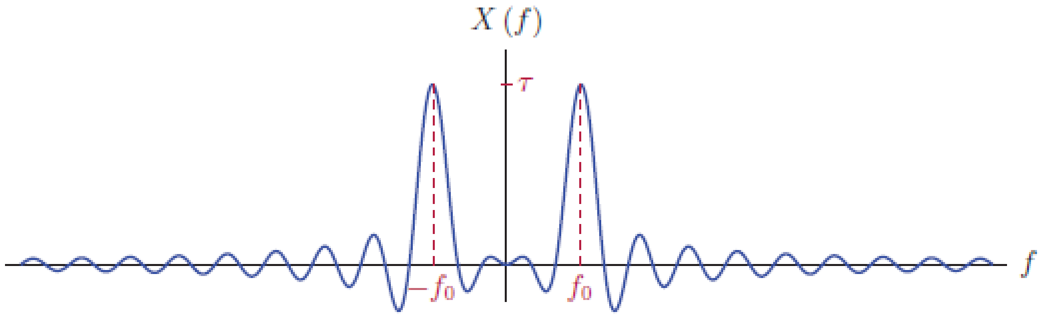
CEDC403: Signals and Systems

Lecture Notes 5: Fourier Analysis for Continuous Time Signals and Systems: Part A



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

Chapter 4

Fourier Analysis for Continuous Time Signals and Systems

Jean-Baptiste Joseph Fourier
(1768-1830)

Born	21 March 1768 in Auxerre, Kingdom of France
Died	16 May 1830 (aged 62) in Paris, Kingdom of France

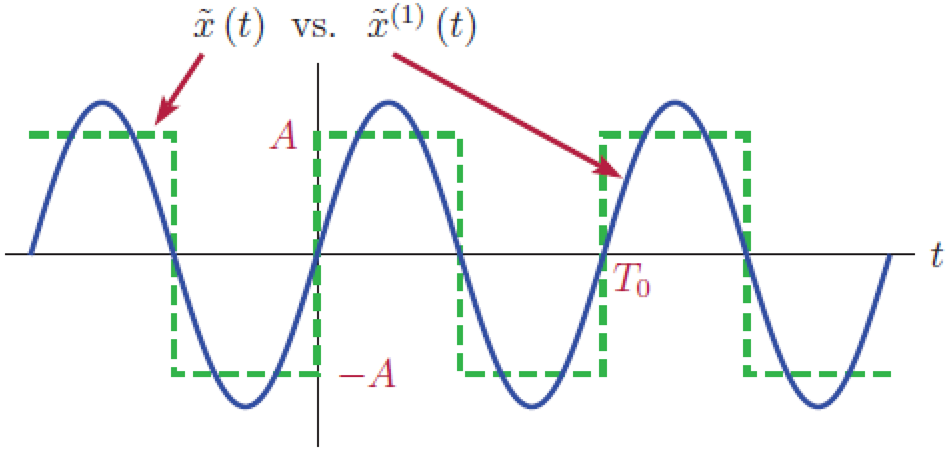
1. Introduction

- Fourier analysis leads to the frequency spectrum of a continuous-time signal.
- The frequency spectrum displays the various sinusoidal components that make up the signal.
- In the frequency domain, linear systems are described by linear algebraic equations that can be easily solved, in contrast to the time-domain representation, where they are described by linear differential equations.
- A weighted summation of Sines and Cosines of different frequencies can be used to represent periodic (Fourier Series), or non-periodic (Fourier Transform) functions.

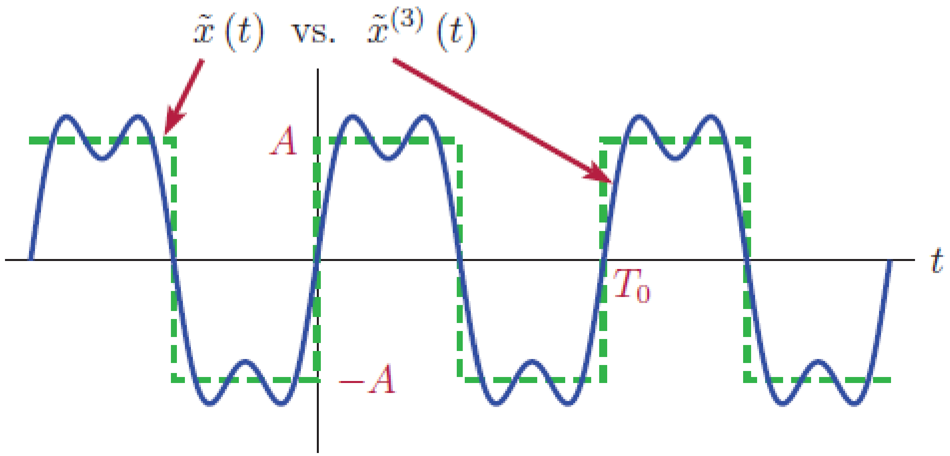
2. Analysis of Periodic Continuous-Time Signals

- We will study methods of expressing periodic continuous-time signals in two different but equivalent formats, namely the trigonometric Fourier series (TFS) and the exponential Fourier series (EFS).

Approximating a periodic signal with trigonometric functions



$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t)$



$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$

- The approximation error $\tilde{\varepsilon}_1(t) = \tilde{x}(t) - \tilde{x}^{(1)}(t) = \tilde{x}(t) - b_1\sin(\omega_0 t)$
- The best value for the coefficient b_1 would be to the value that makes the normalized average power of $\tilde{\varepsilon}_1(t)$ as small as possible.

$$P_\varepsilon = \frac{1}{T_0}\int_0^{T_0}[\tilde{\varepsilon}_1(t)]^2 dt = \frac{1}{T_0}\int_0^{T_0}[\tilde{x}(t) - b_1\sin(\omega_0 t)]^2 dt, \qquad \frac{dP_\varepsilon}{db_1} = 0 \Rightarrow b_1 = \frac{4A}{\pi}$$

$$\tilde{x}^{(1)}(t) = \frac{4A}{\pi}\sin(\omega_0 t), \qquad \tilde{\varepsilon}_1(t) = \tilde{x}(t) - \frac{4A}{\pi}\sin(\omega_0 t)$$

$$\tilde{x}^{(2)}(t) = b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) \qquad \tilde{\varepsilon}_2(t) = \tilde{x}(t) - \tilde{x}^{(2)}(t)$$

$$P_\varepsilon = \frac{1}{T_0}\int_0^{T_0}[\tilde{\varepsilon}_2(t)]^2 dt, \ \frac{dP_\varepsilon}{db_{1,2}} = 0 \Rightarrow \begin{cases} b_1 = \dfrac{4A}{\pi} \\ b_2 = 0 \end{cases} \Rightarrow \tilde{x}^{(2)}(t) = \tilde{x}^{(1)}(t)$$

$$\tilde{x}^{(3)}(t) = b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + b_3\sin(3\omega_0 t)$$

$$P_\varepsilon = \frac{1}{T_0}\int_0^{T_0}[\tilde{\varepsilon}_3(t)]^2 dt, \qquad \frac{dP_\varepsilon}{db_{1,2,3}} = 0 \Rightarrow \begin{cases} b_1 = \dfrac{4A}{\pi} \\ b_2 = 0 \\ b_3 = \dfrac{4A}{3\pi} \end{cases} \qquad \Rightarrow \tilde{\varepsilon}_3(t) = \tilde{x}(t) - \tilde{x}^{(3)}(t)$$

$$\tilde{x}^{(3)}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t)$$

- The normalized average power of the error signal $\tilde{\varepsilon}_3(t)$ seems to be less than that of the error $\tilde{\varepsilon}_1(t)$.

Trigonometric Fourier series (TFS)

$$\tilde{x}(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \ldots + a_k\cos(k\omega_0 t) + \ldots$$
$$+ b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + \ldots + b_k\sin(k\omega_0 t) + \ldots$$

- In a compact notation (trigonometric Fourier Series TFS of the periodic signal $\tilde{x}(t)$):

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t)$$

where $\omega_0 = 2\pi f_0$ is the fundamental frequency in rad/s.

