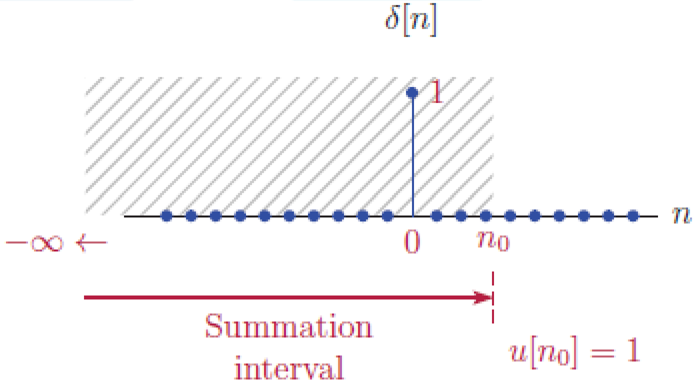
- Relationship between the unit-step signal and the unit-impulse signal:

$$\delta[n] = u[n] - u[n-1]$$

- Conversely, $u[n] = \sum_{k=-\infty}^{n} \delta[k]$

or, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$





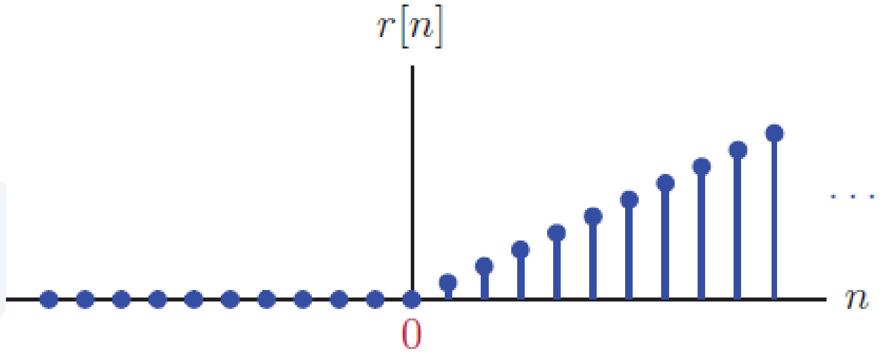
Unit-Ramp Signal

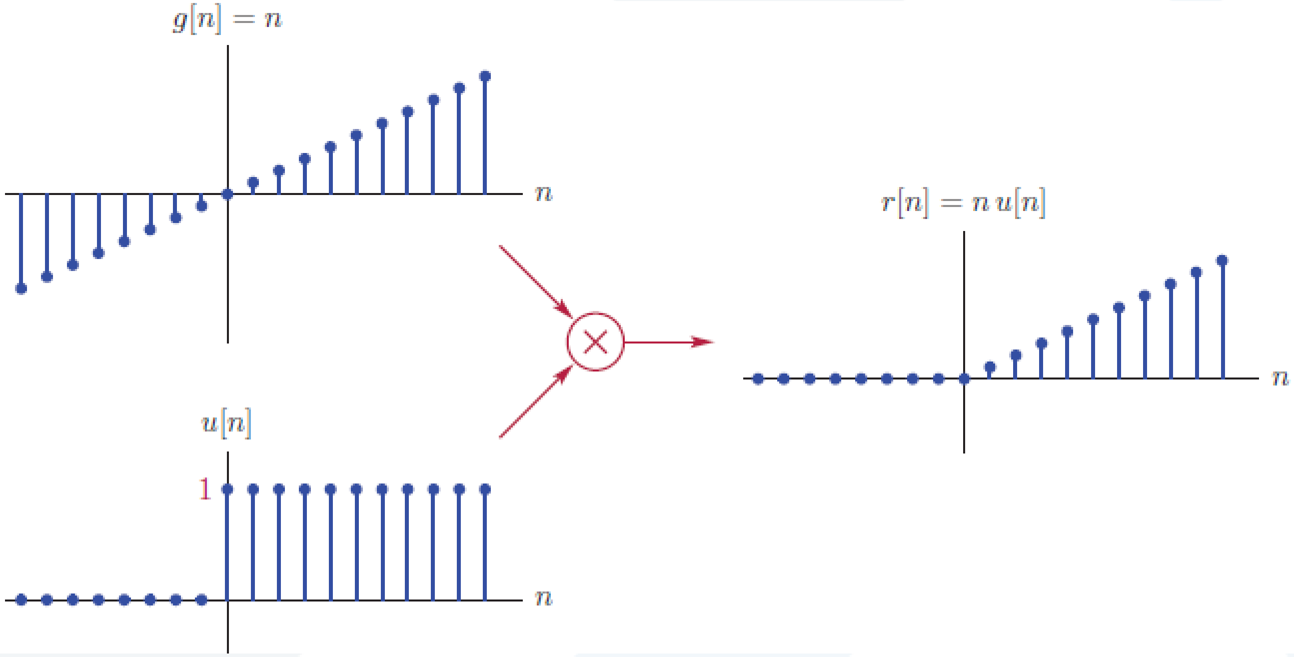
- The unit-ramp signal, denoted r, is defined as:

