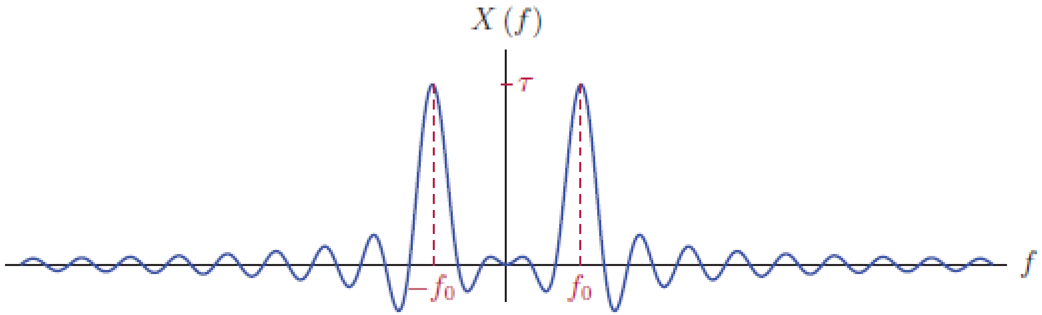
CEDC403: Signals and Systems

Lecture Notes 3: Analyzing Continuous Time Systems in the Time Domain



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

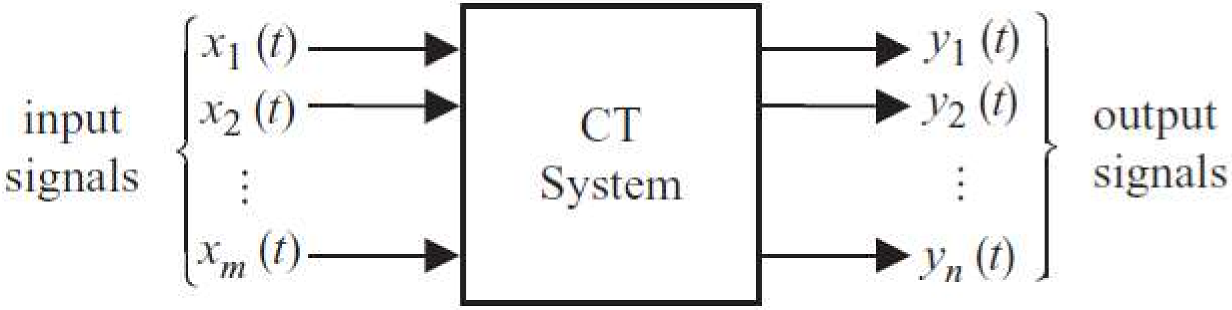
Manara University

Chapter 2

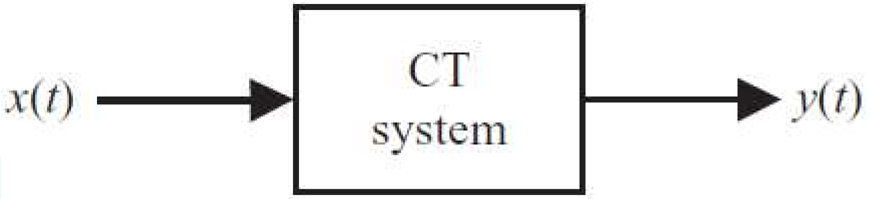
Analyzing Continuous Time Systems in the Time Domain

1. Introduction

- A system is any physical entity that takes in a set of one or more physical signals and, in response, produces a new set of one or more physical signals.
- One representation of a general system is by a block diagram.



Multiple-input, multiple-output (MIMO) CT system



Single-input, single-output CT system

- If we focus our attention on single-input/single-output systems, the interplay between the system and its input and output signals can be graphically illustrated as:

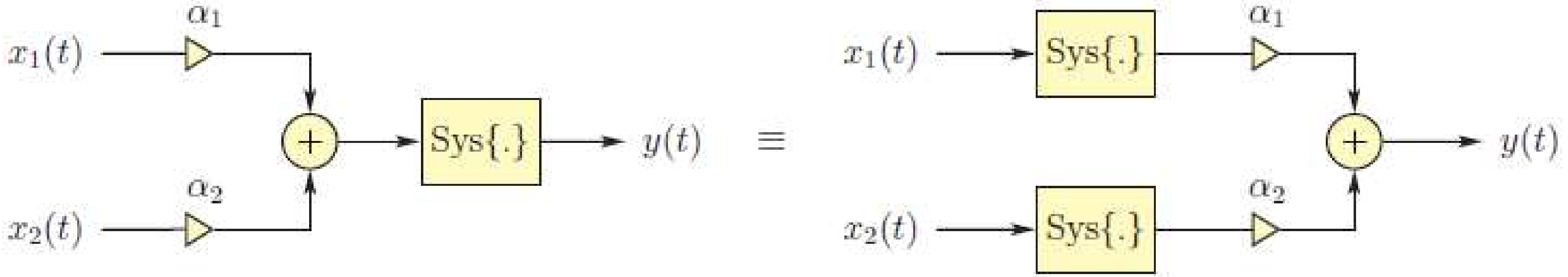
- The input signal is $x(t)$, and the output signal is $y(t)$. The system may be denoted by the equation $y(t) = T\{x(t)\}$, where $T\{.\} = \text{Sys}\{.\}$ indicates a transformation that defines the system in the time domain.
- A very simple example is a system that simply multiplies its input signal by a constant gain factor K to yield an output signal $y(t) = Kx(t)$,
- Or one that delays its input signal by a constant time delay τ $y(t) = x(t-\tau)$,

- Or one that produces an output signal proportional to the square of the input signal $y(t) = K[x(t)]^2$.

2. Basic System Properties

Linearity in continuous-time systems

- A system T is linear, if for all functions x_1 and x_2 and all constants α_1 and α_2, the following condition holds: $T\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 T\{x_1(t)\} + \alpha_2 T\{x_2(t)\}$.



- The linearity property is also referred to as the superposition property.
- Linear systems are much easier to design and analyze than nonlinear systems.

- Example 1: Testing linearity of continuous-time systems

  a. $y(t) = 5x(t)$ √ b. $y(t) = 5x(t) + 3$ ✗

  c. $y(t) = 3[x(t)]^2$ ✗ d. $y(t) = \cos(x(t))$ ✗

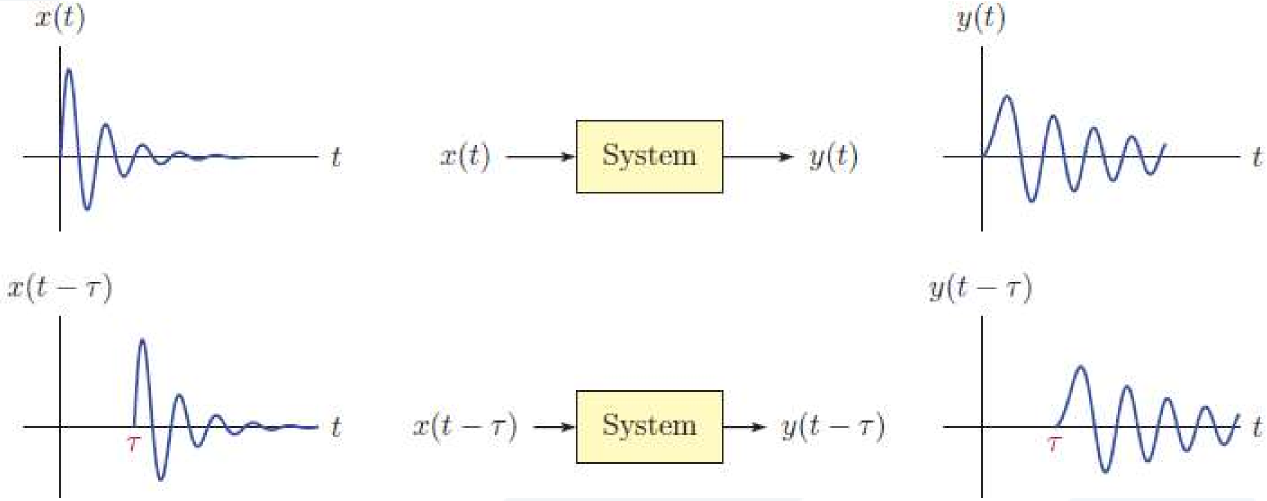
- A direct consequence of the linearity property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time.