- The set of orthogonal basis functions: $\phi_k(t) = \cos(k\omega_0 t), \quad k = 0, 1, 2, \ldots, \infty$

$$\psi_k(t) = \sin(k\omega_0 t) \quad k = 1, 2, \ldots, \infty$$

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k\phi_k(t) + \sum_{k=1}^{\infty} b_k\psi_k(t)$$

- We call the frequencies that are integer multiples of the fundamental frequency the harmonics.
- The frequencies $2\omega_0$, $3\omega_0$, ..., $k\omega_0$ are the second, the third, and the k-th harmonics of the fundamental frequency respectively.
- We need to determine the coefficients: a_0, a_k, and b_k.

$$\int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\cos(k\omega_0 t)\,dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t)\sin(k\omega_0 t)\,dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t)\cos(k\omega_0 t)\,dt = 0$$

Trigonometric Fourier series (TFS)

1. Synthesis equation:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

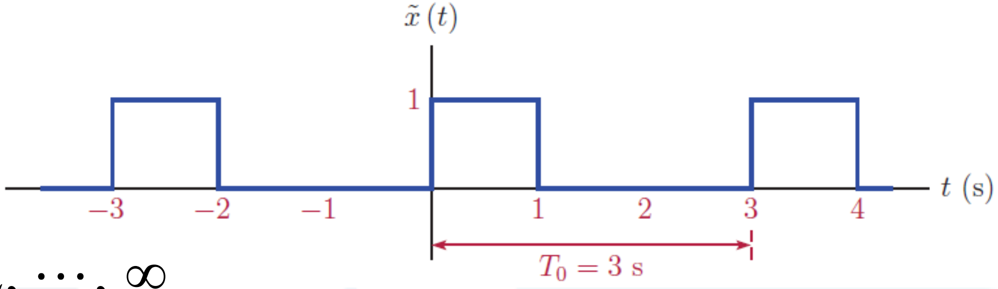
2. Analysis equation:

$$a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\,dt \qquad \text{(dc component)}$$

$$a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\cos(k\omega_0 t)\,dt, \qquad \text{for } k = 1, 2, \cdots, \infty$$

$$b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\sin(k\omega_0 t)\,dt, \qquad \text{for } k = 1, 2, \cdots, \infty$$

- Example 1: Trigonometric Fourier series of a periodic pulse train



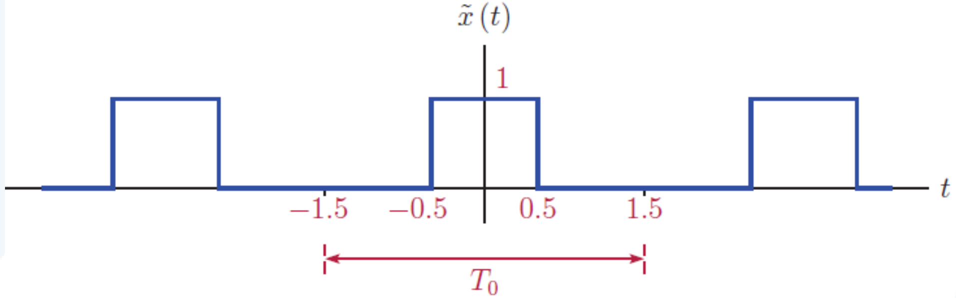
$$a_0 = \frac{1}{3}\int_0^1 (1)\,dt = \frac{1}{3}$$

$$a_k = \frac{2}{3}\int_0^1 (1)\cos\left(\frac{2\pi kt}{3}\right)dt = \frac{\sin(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \cdots, \infty$$

$$b_k = \frac{2}{3}\int_0^1 (1)\sin\left(\frac{2\pi kt}{3}\right)dt = \frac{1-\cos(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \cdots, \infty$$

$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{\sin(2\pi k/3)}{\pi k}\cos\left(\frac{2\pi kt}{3}\right) + \sum_{k=1}^{\infty} \frac{1-\cos(2\pi k/3)}{\pi k}\sin\left(\frac{2\pi kt}{3}\right)$$

- Example 2: Periodic pulse train



$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{2\sin(\pi k/3)}{\pi k}\cos\left(\frac{2\pi kt}{3}\right)$$

Exponential Fourier series (EFS)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Single-tone signals:

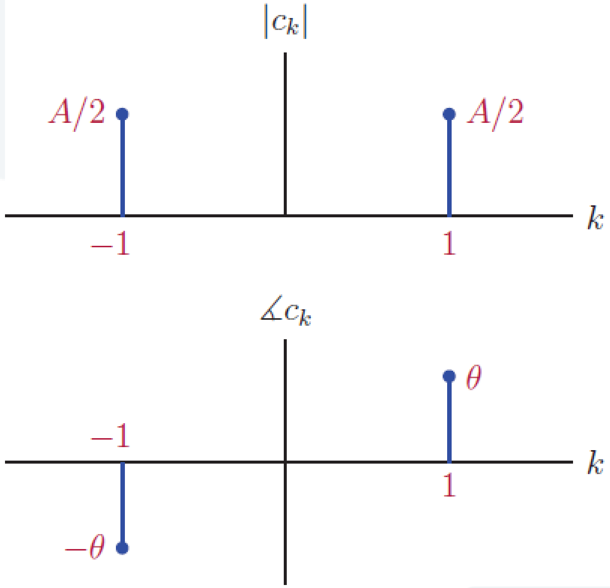
$$\tilde{x}(t) = A\cos(\omega_0 t + \theta) = \frac{A}{2} e^{j(\omega_0 t+\theta)} + \frac{A}{2} e^{-j(\omega_0 t+\theta)} = \frac{A}{2} e^{j\theta} e^{j\omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j\theta}, \quad c_{-1} = \frac{A}{2} e^{-j\theta}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

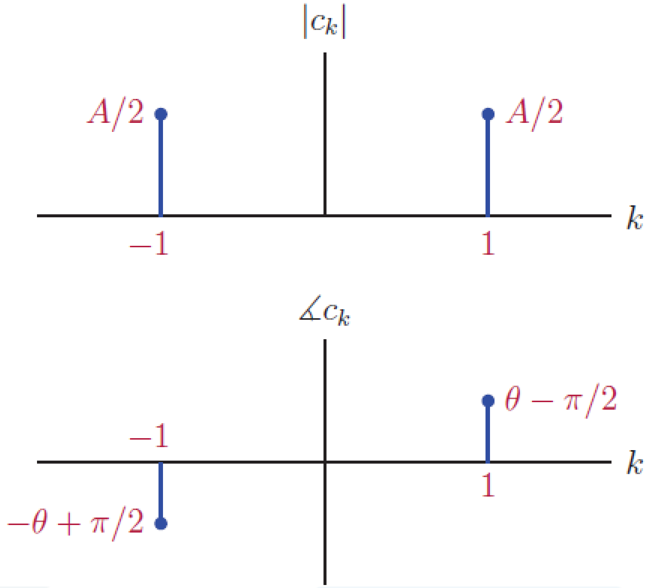
$$\tilde{x}(t) = A\sin(\omega_0 t + \theta) = \frac{A}{2} e^{j(\theta-\pi/2)} e^{j\omega_0 t} + \frac{A}{2} e^{-j(\theta-\pi/2)} e^{-j\omega_0 t}$$

$$c_1 = \tfrac{A}{2} e^{j(\theta-\pi/2)}, \quad c_{-1} = \tfrac{A}{2} e^{-j(\theta-\pi/2)}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

- The EFS representations of the two signals are shown graphically, in the form of a line spectrum.



$$\tilde{x}(t) = A\cos(\omega_0 t + \theta)$$



$$\tilde{x}(t) = A\sin(\omega_0 t + \theta)$$

The general case:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$c_0 = a_0$$

$$c_k + c_{-k} = a_k \quad \text{and} \quad j(c_k - c_{-k}) = b_k, \quad \text{for } k = 1, \cdots, \infty$$

$$c_k = \tfrac{1}{2}(a_k - jb_k) \quad \text{and} \quad c_{-k} = \tfrac{1}{2}(a_k + jb_k), \quad \text{for } k = 1, \cdots, \infty$$

- What if we would like to compute the EFS coefficients of a signal without first having to obtain the TFS coefficients? The exponential basis functions also form an orthogonal set.

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} T_0, & m = k \\ 0, & m \neq k \end{cases}$$

Exponential Fourier series (EFS):

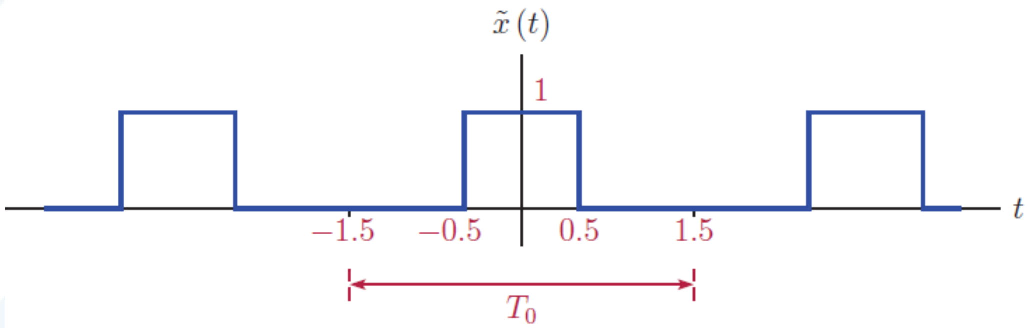
1. Synthesis equation: $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

2. Analysis equation: $$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t)\, e^{-jk\omega_0 t} dt$$

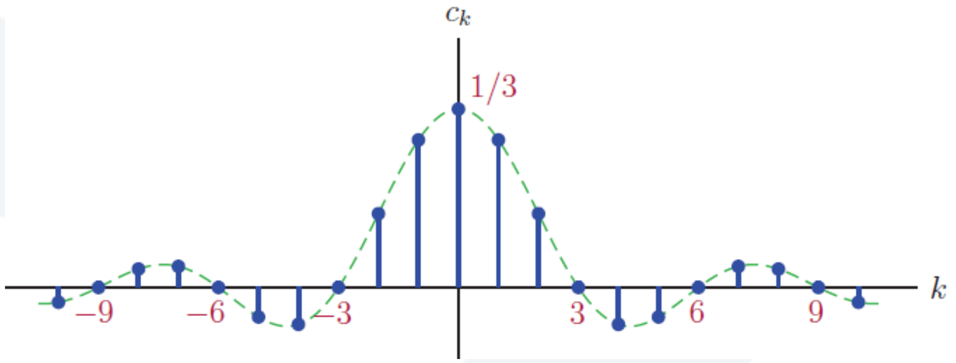
- In general, the coefficients of the EFS representation of a periodic signal $\tilde{x}(t)$ are complex valued.
- They can be graphed in the form of a line spectrum if each coefficient is expressed in polar complex form with its magnitude and phase: $c_k = |c_k| e^{j\theta_k}$
- Example 3: Exponential Fourier series for periodic pulse train