$$r[n] = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

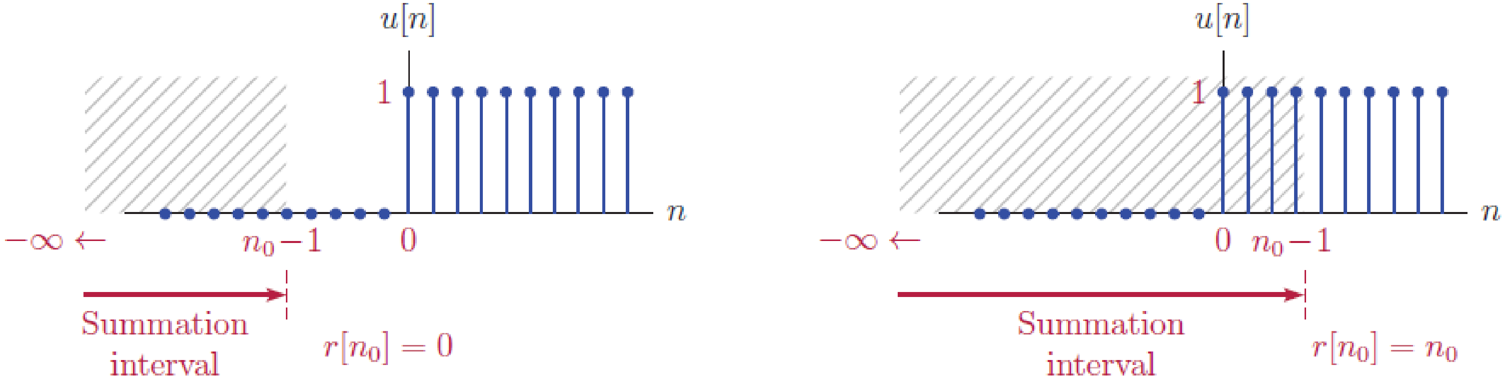
or, equivalently:

$$r[n] = n u[n]$$



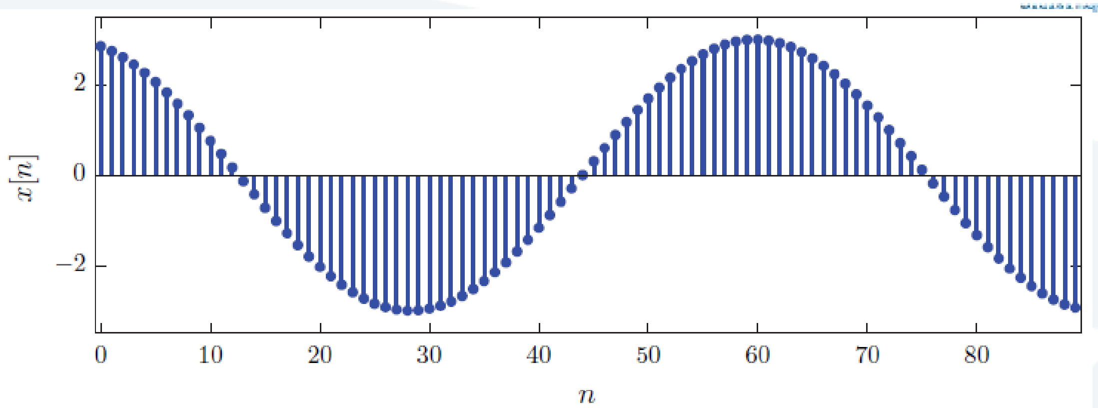


- Constructing a unit-ramp from a unit-step $r[n]=\sum_{n=-\infty}^{n-1} u[k]$

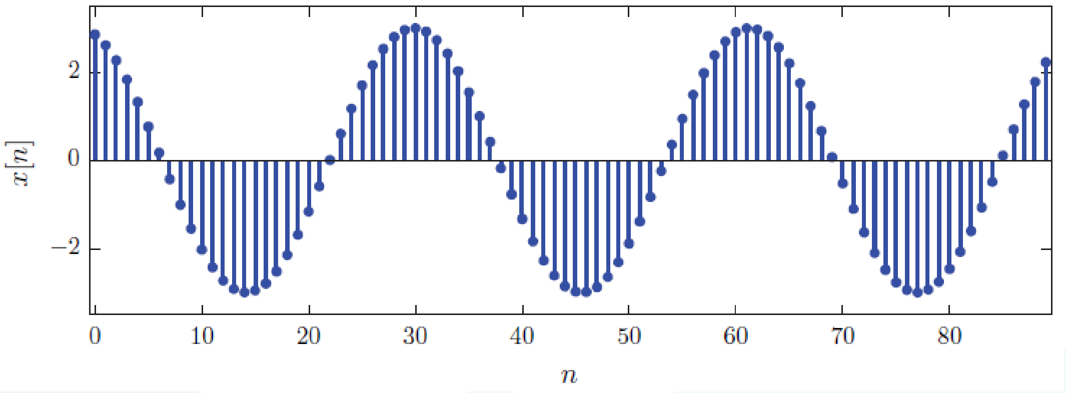


Sinusoidal Signal

- A discrete-time sinusoidal signal is a signal of the form: $x[n]=A\cos(\Omega_0 n+\theta)$ where A is the amplitude of the signal, Ω_0 is the angular frequency (rad), and θ is the initial phase angle (rad). $\Omega_0 = 2\pi F_0$ where F_0 is the normalized frequency (a dimensionless quantity).