  $0 = T\{0x_1(t) + 0x_2(t)\} = 0\,T\{x_1(t)\} + 0\,T\{x_2(t)\} = 0$ (zero-in/zero-out property)

Time Invariance in continuous-time systems

- A system T is said to be time invariant (TI) if, for every function x and every real constant τ, the following condition holds: $T\{x(t)\} = y(t) \Rightarrow T\{x(t-\tau)\} = y(t-\tau)$.
- Example 2: Testing time invariance of continuous-time systems

  a. $y(t) = 5x(t)$ √ b. $y(t) = 3\cos(x(t))$ √ c. $y(t) = 3\cos(t)x(t)$ ✗



Causality in continuous-time systems

- A system T is said to be causal if, for every real constant t_0, $T\{x(t_0)\}$ does not depend on $x(t)$ for some $t > t_0$.
- A causal system is such that the value of its output at any given point in time can depend on the value of its input at only the same or earlier points in time.

- If the independent variable t represents time, a system must be causal in order to be physically realizable. Real-time physical systems are causal, cause before effect.

- Example 3: causal and non causal systems

a. CT time-delay system $y(t) = x(t) + x(t - 0.01) + x(t - 0.02)$ √

b. CT time-forward system $y(t) = x(t) + x(t + 0.1)$ ✗

Stability in continuous-time systems

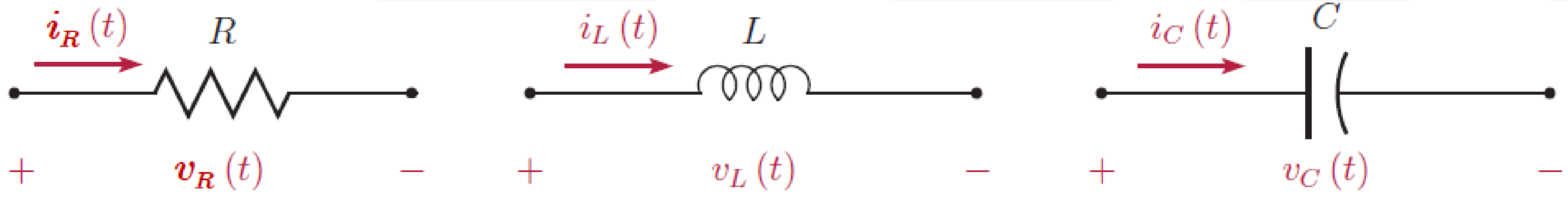
- A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal produce a bounded output signal.

- An input signal $x(t)$ is said to be bounded if an upper bound B_x exists such that $x(t) < B_x < \infty$ for all values of t.

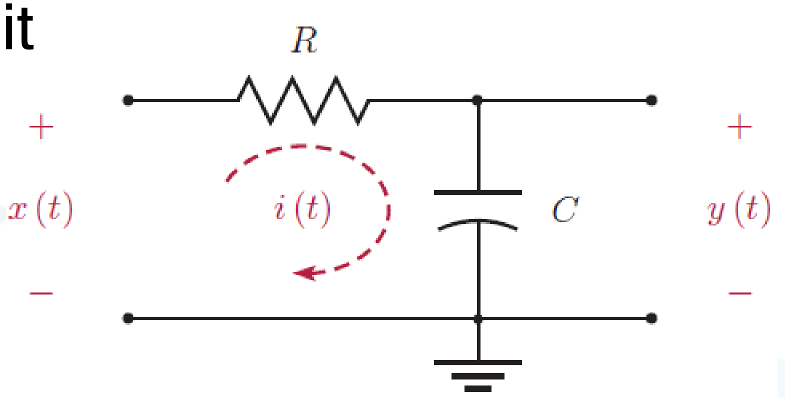
- For stability of a continuous-time system: $x(t) < B_x < \infty \Rightarrow y(t) < B_y < \infty$.

3. Differential Equations for Continuous-Time Systems

- One method of representing the relationship established by a system between its input and output signals is a differential equation (DE).
- model for an ideal resistor is: $v_R(t) = Ri_R(t)$
- model for an ideal inductor is: $v_L(t) = L\dfrac{di_L(t)}{dt}$
- model for an ideal capacitor is: $i_C(t) = C\dfrac{dv_C(t)}{dt}$



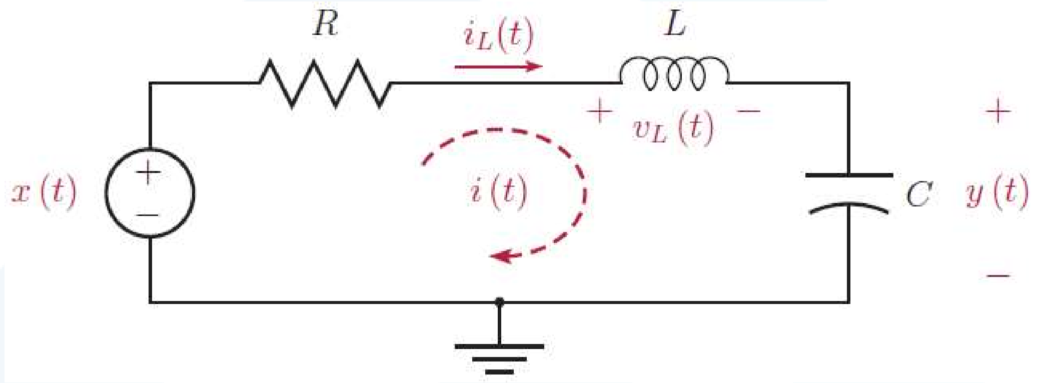
- Example 4: Differential equation for simple RC circuit



$$v_R(t) = Ri(t), \qquad i(t) = C\frac{dy(t)}{dt}$$

$$RC\frac{dy(t)}{dt} + y(t) = x(t) \Rightarrow \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

- Example 5: DE for RLC circuit

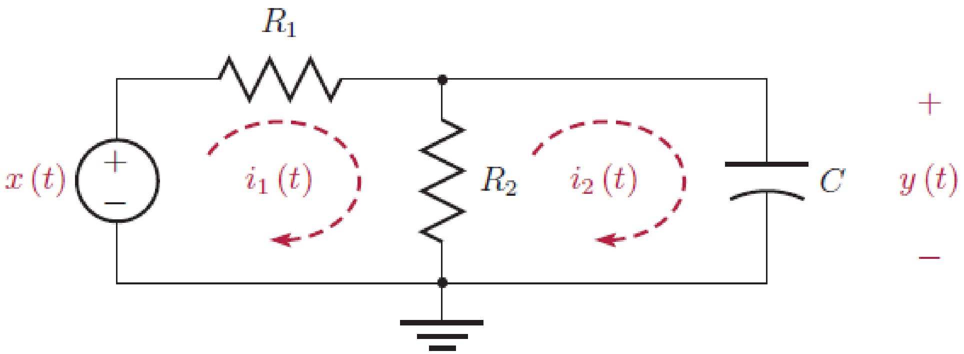


$$v_L(t) = L\frac{di(t)}{dt}, \quad i(t) = C\frac{dy(t)}{dt}$$

$$-x(t) + Ri(t) + v_L(t) + y(t) = 0$$

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

- Example 6: Another RC circuit



$$-x(t) + R_1 i_1(t) + R_2[i_1(t) - i_2(t)] = 0$$

$$R_2[i_2(t) - i_1(t)] + y(t) = 0$$

$$i_2(t) = C\frac{dy(t)}{dt} \Rightarrow i_1(t) = C\frac{dy(t)}{dt} + \frac{1}{R_2}y(t)$$

$$-x(t) + R_1 C\frac{dy(t)}{dt} - \frac{R_1 + R_2}{R_2}y(t) = 0 \Rightarrow \frac{dy(t)}{dt} + \frac{R_1 + R_2}{R_1 R_2 C}y(t) = \frac{1}{R_1 C}x(t)$$