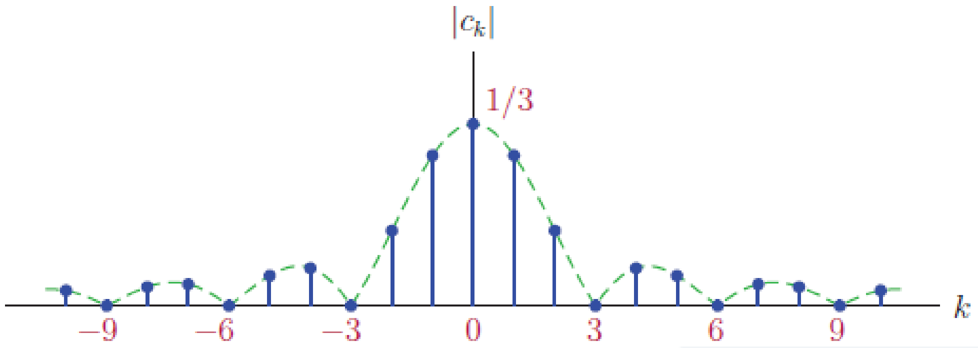
$$c_k = \frac{1}{3}\int_{-0.5}^{0.5} (1)\, e^{-j2\pi kt/3} dt = \frac{\sin(\pi k/3)}{\pi k}$$

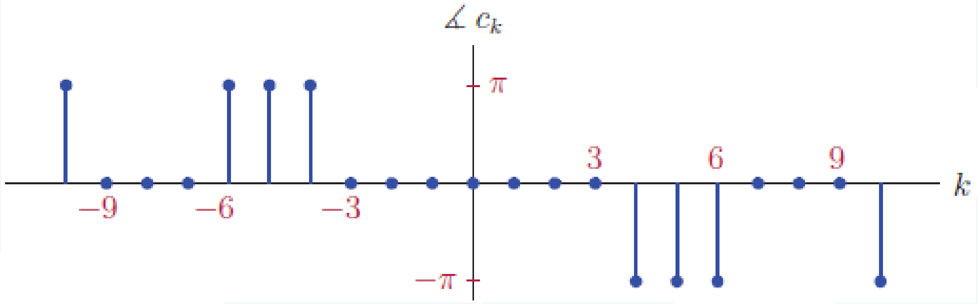
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/3)}{\pi k} e^{j2\pi kt/3}$$



- A line graph of the set of coefficients c_k is useful for illustrating the make-up of the signal $\tilde{x}(t)$ in terms of its harmonics.

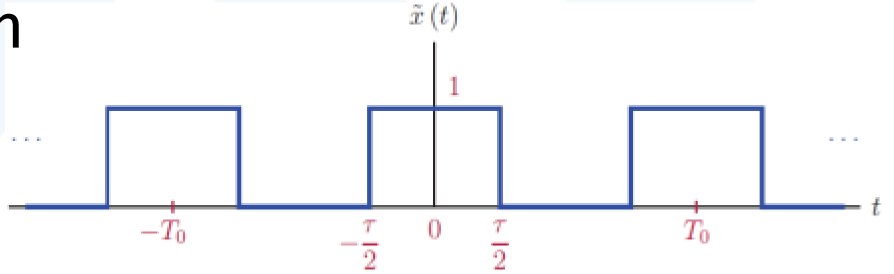






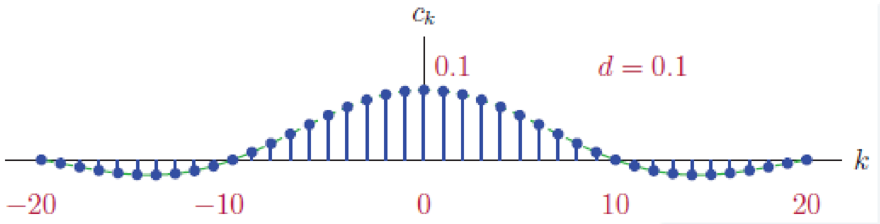
- Example 4: Effects of duty cycle on the spectrum

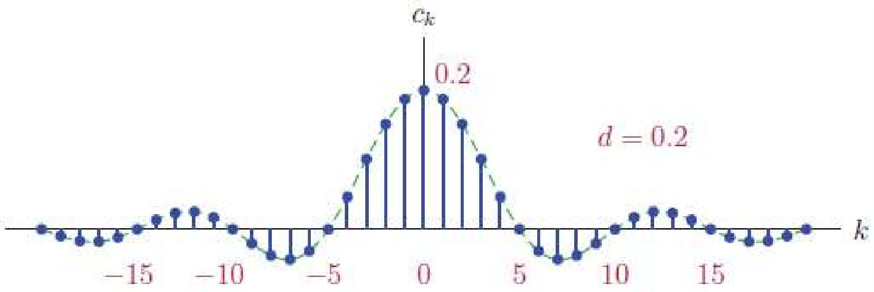
  The duty cycle of a pulse train is defined as the ratio of the pulse-width to the period, $d = \tau / T_0$

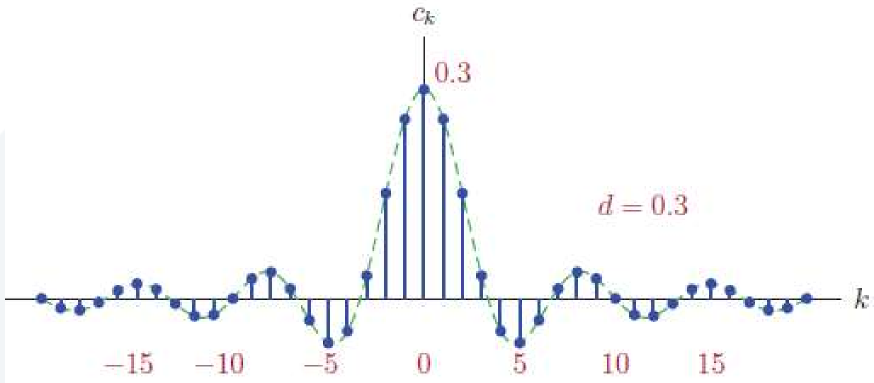


$$c_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2}(1)\,e^{-j2\pi kt/T_0}dt = \frac{\sin(\pi kd)}{\pi k} = d\,\mathrm{sinc}(kd)$$

- Note: Values of coefficients c_k depend only on the duty cycle and not on the period T_0.

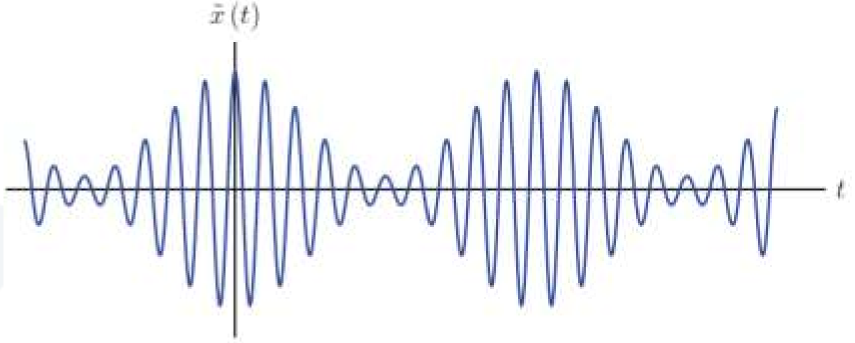






- Example 5: Spectrum of multi-tone signal

$\tilde{x}(t) = \cos(2\pi[10f_0]t) + 0.8\cos(2\pi f_0 t)\cos(2\pi[10f_0]t).$

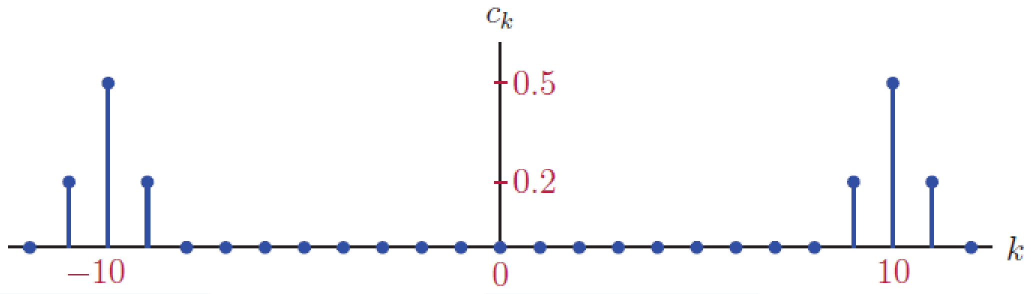


$$\tilde{x}(t) = 0.5e^{j2\pi(10f_0)t} + 0.5e^{-j2\pi(10f_0)t} + 0.2e^{j2\pi(11f_0)t} + 0.2e^{-j2\pi(11f_0)t} + 0.2e^{j2\pi(9f_0)t} + 0.2e^{-j2\pi(9f_0)t}$$

The significant EFS coefficients for the signal $\tilde{x}(t)$ are

$$c_9 = c_{-9} = 0.2, \qquad c_{10} = c_{-10} = 0.5, \qquad c_{11} = c_{-11} = 0.2$$

and all other coefficients are equal to zero.



Existence of Fourier series

- The Question: Is it always possible to determine the Fourier series coefficients?

