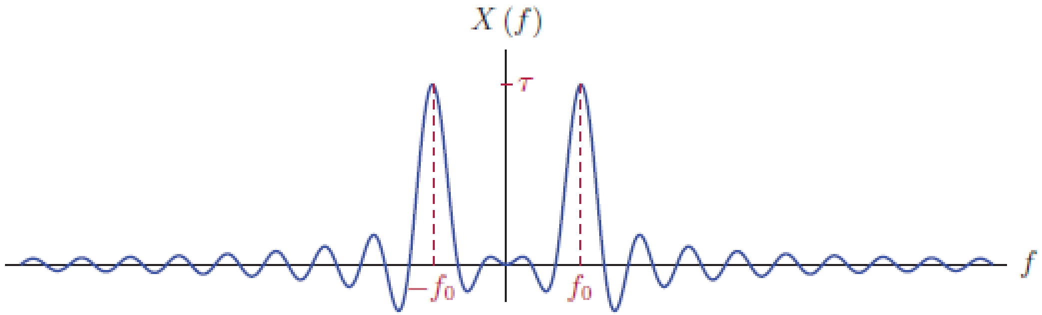
CEDC403: Signals and Systems

Lecture Notes 6: Fourier Analysis for Continuous Time Signals and Systems: Part B



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Chapter 4

Fourier Analysis for Continuous Time Signals and Systems

Properties of Fourier transform

Linearity of the Fourier transform: $\mathcal{F}\{\alpha_1 x(t) + \alpha_2 y(t)\} = \alpha_1 \mathcal{F}\{x(t)\} + \alpha_2 \mathcal{F}\{y(t)\}$

Duality property: $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$

Duality property (using f): $x(t) \xleftrightarrow{\mathcal{F}} X(f) \quad \Rightarrow \quad X(t) \xleftrightarrow{\mathcal{F}} x(-f)$

What is the relationship between $X(t)$ and $x(\omega)$?

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega, \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$\omega \to \tau \Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, e^{j\tau t} d\tau \qquad t \to -\omega \Rightarrow x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, e^{-j\tau\omega} d\tau$$

$$\tau \to t \Rightarrow x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{-jt\omega} dt$$

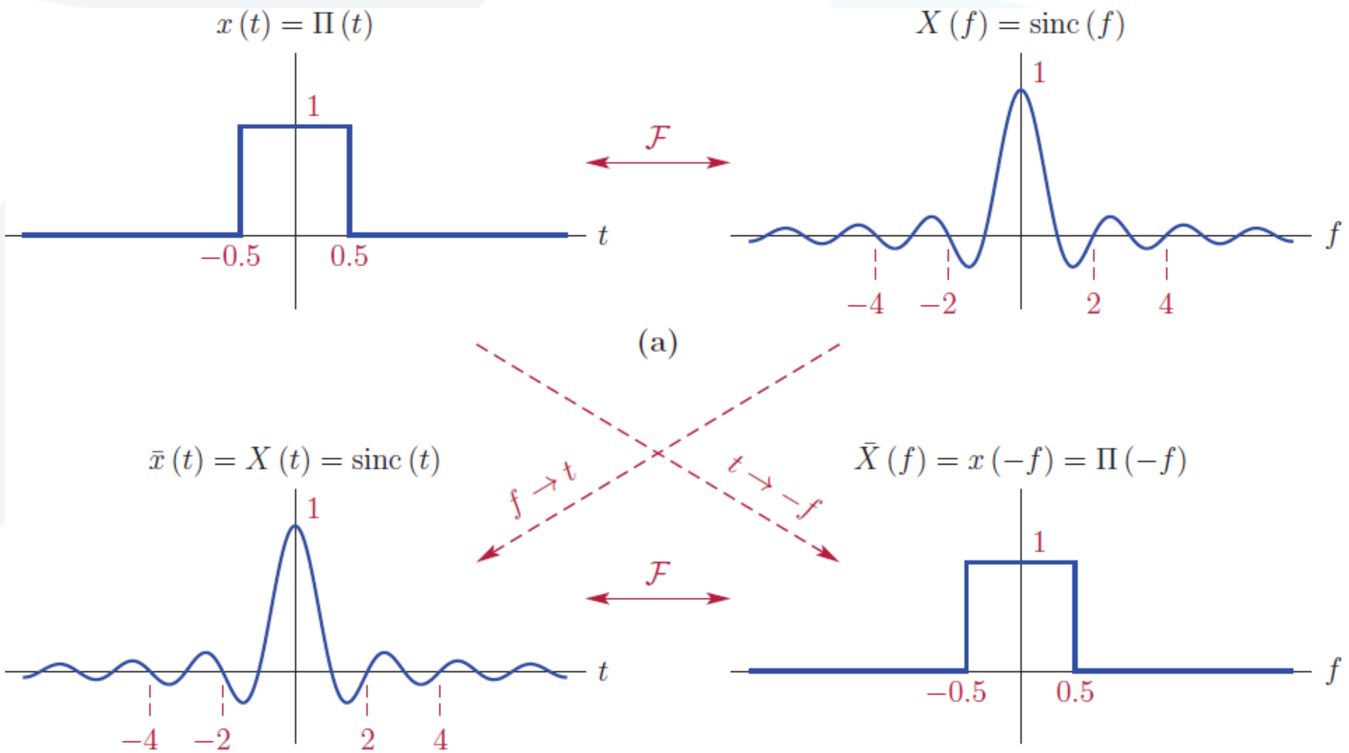
$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-jt\omega} dt \Rightarrow X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