4. Constant-Coefficient Ordinary Differential Equations

- In general, CTLTI systems can be modeled with ordinary differential equations that have constant coefficients.

$$a_N\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_0 y(t) = b_M\frac{d^M x(t)}{dt^N} + b_{M-1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_0 x(t)$$

or it can be expressed in the form: $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

- In general, a constant-coefficient ODE has a family of solutions. In order to find a unique solution for $y(t)$, initial values of the output signal and its first $N-1$ derivatives need to be specified at a time instant $t = t_0$. We need to know:

$$y(t_0), \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}, \cdots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$ to find the solution for $t > t_0$

- The initial conditions in a differential equation description of an LTI system are directly related to the initial values of the energy storage devices in the system, such as initial voltages on capacitors and initial currents through inductors.
- Initial conditions (ICs) also represent the memory of continuous-time systems.

- A system with zero ICs is said to be at rest (initially relaxed).

Solving Linear Differential Equations

Solution of the first-order differential equation

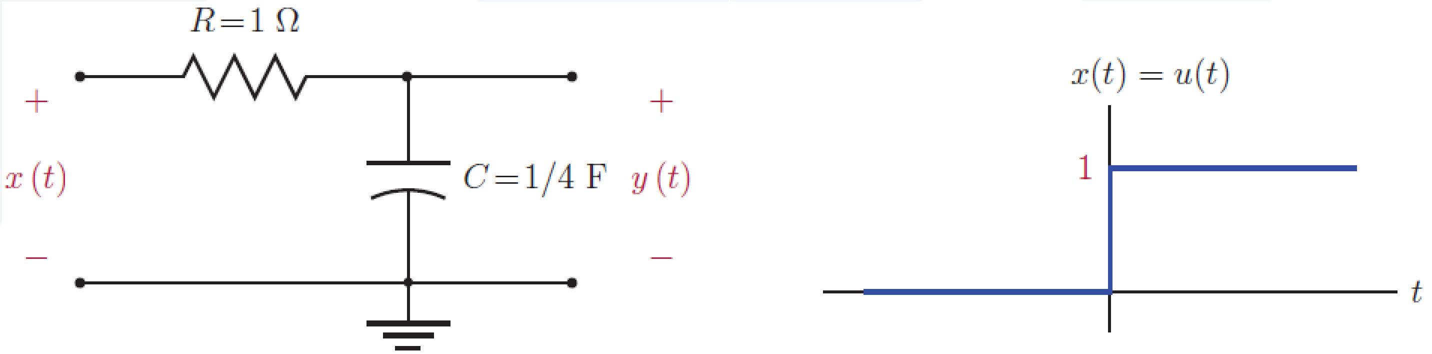
- The differential equation: $\frac{dy(t)}{dt} + \alpha y(t) = r(t), \quad y(t_0)$: specified

  is solved as: $y(t) = e^{-\alpha(t-t_0)} y(t_0) + \int_{t_0}^{t} e^{-\alpha(t-\tau)} r(\tau) d\tau$

- Example 7: Unit-step response of the simple RC circuit ($y(0) = 0$)

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} u(t) \Rightarrow$$

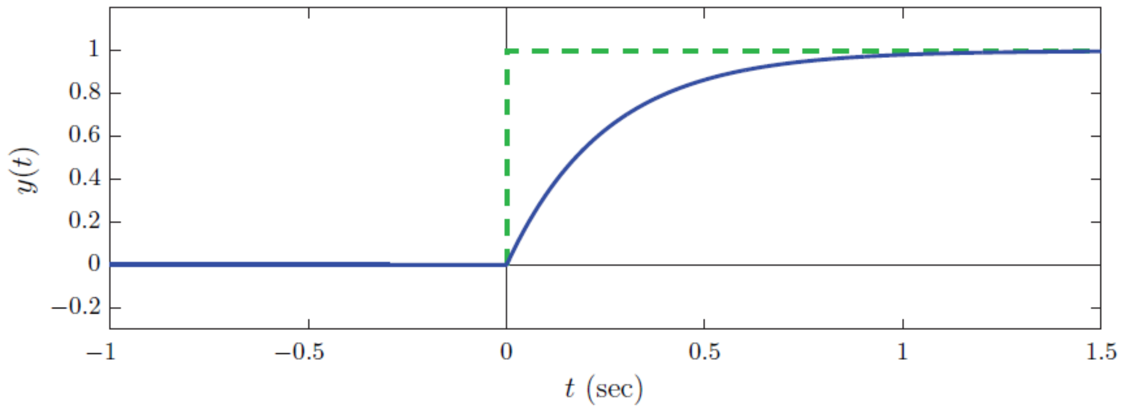
$$\frac{dy(t)}{dt} + 4y(t) = 4u(t)$$



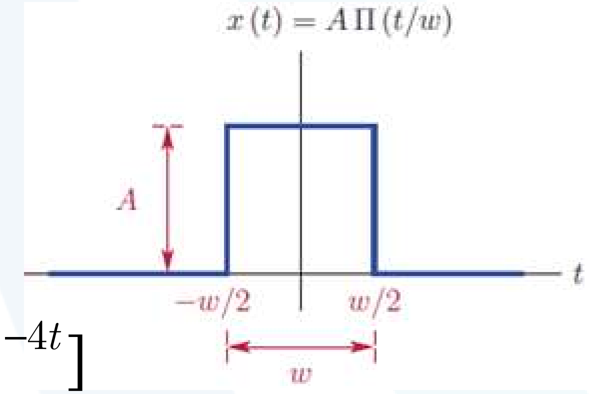
$$y(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} u(\tau)d\tau = \frac{e^{-t/RC}}{RC} \int_0^t e^{\tau/RC} d\tau = 1 - e^{-t/RC}, \quad t \geq 0$$

$$y(t) = (1 - e^{-t/RC})u(t)$$

$$y(t) = (1 - e^{-4t})u(t)$$



- Example 8: Pulse response of the simple RC circuit



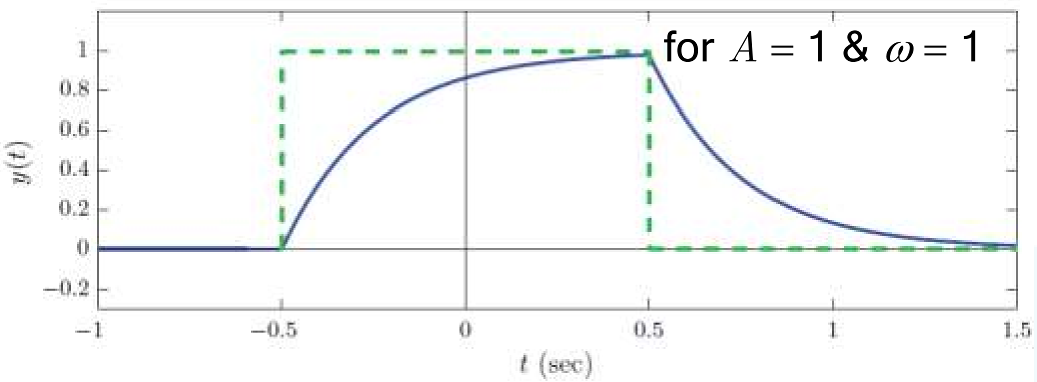
$$\frac{dy(t)}{dt} + 4y(t) = 4A\prod(t/\omega) \Rightarrow y(t) = \int_{-\omega/2}^{t} e^{-4(t-\tau)} 4A\prod(\tau/\omega)d\tau$$