$x[n] = 3\cos(0.1n + \pi/10)$



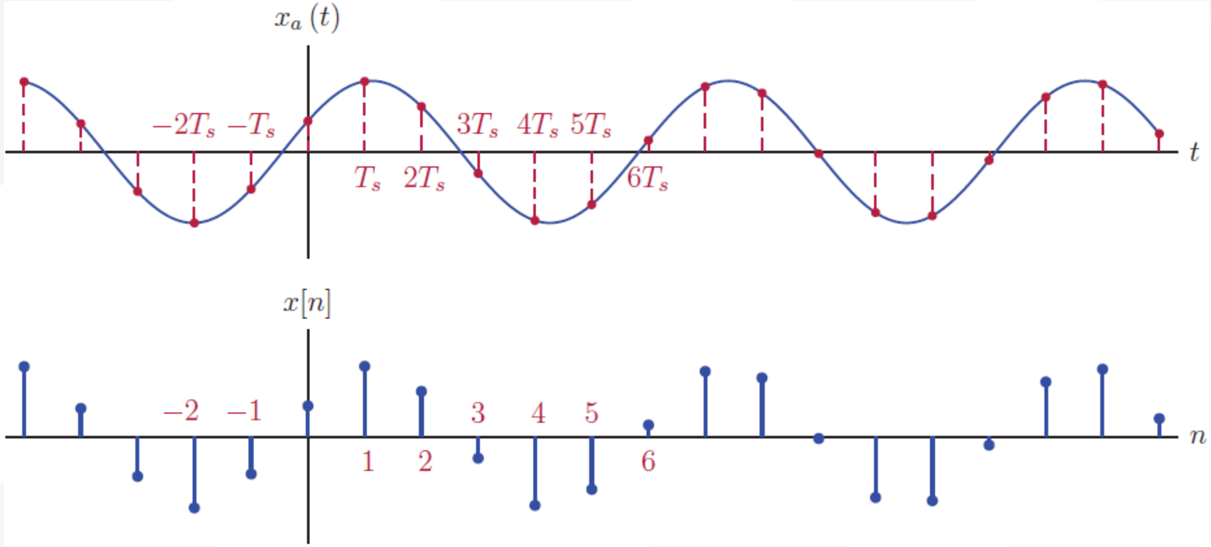
$x[n] = 3\cos(0.2n + \pi/10)$

A fundamental difference between a DT sinusoidal signal and its CT:

- For continuous-time sinusoidal signal $x_a(t) = A\cos(\omega_0 t + \theta)$: ω_0 is in rad/s.
- For discrete-time sinusoidal signal $x[n] = A\cos(\Omega_0 n + \theta)$: Ω_0 is in rad.
- Let us evaluate the amplitude of $x_a(t)$ at time instants that are integer multiples of T_s, and construct a discrete-time signal:

$$x[n] = x_a(nT_s) = A\cos(\omega_0 T_s n + \theta) = A\cos(2\pi f_0 T_s n + \theta)$$

- Since the signal $x_a(t)$ is evaluated at intervals of T_s, the number of samples taken per unit time is $1/T_s$. $x[n] = A\cos\left(2\pi\left[f_0/f_s\right]n + \theta\right) = A\cos(2\pi F_0 n + \theta)$
- The act of constructing a discrete-time signal by evaluating a continuous-time signal at uniform intervals is called sampling.
- The parameters f_s and T_s are referred to as the sampling rate and the sampling interval respectively.



Impulse decomposition for discrete-time signals

- Consider an arbitrary discrete-time signal $x[n]$. Let us define a new signal $x_k[n]$ by:

$$x_k[n] = x[k]\delta[n-k] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$

- The signal $x[n]$ can be reconstructed by: $x[n] = \sum_{k=-\infty}^{\infty} x_k[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Periodic discrete-time signals

- A discrete-time signal is said to be periodic if it satisfies: $x[n] = x[n+N]$

  for all values of the integer index n and for a specific value of $N \neq 0$. The parameter N is referred to as the period of the signal.

- The period of a periodic signal is not unique. That is, a signal that is periodic with period N is also periodic with period kN, for every (strictly) positive integer k, $x[n] = x[n + kN]$.
- The smallest period with which a signal is periodic is called the fundamental period.
- The normalized fundamental frequency of a discrete-time periodic signal is $F_0 = 1/N$.

Periodicity of discrete-time sinusoidal signals

$$\begin{aligned} A\cos(2\pi F_0 n + \theta) &= A\cos(2\pi F_0[n + N] + \theta) \\ &= A\cos(2\pi F_0 n + 2\pi F_0 N + \theta) \end{aligned}$$

$2\pi F_0 N = 2\pi k \Rightarrow N = k/F_0$ $\quad$ N must be an integer value

- Example 6: Periodicity of a discrete-time sinusoidal signal

  Check the periodicity of the following discrete-time signals:

  a. $x[n] = \cos(0.2n)$ b. $x[n] = \cos(0.2\pi n + \pi/5)$

  c. $x[n] = \cos(0.3\pi n - \pi/10)$

  a. $x[n] = \cos(0.2n)$

  $\Omega_0 = 0.2 \Rightarrow F_0 = \Omega_0/2\pi = 0.2/2\pi = 0.1/\pi \Rightarrow N = k/F_0 = 10\pi k$