- Example 13: Fourier transform of the sinc function

$$\mathcal{F}\left\{\frac{1}{2\pi}\Pi\left(\frac{t}{2\pi}\right)\right\} = \text{sinc}(\omega) \Rightarrow$$

$$\mathcal{F}\{\text{sinc}(t)\} = \Pi\left(\frac{-\omega}{2\pi}\right) = \Pi\left(\frac{\omega}{2\pi}\right)$$

$$\mathcal{F}\{\text{sinc}(t)\} = \Pi(f)$$



- Example 14: Transform of a constant-amplitude signal

$$x(t) = 1, \text{ all } t$$

$$\mathcal{F}\{\delta(t)\} = 1 \text{ , all } \omega \quad \Rightarrow \quad \mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega), \quad \mathcal{F}\{1\} = \delta(f) \quad \text{(duality)}$$

- Note 1: The signal $x(t) = 1$ does not satisfy the existence conditions; it is neither absolute integrable nor square integrable. Its FT does not converge. We obtain a function $X(\omega)$ that has the characteristics of a Fourier transform, and that can be used in solving problems in the frequency domain.
- Note 2: the conversion from ω to f using: $X(\omega) = X(f)\big|_{f=\omega/2\pi}$, $\quad X(f) = X(\omega)\big|_{\omega=2\pi f}$ is valid only when the transform does not contains a singularity function.

- Example 15: Fourier transform of the unit-step function

Determine the Fourier transform of the unit-step function $x(t) = u(t)$.

$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{0}^{\infty} e^{-j\omega t} dt$ could not be evaluated