Case 1: $t \leq -\omega/2,\ y(t) = 0$

Case 2: $-\omega/2 < t \leq \omega/2,\ y(t) = 4A\int_{-\omega/2}^{t} e^{-4(t-\tau)}d\tau = A[1 - e^{-2\omega}e^{-4t}]$

Case 3: $t > \omega/2$, $y(t) = 4A\int_{-\omega/2}^{\omega/2} e^{-4(t-\tau)}d\tau = Ae^{-4t}[e^{2\omega} - e^{-2\omega}]$

$$y(t) = \begin{cases} 0, & t < -\frac{\omega}{2} \\ A[1 - e^{-2\omega}e^{-4t}], & -\frac{\omega}{2} < t \le \frac{\omega}{2} \\ Ae^{-4t}[e^{2\omega} - e^{-2\omega}], & t > \frac{\omega}{2} \end{cases}$$



Solution of the general differential equation

- The complete solution of a linear constant coefficient differential equation can be decomposition into:

1. The point of view of Mathematics:

   Homogenous solution $y_h(t)$ + Particular solution $y_p(t)$.

2. The point of view of Engineer: Natural response $y_n(t)$ + Forced response $y_\phi(t)$.

3. The point of view of control engineer:

Zero-input response $y_{zi}(t)$ + Zero-state response $y_{zs}(t)$.

Transient response $y_t(t)$ + Steady state response $y_{ss}(t)$.

- A system represented by a linear ODE, of order N, having constant coefficients, and with input $x(t)$ and output $y(t)$ is LTI if all the ICs are zero.
- In practice, most systems are causal, their response cannot begin before the input. Furthermore, most inputs are also causal, which means they start at $t = 0$ ($t = 0$ is the reference point).
- In respect to the origin, ICs are defined in two forms: Post-initial conditions, defined in $t_0 = 0^+$ and Pre-initial conditions, defined in $t_0 = 0^-$.
- The two sets of ICs are generally different, although in some cases they may be identical.

- In practice, we are likely to know the ICs at $t = 0^-$ rather than at $t = 0^+$.
- The ICs at $t = 0^-$ must be translated to $t = 0^+$ to reflect the effect of applying the input at $t = 0$. Translating ICs for the most general DE is complicated.
- A necessary and sufficient condition for the ICs at $t = 0^+$ to equal the ICs at $t = 0^-$ for a given input is that the right-hand side of the DE, contain no impulses or derivatives of impulses.
- For example, if $M = 0$, then the ICs do not need to be translated as long as there are no impulses in $x(t)$. But if $M = 1$, then any input involving a step discontinuity at $t = 0$ generates an impulse term due to the $dx(t)/dt$ term on the right-hand side, and the ICs at $t = 0^+$ are no longer equal to the ICs at $t = 0^-$.
- Note: The Laplace transform method, circumvents these difficulties.

Homogeneous solution (natural response) & particular solution (forced response)

Homogeneous solution $y_h(t)$ satisfies $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$

Determine the characteristic values $\sum_{k=0}^{N} a_k \alpha^k = 0$ as α_1, α_2, ..., α_N

a. If all α_i are of order 1, $y_h(t) = \sum_{i=1}^{N} c_i e^{\alpha_i t}$

b. If a root α_i is repeated k times (order k), $y_h(t) = \sum_{i=1}^{k} c_i t^{k-i} e^{\alpha_i t} + \sum_{j=k+1}^{N} c_j e^{\alpha_j t}$

- Note: the coefficients c_i or c_j should be determined by the initial conditions at $t = 0^+$ simultaneously with those in the particular solution.
- The particular solution $y_p(t)$ represents any solution of the DE for the given input. It is also called Forced response $y_\phi(t)$.

- A particular solution is usually obtained by assuming an output of the same general form as the input.

Input signal	Particular solution
t^n	$k_n t^n + k_{n-1} t^{n-1} + \dots k_1 t + k_0$ (Constant input is a special case with $n = 0$)
$e^{\alpha t}$	$ke^{\alpha t}$, α is not the characteristic value (c.v.) $k_1 te^{\alpha t} + k_0 e^{\alpha t}$, α is the characteristic value with order 1 $k_k t^k e^{\alpha t} + k_{k-1} t^{k-1} e^{\alpha t} + \dots k_1 te^{\alpha t} + k_0 e^{\alpha t}$, α is the c.v. with order k
$\cos(\omega t)$ or $\sin(\omega t)$	$k_1 \cos(\omega t) + k_2 \sin(\omega t)$

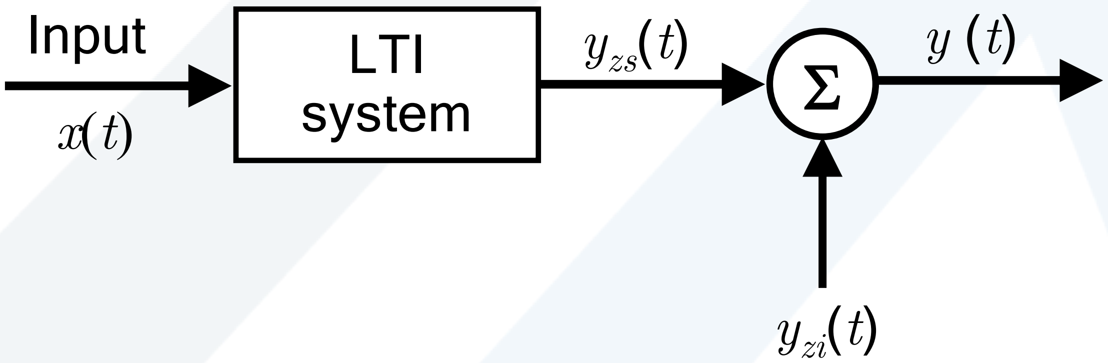
The complete solution $=$ homogeneous solution $+$ particular solution

$$y(t) = y_h(t) + y_p(t) = \underbrace{\sum_k c_k e^{\alpha_k t}}_{natural} + \underbrace{y_p(t)}_{forced}$$

- $y_h(t)$ is called the natural response $y_n(t)$ of the system. It depends on the structure of the system as well as the initial state of the system. It does not depend, on the input signal.
- For a stable system, $y_h(t)$ tends to gradually disappear in time. Because of this, it is also referred to as the transient response of the system.
- $y_p(t)$ depends on the input signal $x(t)$ and the internal structure of the system, but it does not depend on the initial state of the system.
- $y_p(t)$ is the part of the response that remains active after the homogeneous solution gradually becomes smaller and disappears.
- $y_p(t)$ will be linked to the steady-state response of the system.

Zero-input response and Zero-states response

- If the system is not energized for $t < 0$, i.e., the input and the ICs are zero, it is LTI. However, many LTI systems represented by ODE have nonzero ICs.
- Considering the input signal $x(t)$ and the ICs two different inputs, using superposition we have that the complete response of the ODE is composed of a zero-input response, due to the ICs when the input $x(t)$ is zero, and the zero-state response due to the input $x(t)$ with zero ICs.