  Since no value of k would produce an integer value for N, the signal is not periodic.

  b. $x[n] = \cos(0.2\pi n + \pi/5)$

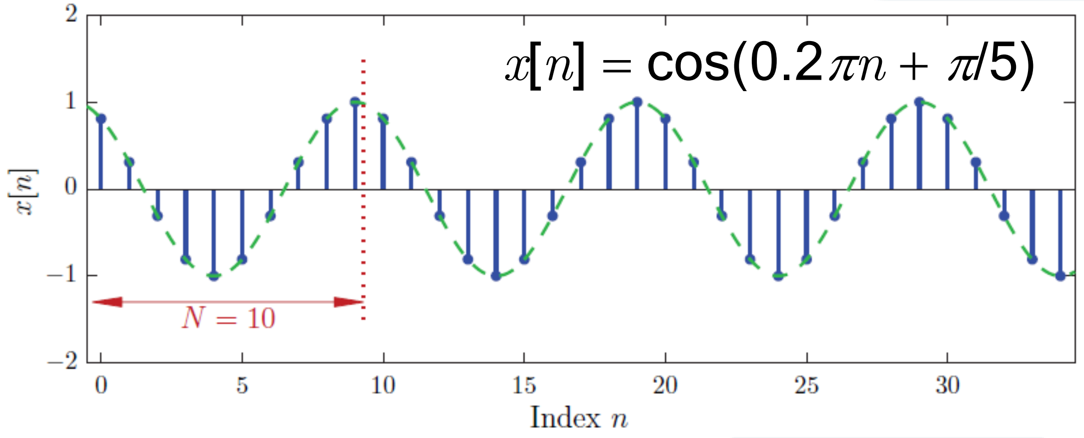
  $\Omega_0 = 0.2\pi \Rightarrow F_0 = \Omega_0/2\pi = 0.2\pi/2\pi = 0.1 \Rightarrow N = k/F_0 = 10k$

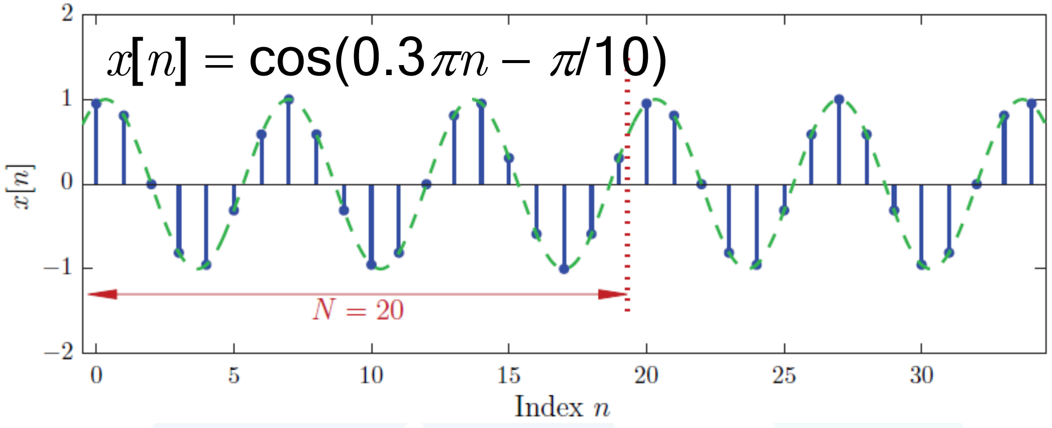
  For $k = 1$ we have $N = 10$ samples as the fundamental period.

c. $x[n] = \cos(0.3\pi n - \pi/10)$

$\Omega_0 = 0.3\pi \Rightarrow F_0 = \Omega_0/2\pi = 0.3\pi/2\pi = 0.15 \Rightarrow N = k/F_0 = k/0.15$

For $k = 3$ we have $N = 20$ samples as the fundamental period.