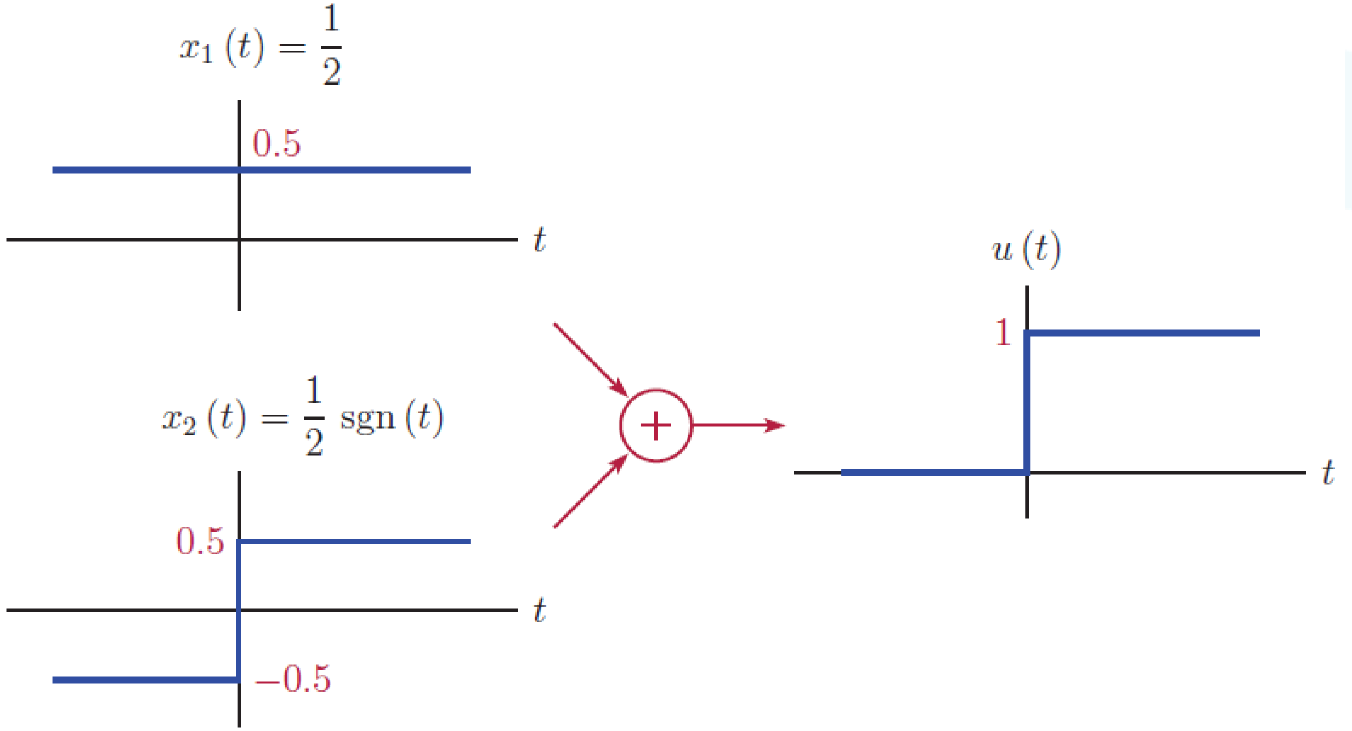
$$u(t) = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\{\tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{sgn}(t)\}$$

$$= \tfrac{1}{2}\mathcal{F}\{1\} + \tfrac{1}{2}\mathcal{F}\{\mathrm{sgn}(t)\}$$

$$\mathcal{F}\{u(t)\} = \pi\,\delta(\omega) + \frac{1}{j\omega}$$

$$\mathcal{F}\{u(t)\} = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$



Symmetry of the Fourier transform

$$x(t)\text{: real, } \mathrm{Im}\{x(t)\} = 0 \Rightarrow X^*(\omega) = X(-\omega)$$

$$x(t)\text{: imag, } \mathrm{Re}\{x(t)\} = 0 \Rightarrow X^*(\omega) = -X(-\omega)$$

Transforms of even and odd signals

- If the real-valued signal $x(t)$ is an even function of time, the resulting Fourier transform $X(\omega)$ is real-valued for all ω.

$$x(-t) = x(t), \text{ for all } t \Rightarrow \mathrm{Im}\{X(\omega)\} = 0, \text{ for all } \omega$$

- If the real-valued signal $x(t)$ has odd-symmetry, the resulting Fourier transform $X(\omega)$ is purely imaginary.

$$x(-t) = -x(t), \text{ for all } t \Rightarrow \mathrm{Re}\{X(\omega)\} = 0, \text{ for all } \omega$$

Time shifting $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad x(t-\tau) \xleftrightarrow{\mathcal{F}} X(\omega)e^{-j\omega\tau}$

Frequency shifting $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad x(t)e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$