Zero-input response $y_{zi}(t)$

- System response to the non-zero initial states.
- The response is part of homogeneous solution.

If all α_k are of order 1, $y_{zi}(t) = \sum_{k=1}^{N} c_{zik} e^{\alpha_k t}, c_{zik}$ could be determined by the ICs $\frac{d^k y}{dt^k}(0^-)$

Zero-state response $y_{zs}(t)$

- System response to the external input.
- The response includes part of the homogenous solution and particular solution.

If all α_k are of order 1, $y_{zs}(t) = \sum_{k=1}^{N} c_{zsk} e^{\alpha_k t} + y_p(t)$

c_{zsk} could be determined by the states changes at time 0, i.e. $\frac{d^k y}{dt^k}(0^+) - \frac{d^k y}{dt^k}(0^-)$

- Note: no impulses or derivatives of impulses in the right-hand side of the DE.

$$\frac{d^k y}{dt^k}(0^+) - \frac{d^k y}{dt^k}(0^-) = 0$$

The complete solution = Zero-input response + Zero-state response

$$y(t) = y_{zi}(t) + y_{zs}(t) = \underbrace{\sum_k c_{zik} e^{\alpha_k t}}_{zero-input} + \underbrace{\sum_k c_{zsk} e^{\alpha_k t} + y_p(t)}_{zero-state} = \underbrace{\sum_k c_k e^{\alpha_k t}}_{natural} + \underbrace{y_p(t)}_{forced}$$

- Note: the natural response = zero-input response + part of zero-state response

$$y^{(j)}(0^-) = y_{zi}^{(j)}(0^-) + y_{zs}^{(j)}(0^-) = y_{zi}^{(j)}(0^-), \quad j = 0, 1, \ldots, n-1$$

$$y^{(j)}(0^+) = y_{zi}^{(j)}(0^+) + y_{zs}^{(j)}(0^+), \quad j = 0, 1, \ldots, n-1$$

$$y_{zi}^{(j)}(0^+) = y_{zi}^{(j)}(0^-) = y^{(j)}(0^-), \quad j = 0, 1, \ldots, n-1$$

- Example 9: Response of the first-order system for sinusoidal input (RC circuit)

  The initial value of the output signal is $y(0^-) = 5$. Determine the output signal in response to a sinusoidal input signal in the form $x(t) = 5\cos(8t)$.

$$\frac{dy(t)}{dt} + 4y(t) = 4x(t) \qquad y_h(t) = ce^{-4t},\ t \geq 0$$

$$y_p(t) = a\cos(8t) + b\sin(8t) \Rightarrow \frac{dy_p(t)}{dt} = -8a\sin(8t) + 8b\cos(8t)$$

$$-8a\sin(8t) + 8b\cos(8t) + 4a\cos(8t) + 4b\sin(8t) = 20\cos(8t) \Rightarrow a = 1,\ b = 2$$

$$y(t) = ce^{-4t} + \cos(8t) + 2\sin(8t),\ t \geq 0$$

$$y(0^+) = y(0^-) = 5 \Rightarrow c = 4 \Rightarrow y(t) = \underbrace{4e^{-4t}}_{y_n(t)} + \underbrace{\cos(8t) + 2\sin(8t)}_{y_p(t)},\ t \geq 0$$

$$\frac{dy_{zi}(t)}{dt} + 4y_{zi}(t) = 0,\ y_{zi}(0^+) = y_{zi}(0^-) = 5 \Rightarrow y_{zi}(t) = 5e^{-4t},\ t \geq 0$$

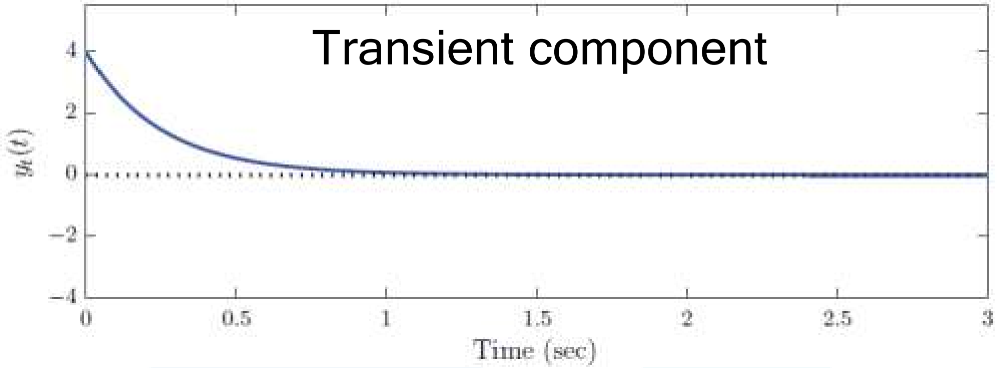
$$\frac{dy_{zs}(t)}{dt} + 4y_{zs}(t) = 5\cos(8t),\ y_{zs}(0^{+}) = y(0^{+}) - y(0^{-}) = 0$$

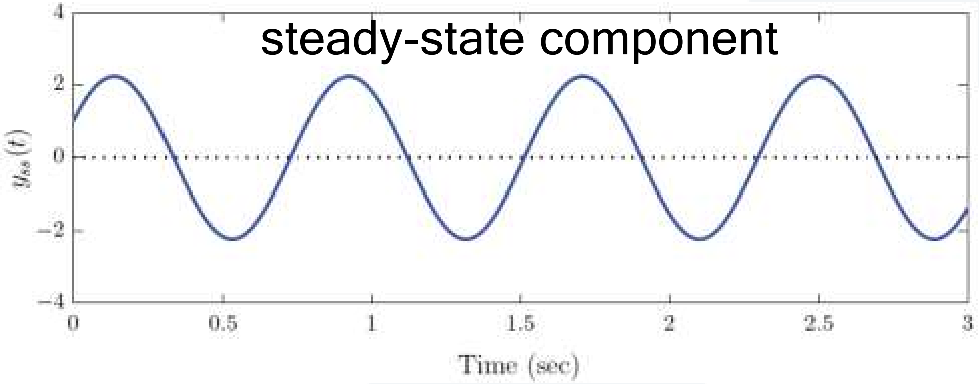
$$y_{zs}(t) = \alpha e^{-4t} + \cos(8t) + 2\sin(8t)$$

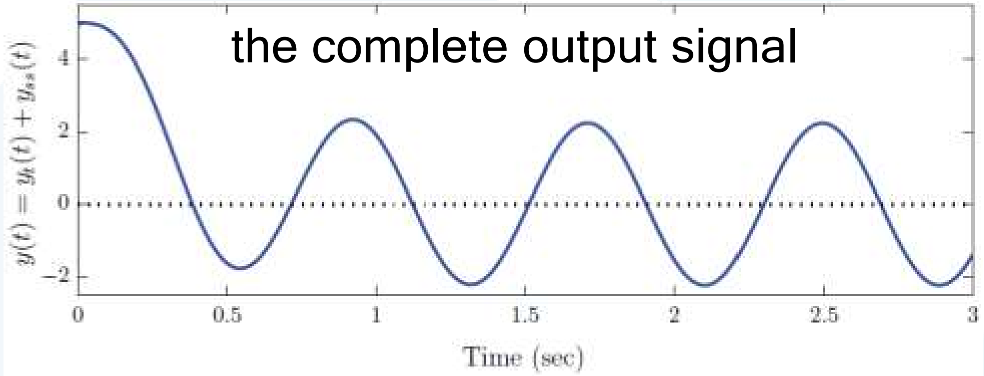
$$y_{zs}(t) = -e^{-4t} + \cos(8t) + 2\sin(8t),\ t \geq 0$$