  The Dirichlet conditions (for determining if the Fourier series converges) for the periodic function $\tilde{x}$:

  1. Over a single period, $\tilde{x}$ is absolutely integrable $\int_0^{T_0} |\tilde{x}(t)|\, dt < \infty$
  2. Over a single period, $\tilde{x}$ has a finite number of maxima and minima (i.e., $\tilde{x}$ is of bounded variation); and
  3. Over any finite interval, $\tilde{x}$ has a finite number of discontinuities, each of which is finite.

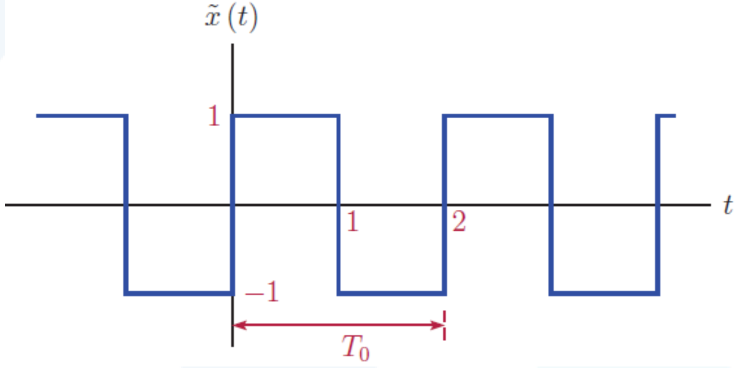
- If a periodic function $\tilde{x}$ satisfies the Dirichlet conditions, then:

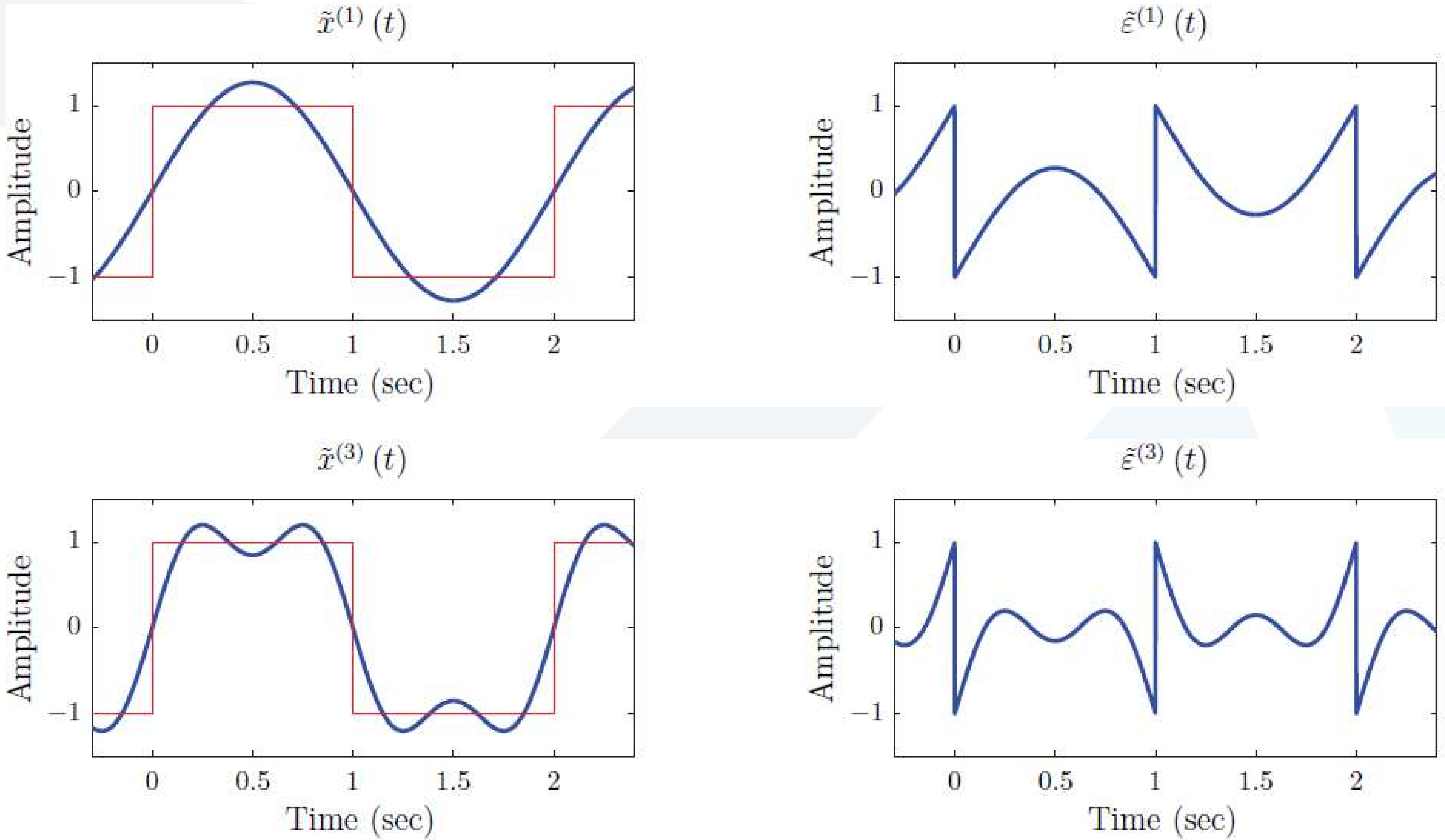
1. The Fourier series converges everywhere to $\tilde{x}$, except at the points of discontinuity of $\tilde{x}$; and
2. At each point of discontinuity of $\tilde{x}(t)$, the Fourier series converges to ½[$\tilde{x}(t^{+}) + \tilde{x}(t^{-})$], where $\tilde{x}(t^{+})$ and $\tilde{x}(t^{-})$ denote the values of the function $\tilde{x}$ on the left- and right-hand sides of the discontinuity, respectively.
3. If $\tilde{x}(t)$ is continuous everywhere, then the series converges absolutely and uniformly.

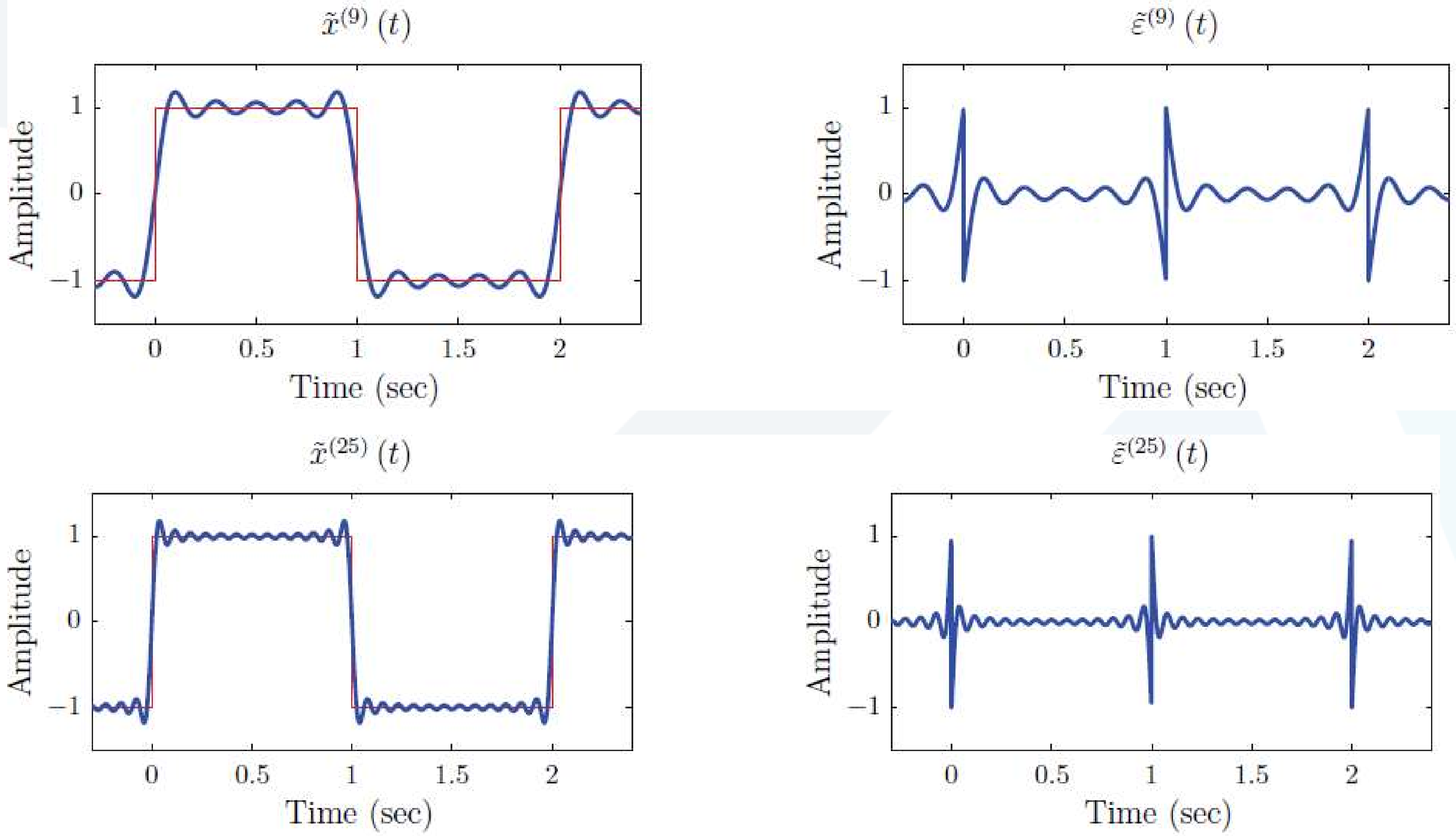
Gibbs phenomenon

$$a_k = 0 \text{ and } b_k = \begin{cases} \frac{4}{\pi k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} \qquad \tilde{x}^{(m)}(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{m} \frac{4}{\pi k} \sin(\pi k t)$$

$$\tilde{\varepsilon}^{(m)}(t) = \tilde{x}(t) - \tilde{x}^{(m)}(t)$$




$\tilde{x}^{(1)}(t)$
$\tilde{\varepsilon}^{(1)}(t)$
$\tilde{x}^{(3)}(t)$
$\tilde{\varepsilon}^{(3)}(t)$
Amplitude
Time (sec)
1
0
−1
0
0.5
1
1.5
2



- This behavior is known as Gibbs phenomenon.