Modulation property $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow$

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

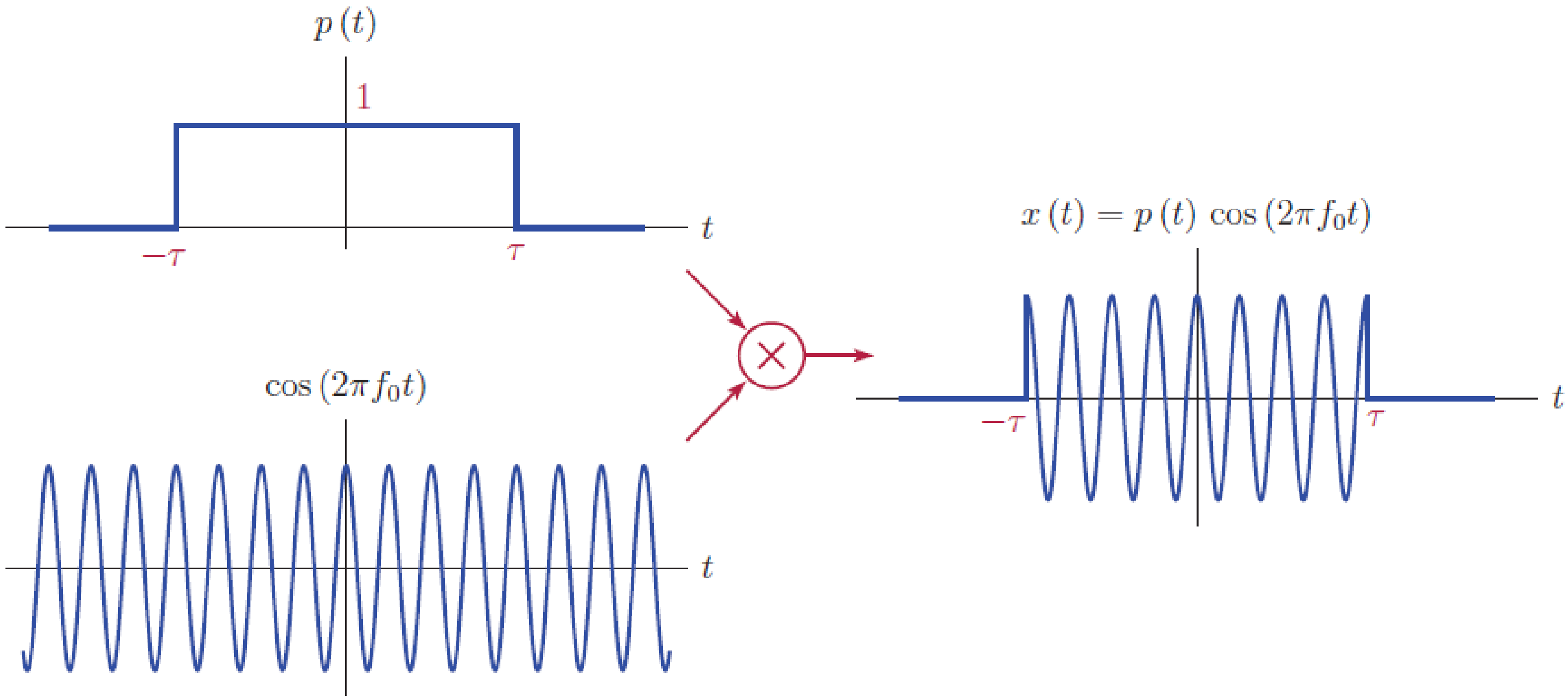
$$x(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\left[X(\omega-\omega_0)e^{-j\pi/2}+X(\omega+\omega_0)e^{j\pi/2}\right]$$