$$y(t) = \underbrace{5e^{-4t}}_{y_{zi}(t)} + \underbrace{-e^{-4t} + \cos(8t) + 2\sin(8t)}_{y_{zs}(t)},\ t \geq 0$$

$$y(t) = \underbrace{4e^{-4t}}_{y_t(t)} + \underbrace{\cos(8t) + 2\sin(8t)}_{y_{ss}(t)},\ t \geq 0$$







$$\frac{dy(t)}{dt} + 4y(t) = 4x(t)$$

$x(t) = 5\cos(8t)$,

$y(0^-) = 5$

Total response

$$y(t) = 4e^{-4t} + \cos(8t) + 2\sin(8t),\ t \geq 0$$

Zero input response

$$y_{zi}(t) = 5e^{-4t},\ t \geq 0$$

Zero state response

$$y_{zs}(t) = -e^{-4t} + \cos(8t) + 2\sin(8t),\ t \geq 0$$

Natural response

$$y_n(t) = 4e^{-4t},\ t \geq 0$$

Forced response

$$y_p(t) = \cos(8t) + 2\sin(8t),\ t \geq 0$$

Transient response

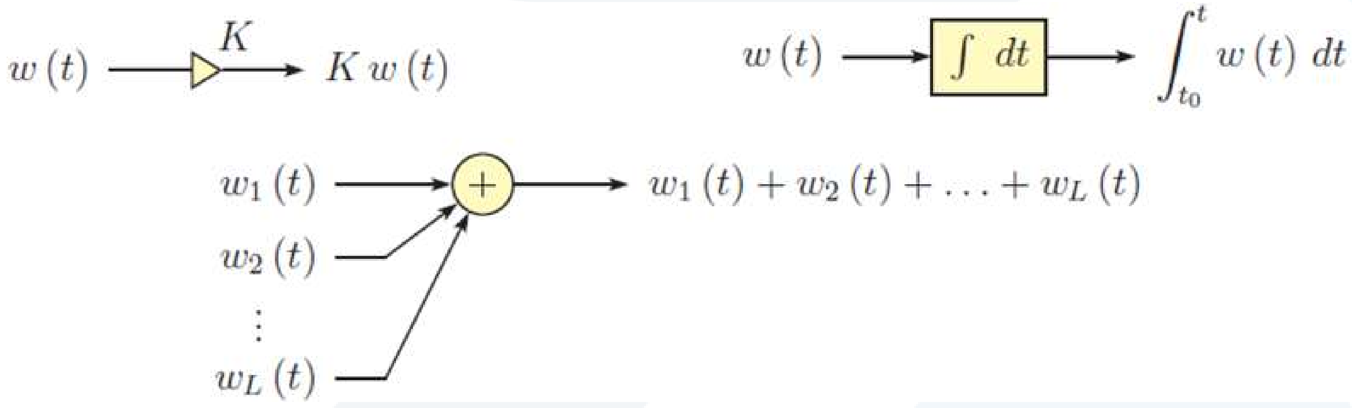
$$y_t(t) = 4e^{-4t},\ t \geq 0$$

Steady state response

$$y_{ss}(t) = \cos(8t) + 2\sin(8t),\ t \geq 0$$

5. Block Diagram Representation of Continuous-Time Systems

- Block diagrams for CT systems are constructed using three types of components, namely constant-gain amplifiers, signal adders and integrators.

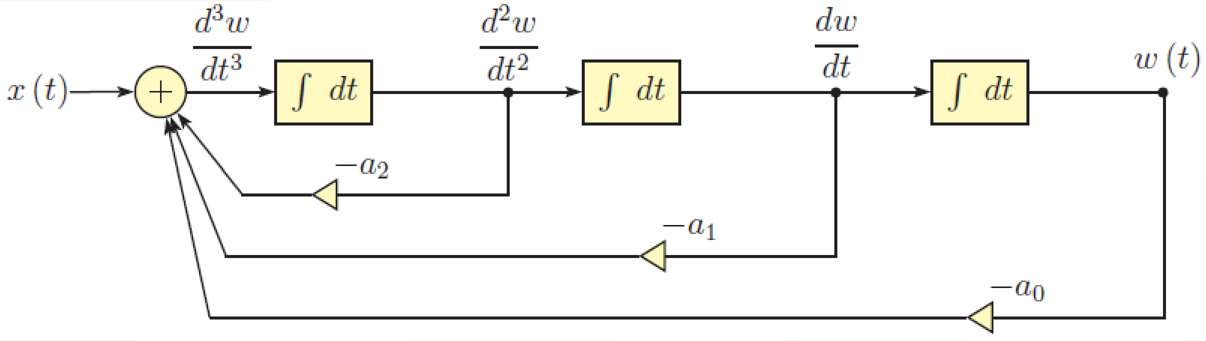


- Finding a block diagram from a DE is best explained with an example.

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0x$$

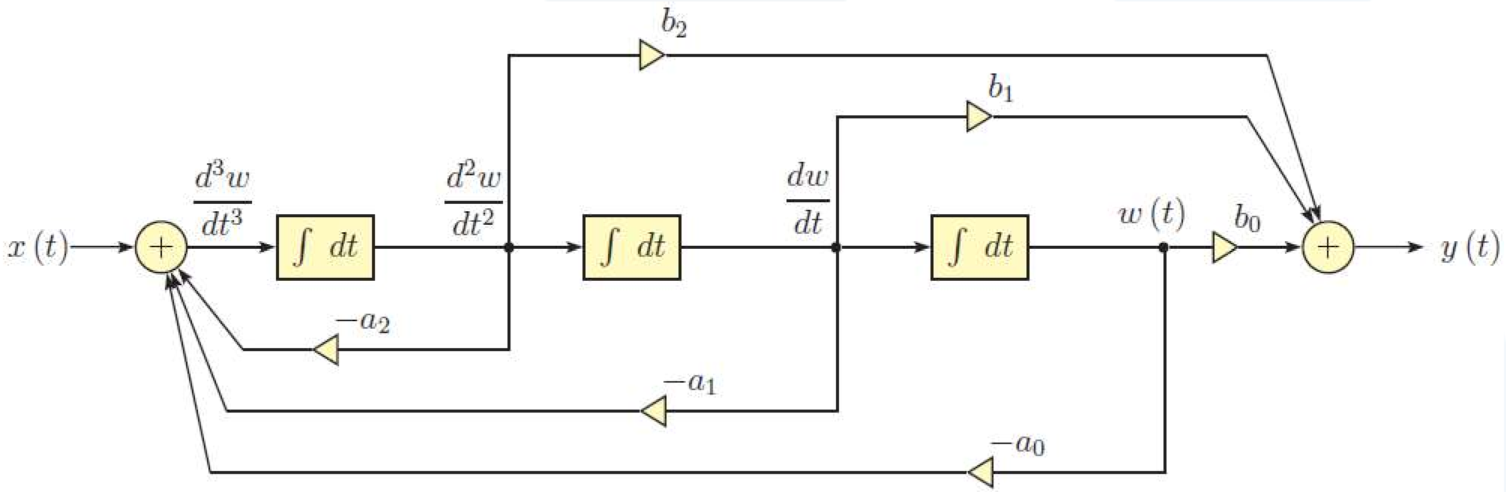
- We will introduce an intermediate variable $w(t)$

$$\frac{d^3w}{dt^3}+a_2\frac{d^2w}{dt^2}+a_1\frac{dw}{dt}+a_0w=x \Rightarrow \frac{d^3w}{dt^3}=x-a_2\frac{d^2w}{dt^2}-a_1\frac{dw}{dt}-a_0w$$