- One way to explain the reason for the Gibbs phenomenon would be to link it to the inability of sinusoidal basis functions that are continuous at every point to approximate a discontinuity in the signal.

Properties of Fourier series

Linearity $\alpha_1\tilde{x}(t)+\alpha_2\tilde{y}(t)=\sum_{k=-\infty}^{\infty}[\alpha_1 c_k+\alpha_2 d_k]e^{jk\omega_0 t}$

Symmetry of Fourier series

$$\tilde{x}(t)\text{: real, }\mathrm{Im}\{\tilde{x}(t)\}=0 \Rightarrow c_{-k}=c_k^*,\qquad \tilde{x}(t)\text{: imag, }\mathrm{Re}\{\tilde{x}(t)\}=0 \Rightarrow c_{-k}=-c_k^*$$

Fourier series for even and odd signals

- If the real-valued signal $\tilde{x}(t)$ is an even function of time, the resulting EFS spectrum c_k is real-valued for all k. $\tilde{x}(-t)=\tilde{x}(t)$, for all t $\Rightarrow \mathrm{Im}\{c_k\}=0$, for all k

- If the real-valued signal $\tilde{x}(t)$ has odd-symmetry, the resulting EFS spectrum is purely imaginary. $\tilde{x}(-t) = -\tilde{x}(t)$, for all $t \Rightarrow \mathrm{Re}\{c_k\} = 0$, for all k

Time shifting $\quad \tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow \tilde{x}(t-\tau) = \sum_{k=-\infty}^{\infty} [c_k e^{-jk\omega_0 \tau}] e^{jk\omega_0 t}$

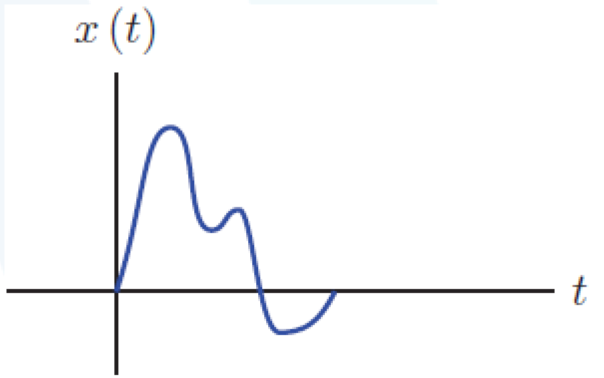
3. Analysis of Non-Periodic Continuous-Time Signals

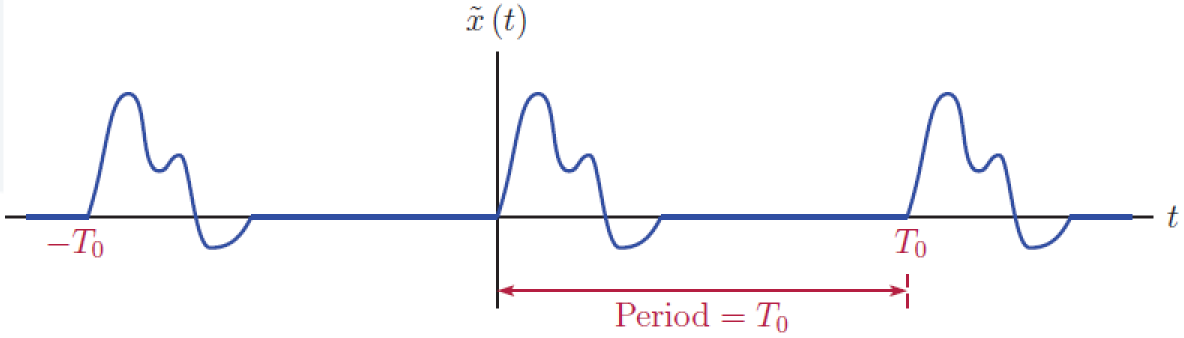
Fourier transform

- Consider the non-periodic signal $x(t)$

  What frequencies are contained in this signal?

- Let us construct a periodic extension $\tilde{x}(t)$ of the signal $x(t)$ by repeating it at intervals of T_0.





$$\tilde{x}(t) = \cdots + x(t+T_0) + x(t) + x(t-T_0) + x(t-2T_0) + \cdots = \sum_{k=-\infty}^{\infty} x(t-kT_0)$$

- Since $\tilde{x}(t)$ is periodic, it can be analyzed in the frequency domain.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \qquad c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t} dt$$

- Realizing that $\tilde{x}(t) = x(t)$ within the span $-T_0/2 < t < T_0/2$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(\tau)\, e^{-jk(2\pi/T_0)\tau} d\tau \right) e^{jk(2\pi/T_0)t} \qquad \lim_{T_0\to\infty} [\tilde{x}(t)] = x(t)$$

- As $T_0 \to \infty$ implies that $\Delta\omega = 2\pi/T_0 \to 0$ (we switch to the notation $\Delta\omega$ instead of ω_0 to emphasize the infinitesimal nature of the fundamental frequency).

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\Delta\omega t} dt$$

where c_k is the contribution of the complex exponential at the frequency $\omega = k\Delta\omega$. Each individual coefficient c_k is very small in magnitude, and in the limit we have $c_k \to 0$ when $T_0 \to \infty$. In addition, successive harmonics $k\Delta\omega$ are very close to each other due to infinitesimally small $\Delta\omega$.

$$c_k T_0 = \int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\Delta\omega t} dt$$

$$X(\omega) = \lim_{T_0\to\infty} [c_k T_0] = \lim_{T_0\to\infty} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\Delta\omega t} dt = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$X(\omega)$ is the Fourier transform of the non-periodic signal $x(t)$.

Fourier transform for continuous-time signals:

1. Synthesis equation: (Inverse transform)

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df \qquad \text{(using } f\text{)}$$

2. Analysis equation: (Forward transform)

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt \qquad \text{(using } f\text{)}$$

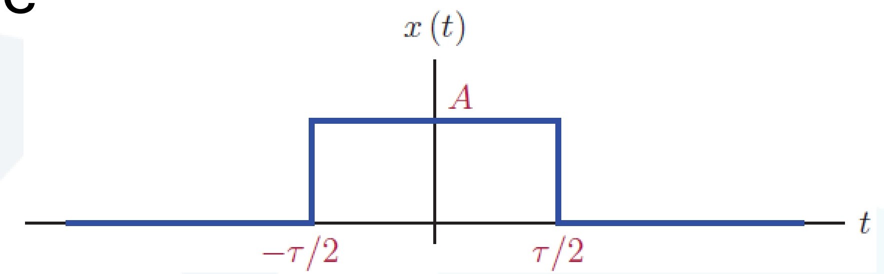
Existence of Fourier transform

- The FT integral may or may not converge for a given signal $x(t)$.
- The Dirichlet conditions (for determining if the Fourier transform converges) for the function x:
  1. The function x is absolutely integrable $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$
  2. On any finite interval x has a finite number of maxima and minima (i.e. x is of bounded variation); and
  3. On any finite interval, x has a finite number of discontinuities, and each discontinuity is itself finite.
- All energy signals have Fourier transforms.

Fourier transforms of some signals

Example 6: Fourier transform of a rectangular pulse

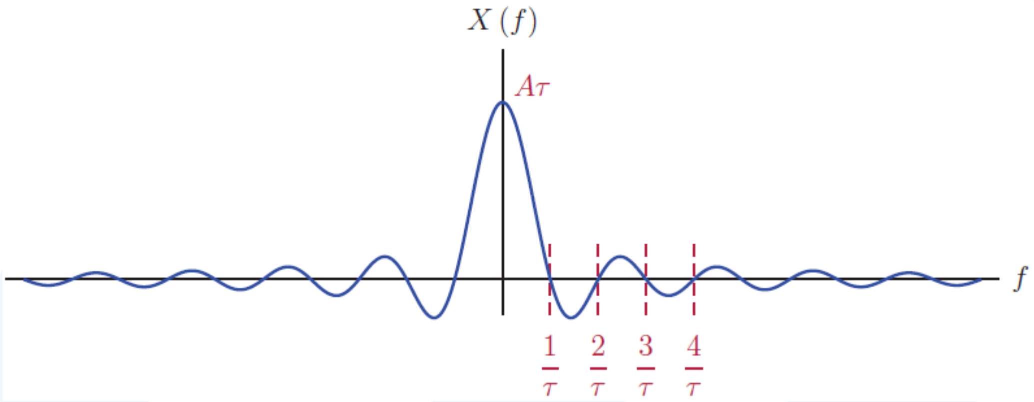
$$x(t) = A\Pi\left(\frac{t}{\tau}\right)$$