- Example 7: Periodicity of a multi-tone discrete-time sinusoidal signal

Comment on the periodicity of the two-tone discrete-time signal:

$$x[n] = 2\cos(0.4\pi n) + 1.5\sin(0.48\pi n)$$

$x[n] = x_1[n] + x_2[n]$

$x_1[n] = 2\cos(\Omega_1 n)$

$\Omega_1 = 0.4\pi \Rightarrow F_1 = \Omega_1/2\pi = 0.4\pi/2\pi = 0.2$

$\Rightarrow N = k_1/F_1 = 5k_1$

For $k_1 = 1$ we have $N_1 = 5$ samples as the fundamental period.

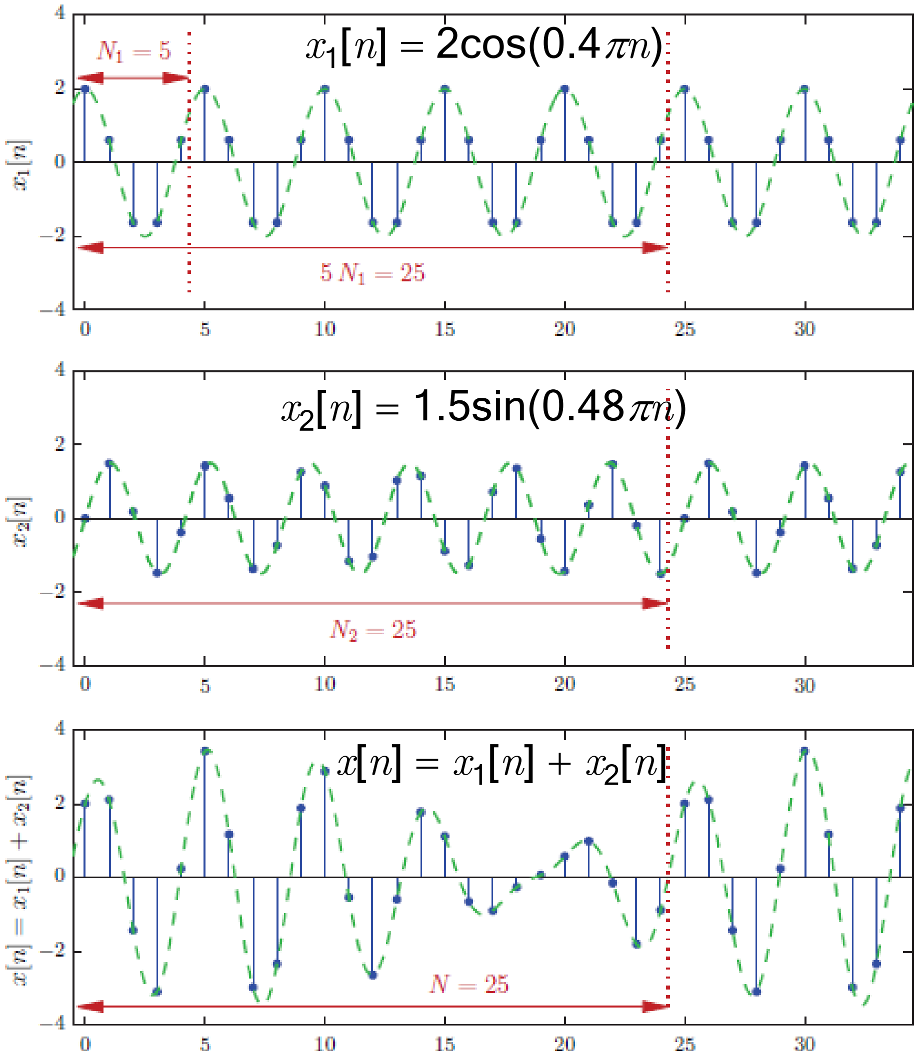
$x_2[n] = 1.5\cos(\Omega_2 n)$

$\Omega_2 = 0.48\pi \Rightarrow F_2 = \Omega_2/2\pi = 0.48\pi/2\pi = 0.24$

$\Rightarrow N_2 = k_2/F_2 = k_2/0.24$

For $k_2 = 6$ we have $N_2 = 25$ samples as the fundamental period.