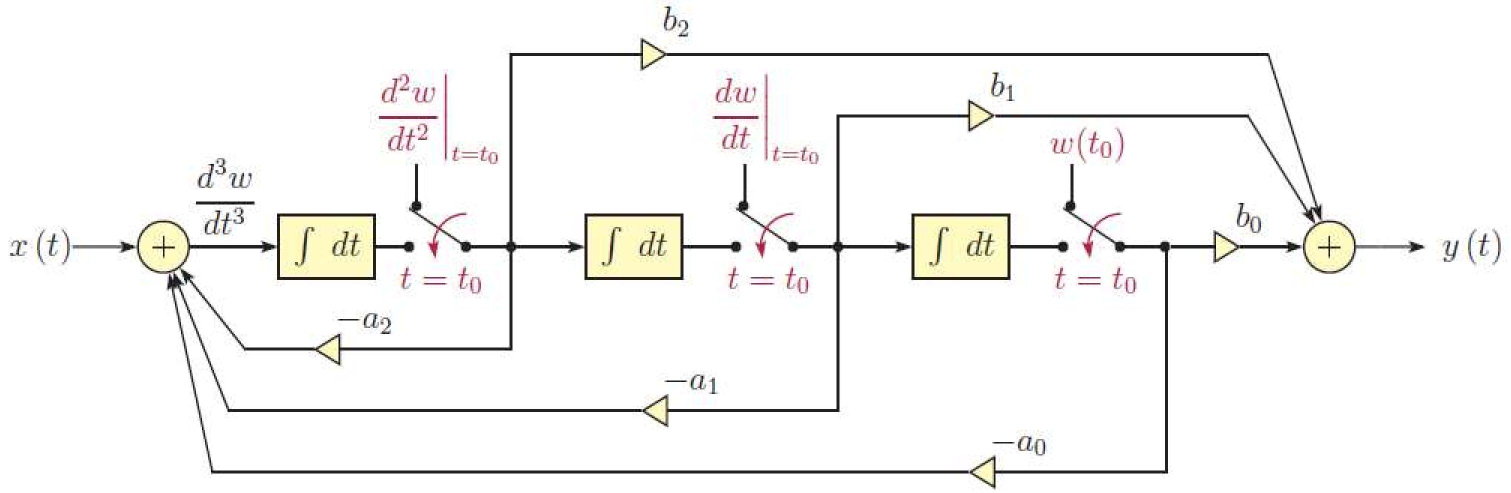
- The output signal $y(t)$ can be expressed in terms of $w(t)$ as:

$$y=b_2\frac{d^2w}{dt^2}+b_1\frac{dw}{dt}+b_0w$$



Imposing initial conditions

- Initial values of $y(t)$ and its first $N-1$ derivatives need to be converted to corresponding initial values of $w(t)$ and its first $N-1$ derivatives.



- Example 10: Block diagram for continuous-time system

$$\frac{d^3y}{dt^3}+5\frac{d^2y}{dt^2}+17\frac{dy}{dt}+13y=x+2\frac{dx}{dt}$$

with the input signal $x(t) = \cos(20\pi t)$ and subject to initial conditions:

$$y(0) = 1, \quad \left.\frac{dy}{dt}\right|_{t=0} = 2, \quad \left.\frac{d^2y}{dt^2}\right|_{t=0} = -4$$

$$\frac{d^3w}{dt^3} + 5\frac{d^2w}{dt^2} + 17\frac{dw}{dt} + 13w = x, \quad y = w + 2\frac{dw}{dt}$$

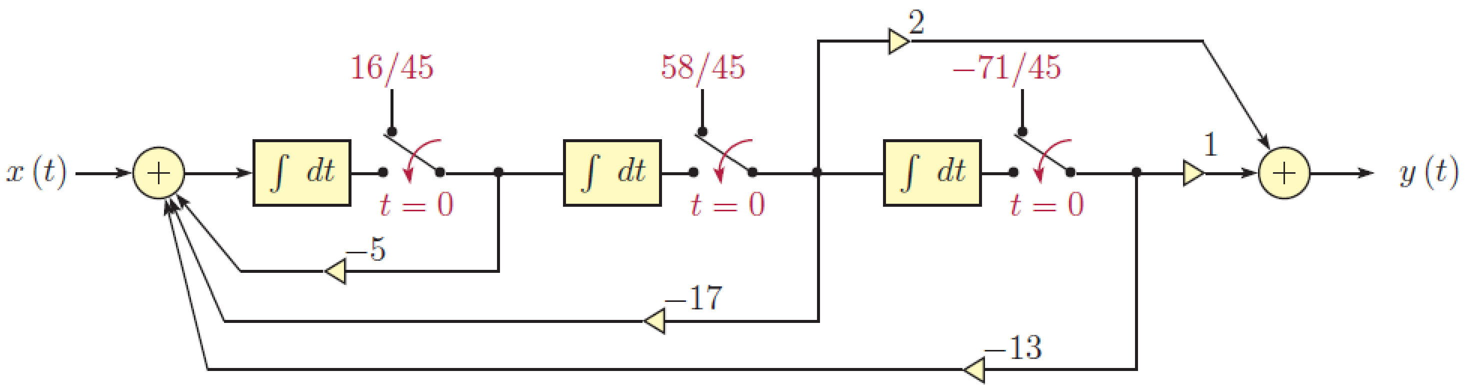
$$y(0) = 1 = w(0) + 2\left.\frac{dw}{dt}\right|_{t=0}, \quad \left.\frac{dy}{dt}\right|_{t=0} = 2 = \left.\frac{dw}{dt}\right|_{t=0} + 2\left.\frac{d^2w}{dt^2}\right|_{t=0},$$

$$\left.\frac{d^2y}{dt^2}\right|_{t=0} = -4 = \left.\frac{d^2w}{dt^2}\right|_{t=0} + 2\left.\frac{d^3w}{dt^3}\right|_{t=0}$$

$$\left.\frac{d^3w}{dt^3}\right|_{t=0} = x(0) - 5\left.\frac{d^2w}{dt^2}\right|_{t=0} - 17\left.\frac{dw}{dt}\right|_{t=0} - 13w(0)$$

$x(0) = 1$. Solving Equations, the initial values of integrator outputs are:

$$w(0)=\frac{-71}{45},\qquad \left.\frac{dw}{dt}\right|_{t=0}=\frac{58}{45},\qquad \left.\frac{d^2w}{dt^2}\right|_{t=0}=\frac{16}{45}$$



6. Impulse Response and Convolution

Convolution operation for CTLTI systems

- The (CT) convolution of the functions x and h, denoted $x * h$, is defined as the function:

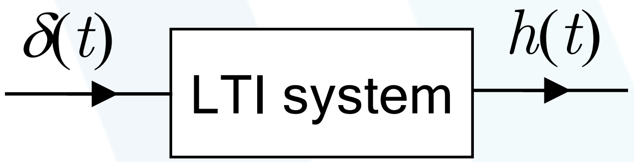
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Properties of Convolution

- Is commutative. For any two functions x and h, $x * h = h * x$.
- Is associative. For any functions x, h_1, and h_2, $(x * h_1) * h_2 = x * (h_1 * h_2)$.
- Is distributive with respect to addition. For any functions x, h_1, and h_2, $x * (h_1 + h_2) = x * h_1 + x * h_2$.
- For any function x, $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$
- Moreover, δ is the convolutional identity. That is, for any function x, $x * \delta = x$.

Impulse response of a CTLTI system

- The response h of a system T to the input δ is called the impulse response of the system (i.e., $h = T\delta$).