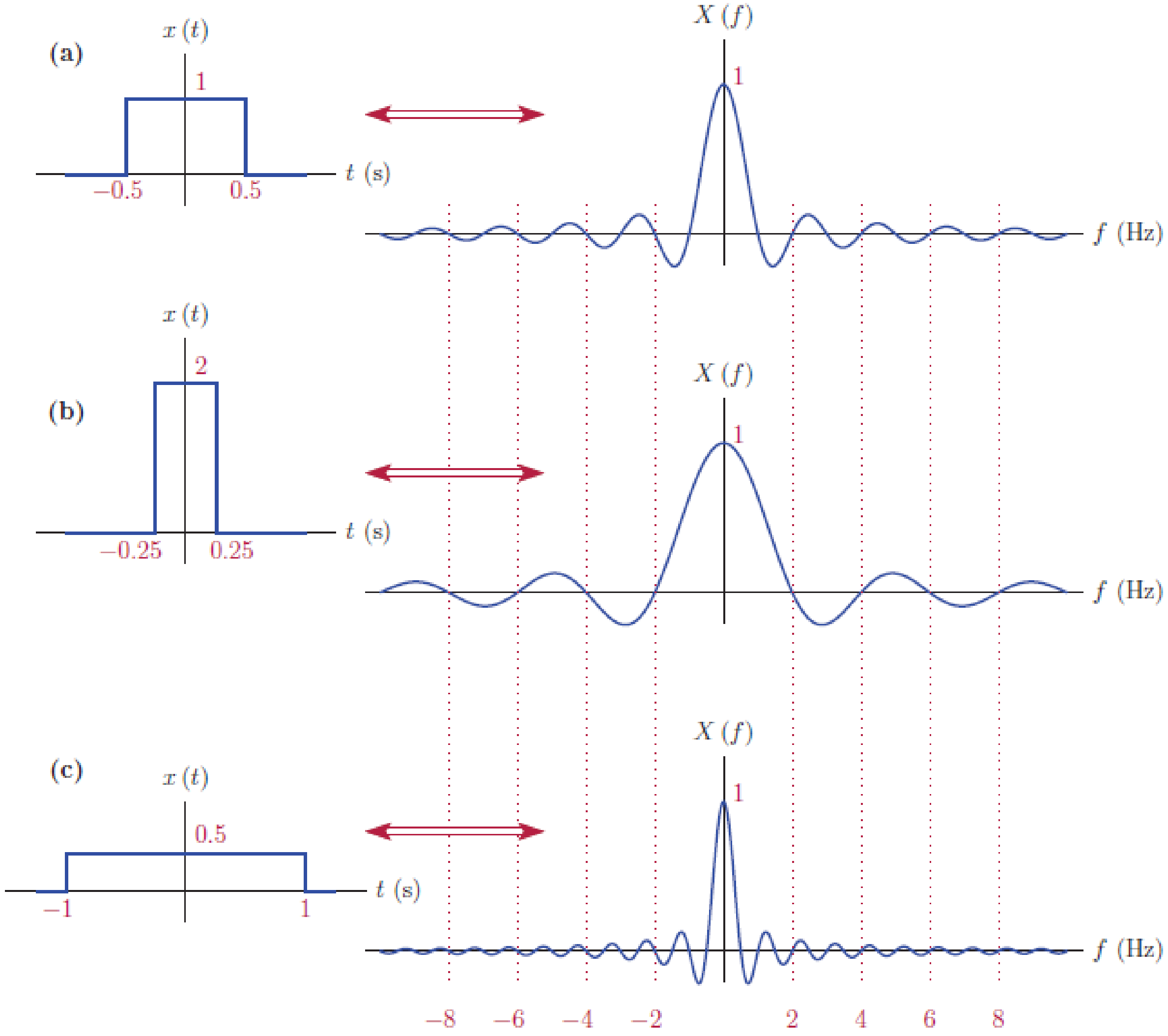
$$X(\omega) = \int_{-\tau/2}^{\tau/2} (A)\, e^{-j\omega t} dt = A \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\tau/2}^{\tau/2} = \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau \,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau \,\mathrm{sinc}(f\tau)$$



- Effects of changing the pulse width on the frequency spectrum:


(a)
$x(t)$
1
−0.5
0.5
t (s)
$X(f)$
1
f (Hz)
(b)
$x(t)$
2
−0.25
0.25
t (s)
$X(f)$
1
f (Hz)
(c)
$x(t)$
0.5
−1
1
t (s)
$X(f)$
1
f (Hz)
−8
−6
−4
−2
2
4
6
8

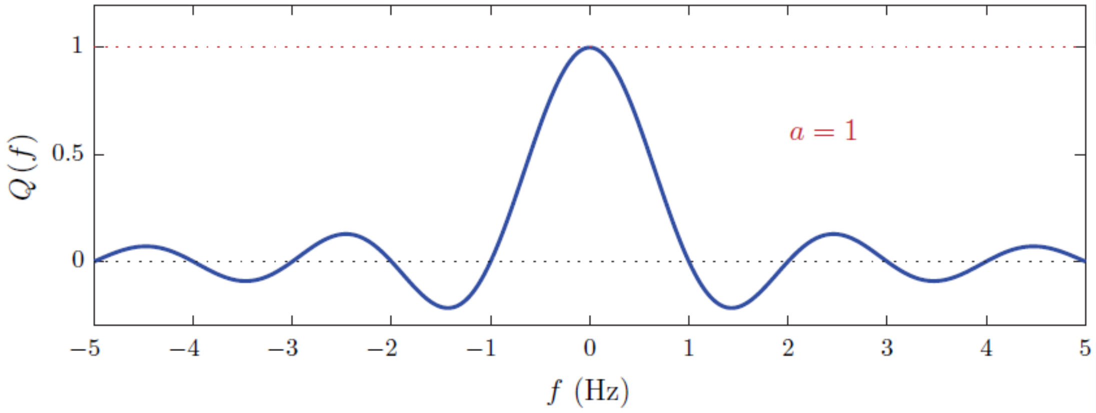
Example 7: Transform of the unit-impulse function

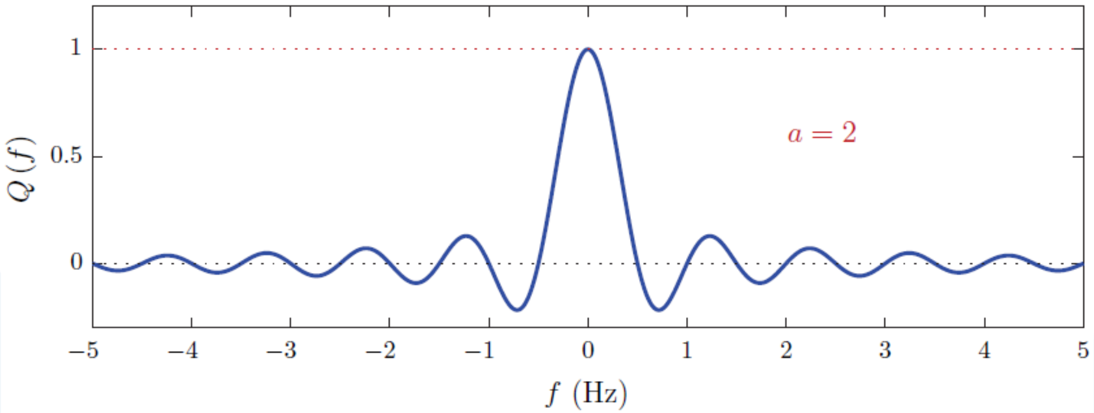
$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

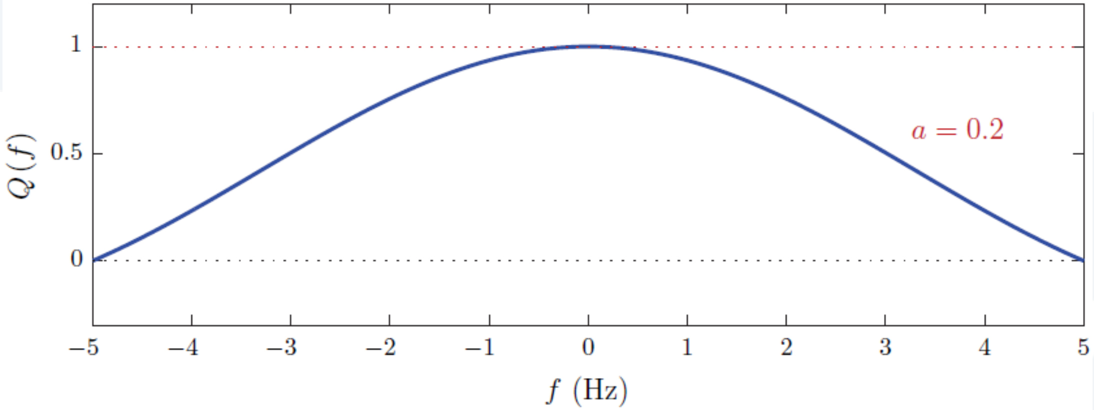
$$q(t) = \frac{1}{a}\Pi\left(\frac{t}{a}\right) \Rightarrow \delta(t) = \lim_{a\to 0} q(t)$$