$\Rightarrow N = 25$



Energy and power definitions

- The energy of a discrete time signal $x[n]$ is given by $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
- The average power of a discrete time signal $x[n]$ is given by:

  periodic complex signal $P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$

  non-periodic complex signal $P_x = \lim_{M\to\infty} \frac{1}{2M+1}\sum_{n=-M}^{M} |x[n]|^2$
- Energy signals are those that have finite energy and zero power, i.e., $E_x < \infty$, and $P_x = 0$.
- Power signals are those that have finite power and infinite energy, i.e., $E_x \to \infty$, and $P_x < \infty$.

- Example 8: Energy and power signals

Determine whether the sequence $x[n] = a^n u[n]$ is an energy signal or a power signal or neither for the following cases: (a) $|a| < 1$, (b) $|a| = 1$, (c) $|a| > 1$.

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} \left|a^{2n}\right|, \qquad P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} |x[n]|^2 = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left|a^{2n}\right|$$

(a) $$E_x = \sum_{n=0}^{\infty} \left|a^{2n}\right| = \frac{1}{1-|a|^2} < \infty,$$

$$P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left|a^{2n}\right| = \lim_{M\to\infty} \frac{1}{2M+1} \frac{1-|a|^{2(M+1)}}{1-|a|^2} = 0$$

The signal $x[n] = a^n u[n]$ is an energy signal for $|a| < 1$.

(b) $$E_x = \sum_{n=0}^{\infty} \left| a^{2n} \right| \to \infty,$$

$$P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left| a^{2n} \right| = \lim_{M\to\infty} \frac{M+1}{2M+1} = \frac{1}{2}$$

The signal $x[n] = a^n u[n]$ is an power signal for $|a| = 1$.

(c) $$E_x = \sum_{n=0}^{\infty} \left| a^{2n} \right| \to \infty,$$

$$P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left| a^{2n} \right| = \lim_{M\to\infty} \frac{1}{2M+1} \frac{|a|^{2(M+1)} - 1}{|a|^2 - 1} \to \infty$$

The signal $x[n] = a^n u[n]$ is neither an energy signal nor a power signal for $|a| = 1$.

Symmetry properties

Even and odd symmetry

- A real-valued signal is said to have even symmetry if it has the property: $x[-n] = x[n]$ for all values of n.
- A real-valued signal is said to have odd symmetry if it has the property: $x[-n] = -x[n]$ for all values of n.

Decomposition into even and odd components

- Every real-valued signal $x[n]$ has a unique representation of the form: $x[n] = x_e[n] + x_o[n]$; where the signals x_e and x_o are even and odd, respectively.
- In particular, the signals x_e and x_o are given by:

$$x_e[n] = ½(x[n] + x[-n]) \text{ and } x_o[n] = ½(x[n] - x[-n])$$

Symmetry properties for complex signals

- A complex-valued signal is said to have conjugate symmetric if it has the property: $x[-n] = x^*[n]$ for all values of n.
- A complex-valued signal is said to have conjugate antisymmetric if it has the property: $x[-n] = -x^*[n]$ for all values of n.

Decomposition of complex signals

- Every complex-valued signal $x[n]$ has a unique representation of the form: $x[n] = x_E[n] + x_O[n]$; where the signals x_E and x_O are conjugate symmetric and conjugate antisymmetric, respectively.
- In particular, the signals x_E and x_O are given by:

$$x_E[n] = ½(x[n] + x^*[-n]) \text{ and } x_O[n] = ½(x[n] - x^*[-n])$$