- For any LTI system with input x, output y, and impulse response h, the following relationship holds: $y = x * h$.
- LTI system is completely characterized by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.

$$\delta(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ h(t)\end{array}} \longrightarrow h(t)$$

$$x(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ h(t)\end{array}} \longrightarrow y(t) = x(t) * h(t)$$

Step Response of a CTLTI system

- The response $s(t)$ of a system T to the input $u(t)$ is called the step response of the system.

$$s(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(t-\tau)d\tau$$

- The impulse response h and step response s of a LTI system are related as:

$$h(t) = \frac{ds(t)}{dt}$$

- Example 11: Impulse response of the simple RC circuit

  Consider the RC circuit. Let the element values be $R = 1\ \Omega$ and $C = 1/4$ F. Assume $y(0) = 0$. Determine the impulse response of the system.

First method: using differential equation $y(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} x(\tau) d\tau$

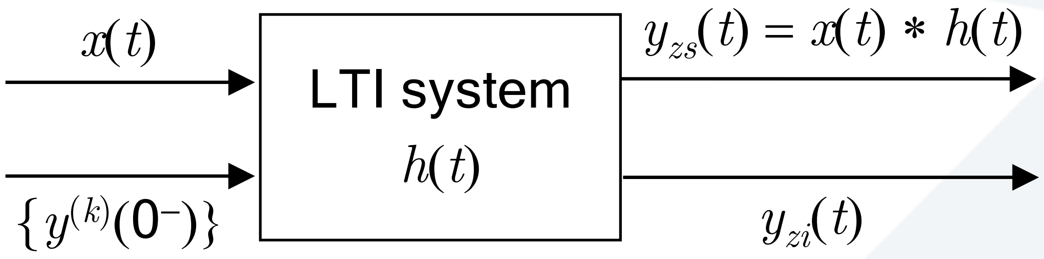
Setting $x(t) = \delta(t)$ $\quad h(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} \delta(\tau) d\tau = \frac{1}{RC} e^{-t/RC} u(t)$

Second method: unit-step response of the system

$$s(t) = (1 - e^{-t/RC})u(t) \Rightarrow h(t) = \frac{ds(t)}{dt} = \frac{1}{RC} e^{-t/RC} u(t) = 4e^{-4t} u(t)$$

Linearity properties of zero-input and zero-state response

- Zero-state response is linear with the input.
- Zero-input response is linear with the initial state.



$$y(t) = y_{zi}(t) + y_{zs}(t) = \underbrace{\sum_k c_{zik} e^{\alpha_k t}}_{zero-input} + \underbrace{x(t) * h(t)}_{zero-state}$$

- Notes:

1. For LTI systems, the excitation and initial states can be thought of as two separate inputs.
2. When the ICs are not zero, there is no linear relationship between the complete response of the system and the external excitation.

3. The impulse response $h(t)$ of an LTIC system is the zero-state output of the system when a unit impulse $\delta(t)$ is applied at the input.

$$a_N \frac{d^N h(t)}{dt^N} + a_{N-1} \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_0 h(t) = \delta(t)$$

$$h^{(n-1)}(0^+) = 1/a_N,\ h^{(j)}(0^+) = 0, \qquad j = 0, 1, \ \ldots, \ n-2$$

- Example 12: Determine the impulse response of the LTIC system given by the following differential equation: $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = x(t)$

  $$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \delta(t), \qquad \dot{h}(0^+) = 1,\ h(0^+) = 0$$

  For $t > 0$, the DE is given by the following homogeneous equation:

  $$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = 0, \qquad \dot{h}(0^+) = 1,\ h(0^+) = 0$$

  $$h(t) = (e^{-2t} - e^{-3t})u(t)$$

- Example 13: Determine the impulse response of the LTIC system given by the following differential equation: $\ddot{y}(t)+5\dot{y}(t)+6y(t)=\ddot{x}(t)+2\dot{x}(t)+3x(t)$

Suppose $h_1(t)$ satisfies: $\ddot{h}_1(t)+5\dot{h}_1(t)+6h_1(t)=\delta(t)$

Due to the differentiation property and linearity of the LTIC system, the impulse response satisfies: $h(t)=\ddot{h}_1(t)+2\dot{h}_1(t)+3h_1(t)$

$$h_1(t)=(e^{-2t}-e^{-3t})u(t)$$

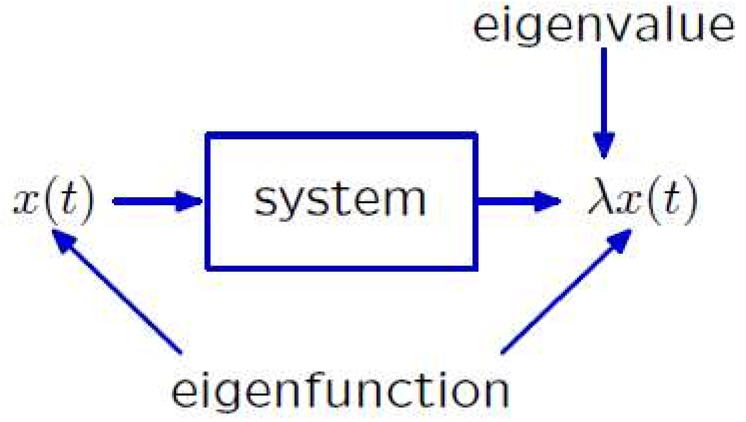
$$\dot{h}_1(t)=(-2e^{-2t}+3e^{-3t})u(t)+(e^{-2t}-e^{-2t})\delta(t)=(-2e^{-2t}+3e^{-3t})u(t)$$

$$\ddot{h}_1(t)=(4e^{-2t}-9e^{-3t})u(t)+(-2e^{-2t}+3e^{-3t})\delta(t)=(4e^{-2t}-9e^{-3t})u(t)+\delta(t)$$

$$h(t)=(4e^{-2t}-9e^{-3t})u(t)+\delta(t)+2(-2e^{-2t}+3e^{-3t})u(t)+3(e^{-2t}-e^{-3t})u(t)$$

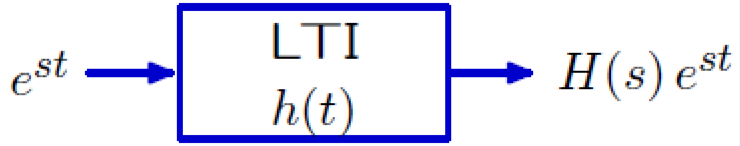
$$h(t)=\delta(t)+(3e^{-2t}-6e^{-3t})u(t)$$

Eigenfunctions of CTLTI system



- If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.

- Complex exponential are eigenfunctions of LTI systems.



$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st}$$

where s is a complex constant.

- We refer to H as the transfer function of the system.

Causality and Stability in Continuous-Time Systems

- For CTLTI systems the causality property can be related to the impulse response of the system $h(t) = 0$ for all $t < 0$.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- For a CTLTI system to be stable, its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} \left|h(\tau)\right| d\tau < \infty$$

- Example 14: Stability of a first-order continuous-time system

  Evaluate the stability of the first-order CTLTI system described by the DE:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

The step response of the system is when $x(t) = u(t)$

$$\frac{dy(t)}{dt} + ay(t) = u(t) \Rightarrow y(t) = ce^{-at} + \frac{1}{a}$$

$y(0) = 0$. (We take the initial value to be zero since the system is specified to be CTLTI. Non-zero initial conditions cannot be linear: Based on a zero input signal must produce a zero output signal).

$$y(0) = 0 \Rightarrow 0 = c + 1/a \Rightarrow c = -1/a$$

$$s(t) = \frac{1}{a}(1 - e^{-at})u(t) \qquad h(t) = \frac{ds(t)}{dt} = s(t) = e^{-at}u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{-at}\, dt = \frac{1}{a}$$ Thus the system is stable if $a > 0$.