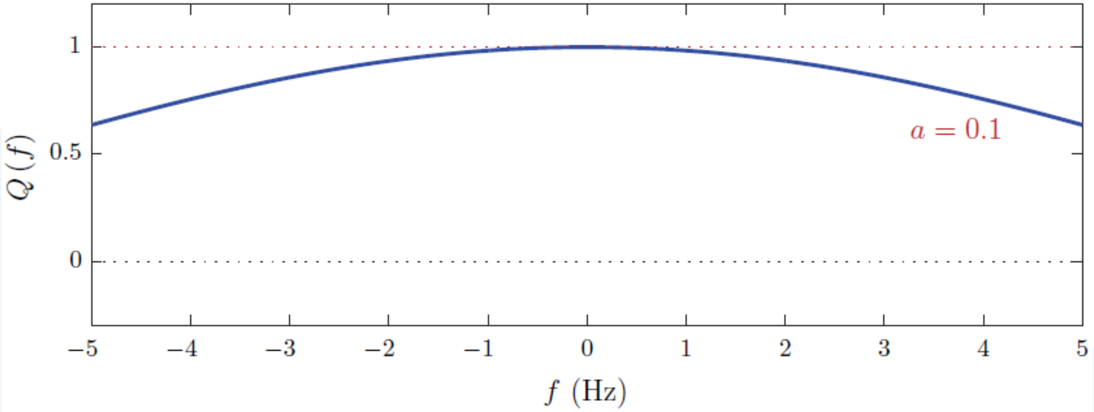
$$Q(f) = \mathcal{F}\{q(t)\} = \mathrm{sinc}\,(fa)$$

$$\mathcal{F}\{\delta(t)\} = \lim_{a\to 0}\{Q(f)\} = \lim_{a\to 0}\{\mathrm{sinc}(fa)\} = 1$$



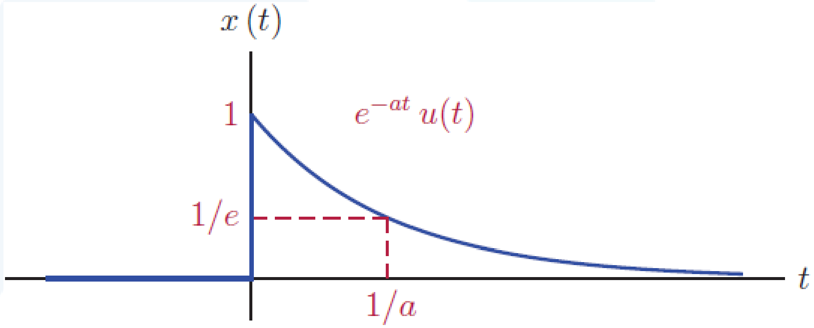






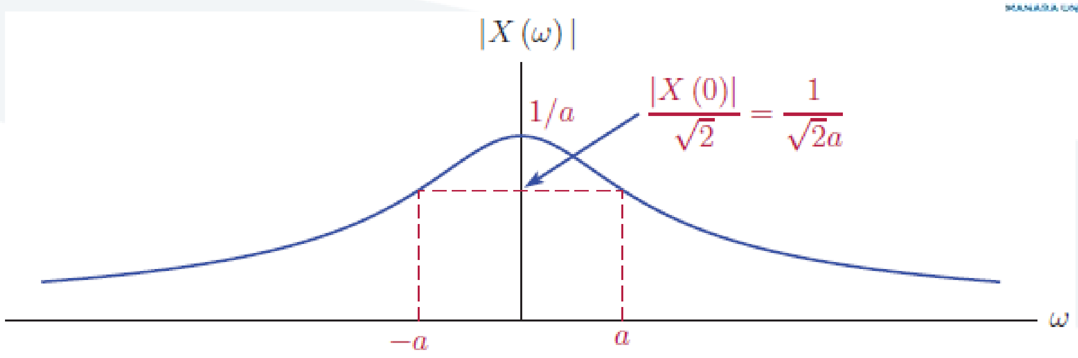
Example 8: Fourier transform of a right-sided exponential signal

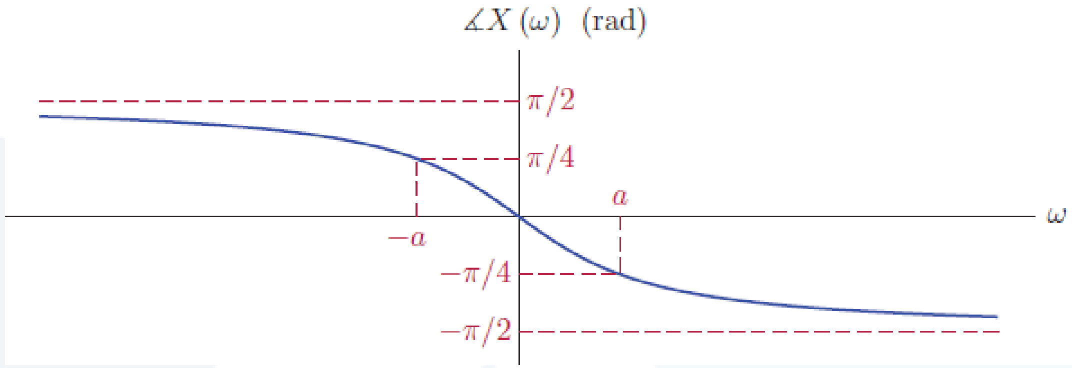
$$x(t) = e^{-at}u(t),\ a > 0$$



$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt = \frac{1}{a + j\omega}$$

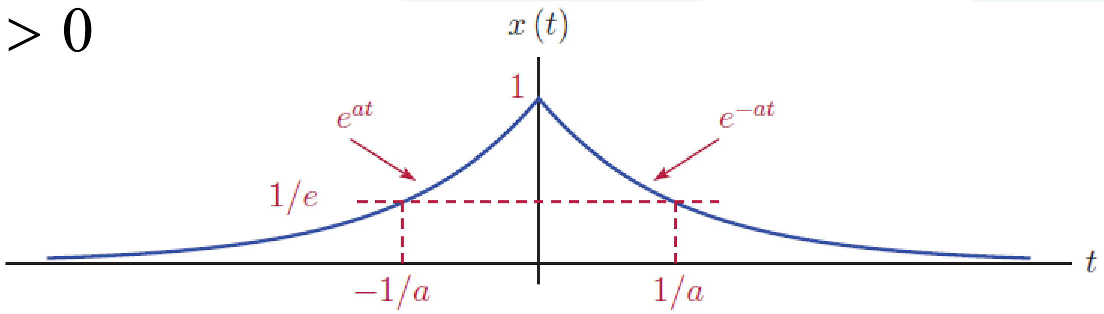
$$|X(\omega)| = \left|\frac{1}{a + j\omega}\right| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \theta(\omega) = -\tan^{-1}(\omega/a)$$

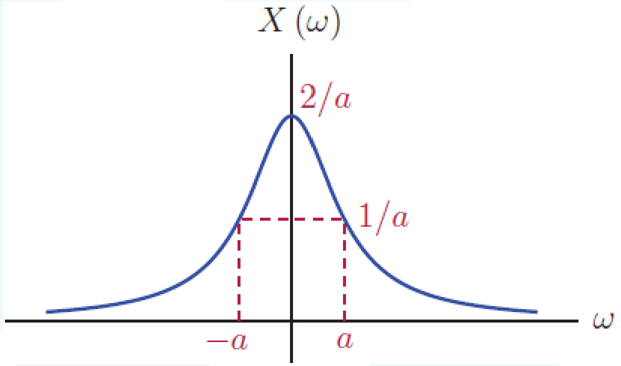




- Example 9: Fourier transform of a two-sided exponential signal

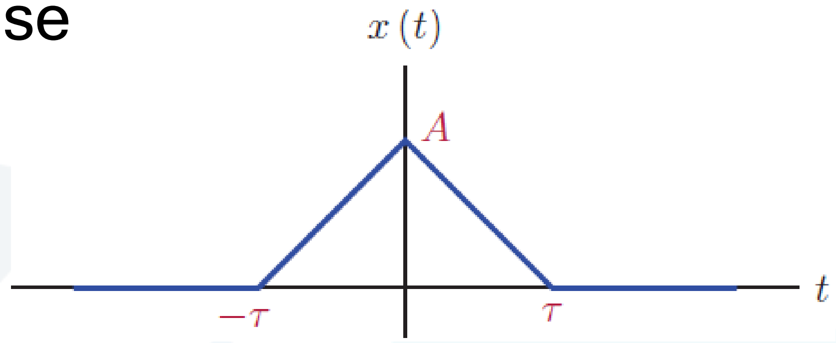
$$x(t) = e^{-a|t|},\ a > 0$$





$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt = \frac{2a}{a^2 + \omega^2}$$

- Example 10: Fourier transform of a triangular pulse



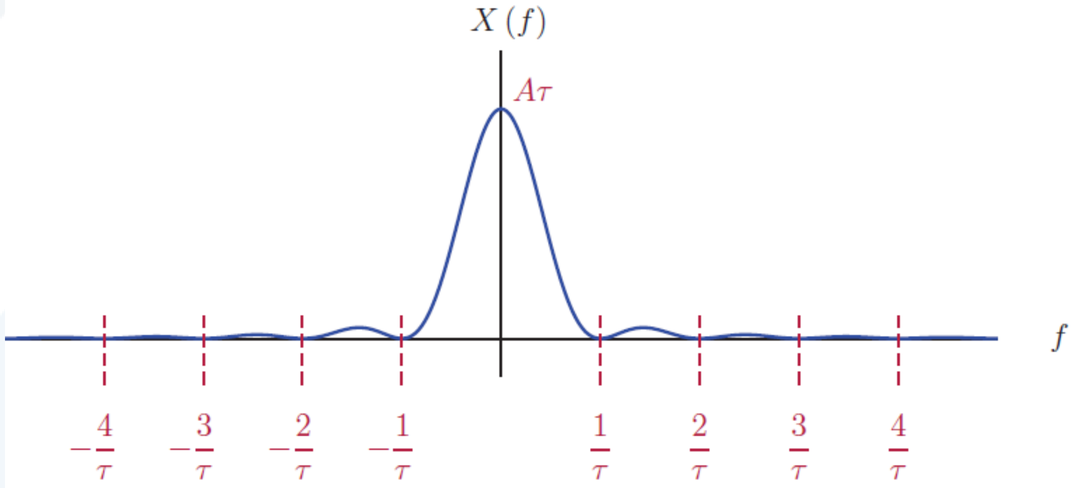
$$x(t) = A\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} A + At/\tau, & -\tau < t < 0 \\ A - At/\tau, & 0 < t < \tau \\ 0, & |t| \geq \tau \end{cases}$$

$$X(\omega) = \int_{-\tau}^{0} (A + At/\tau)\, e^{-j\omega t} dt + \int_{0}^{\tau} (A - At/\tau)\, e^{-j\omega t} dt = \frac{2A}{\omega^2 \tau}[1 - \cos(\omega\tau)]$$

$$\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \frac{\sin(\omega\tau/2)}{\omega\tau/2} = \frac{2}{\omega\tau}\sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau\, \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau\, \operatorname{sinc}^2(f\tau)$$



The Generalized Fourier Transform

- There are some important practical signals that do not have Fourier transforms in the strict sense.
- Because these signals are so important, the Fourier transform has been "generalized" to include them.

- Example 11: Fourier transform of constant-amplitude signal

$$x(t) = A, \text{ all } t$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} A e^{-j\omega t} dt = A \int_{-\infty}^{\infty} e^{-j\omega t} dt$$

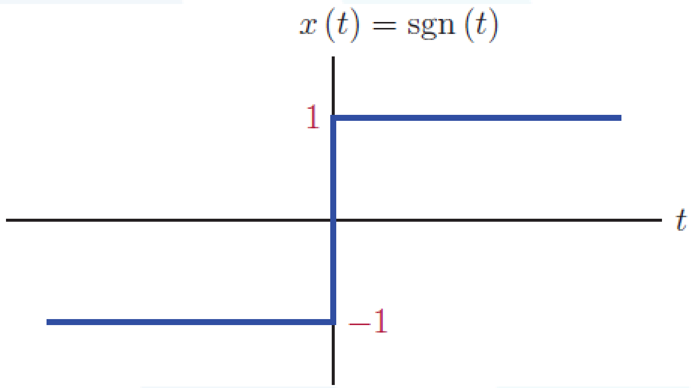
The integral does not converge. Therefore, the Fourier transform does not exist. Let us define an intermediate signal $x_a(t) = A e^{-a|t|}$, $a > 0$.

$$X_a(\omega) = A\frac{2a}{a^2+\omega^2} \qquad \text{For } \omega \neq 0,\ \lim_{a\to 0}\left[A\frac{2a}{a^2+\omega^2}\right] = 0$$

$$\int_{-\infty}^{\infty} A\frac{2a}{a^2+\omega^2}\,d\omega = 2\pi A$$

The area under the function is $2\pi A$ and is independent of the value of a. Therefore the Fourier transform of the constant A is a function that is zero for $\omega \neq 0$ and has an area of $2\pi A$. Therefor $\mathcal{F}(A) = 2\pi A\delta(\omega)$

- Example 12: Fourier transform of the signum function



$$x(t) = \text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases}$$

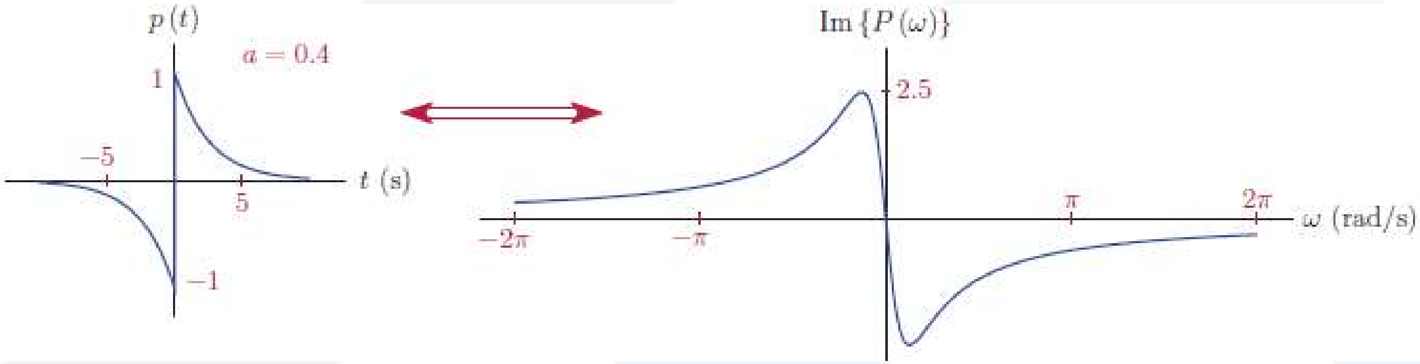
$$X(\omega) = \int_{-\infty}^{0} (-1)\,e^{-j\omega t}dt + \int_{0}^{\infty} (1)\,e^{-j\omega t}dt$$

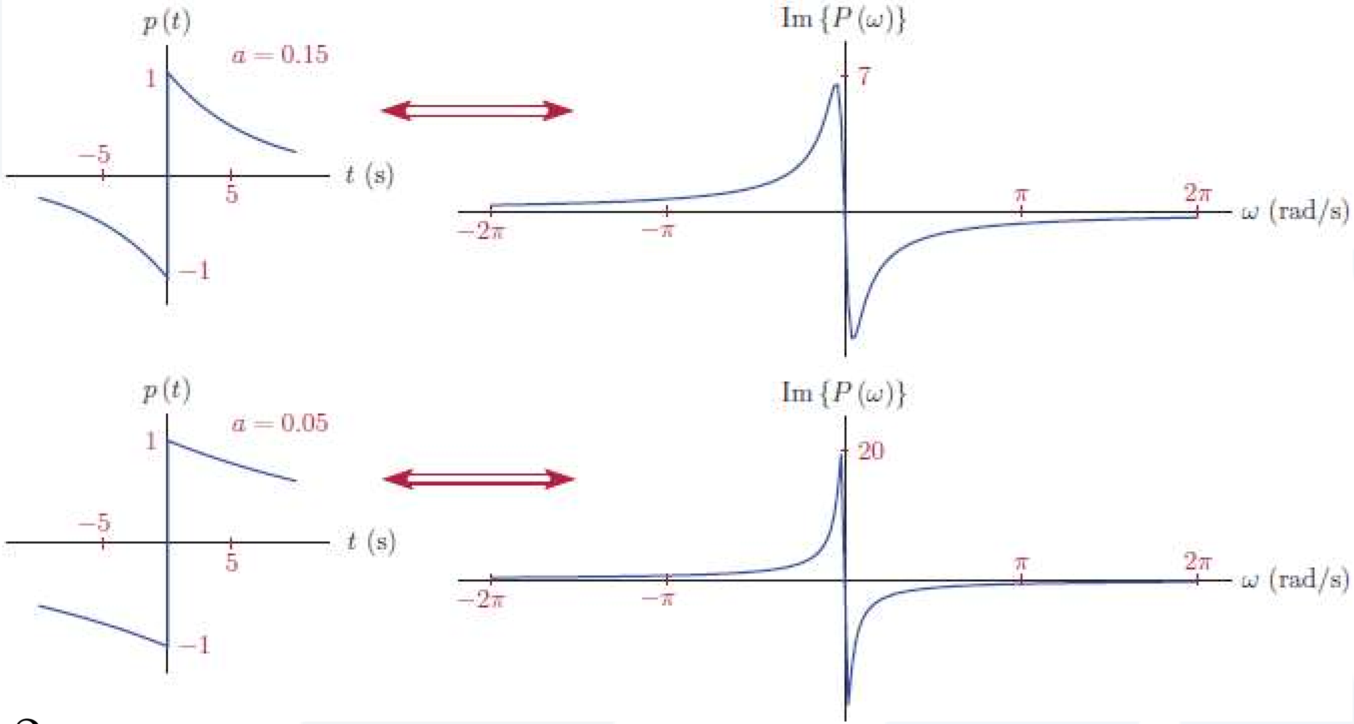
The two integrals cannot be evaluated. Instead, we will define an intermediate signal $p(t)$ as:

$$p(t) = \begin{cases} -e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}, \text{ where } a \geq 0$$

$$P(\omega) = \int_{-\infty}^{0} (-e^{at})\, e^{-j\omega t} dt + \int_{0}^{\infty} (e^{-at})\, e^{-j\omega t} dt = \frac{-j2\omega}{a^2 + \omega^2}$$

$$X(\omega) = \mathcal{F}\{\text{sgn}(t)\} = \lim_{a \to 0} \frac{-j2\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$





$$|X(\omega)| = \frac{2}{|\omega|}$$

$$\theta(\omega) = \begin{cases} \frac{\pi}{2}, & \omega < 0 \\ -\frac{\pi}{2}, & \omega > 0 \end{cases}$$