- Example 16: Modulated pulse

$$x(t)=\begin{cases}\cos(2\pi f_0 t), & |t|<\tau \\ 0, & |t|>\tau\end{cases}$$

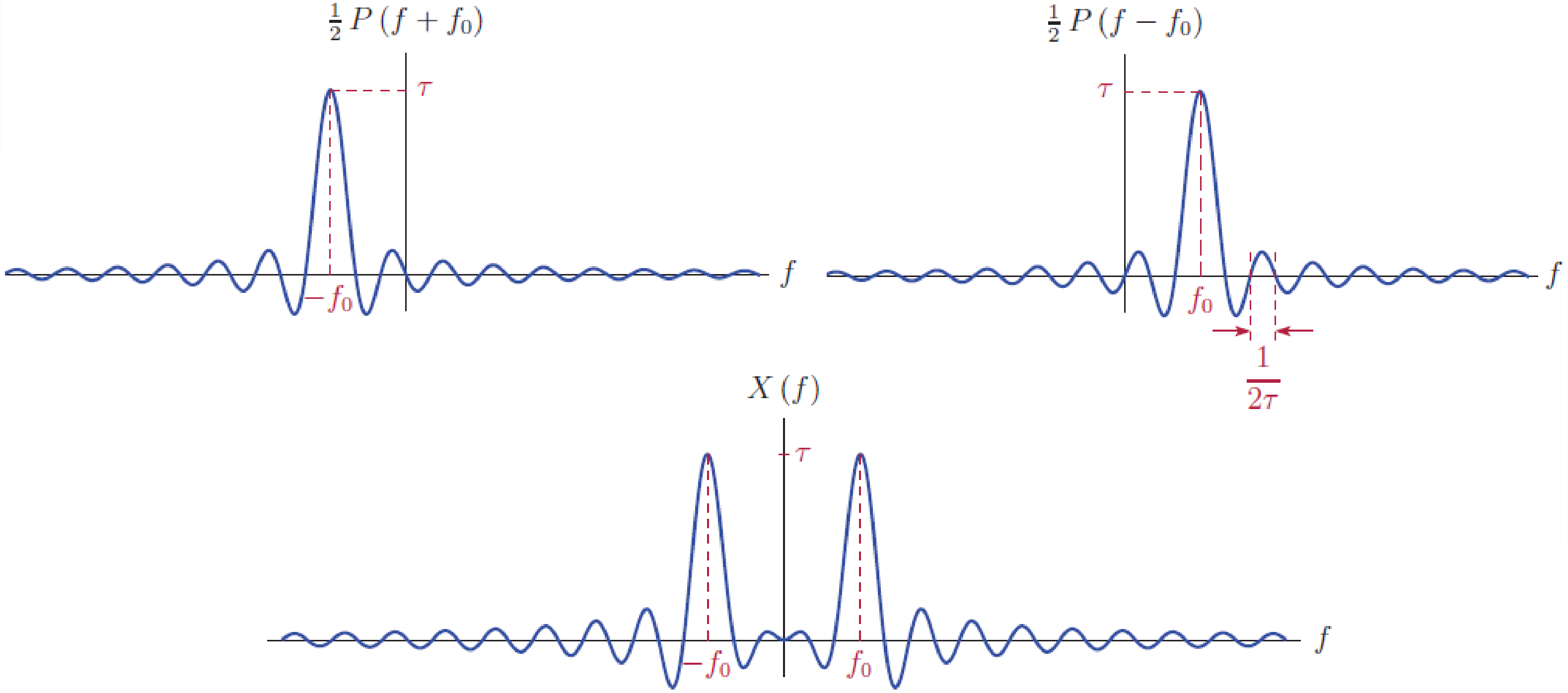
Using the rectangular pulse $p(t)$, the signal $x(t)$ can be expressed as:
$x(t)=p(t)\cos(2\pi f_0 t)$

where $p(t) = \Pi\left(\frac{t}{2\tau}\right)$



$$P(f) = 2\tau \operatorname{sinc}(2\tau f)$$

$$X(f) = \frac{1}{2}\left[P(f - f_0) + P(f + f_0)\right] = \tau \operatorname{sinc}\left(2\tau(f + f_0)\right) + \tau \operatorname{sinc}\left(2\tau(f - f_0)\right)$$



Time and frequency scaling

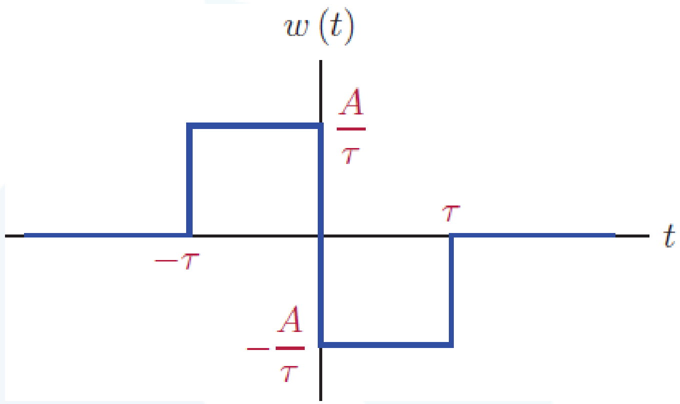
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{\omega}{a})$$

The parameter a is any non-zero and real-valued constant.

Differentiation in the time domain

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow \frac{d^n}{dt^n}[x(t)] \xleftrightarrow{\mathcal{F}} (j\omega)^n X(\omega), \quad \frac{d^n}{dt^n}[x(t)] \xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f)$$

- **Example 17:** Triangular pulse revisited



$$x(t) = A\Lambda(t/\tau)$$

$$w(t) = \frac{dx(t)}{dt} = \frac{A}{\tau}\left[\Pi\left(\frac{t+\tau/2}{\tau}\right) - \Pi\left(\frac{t-\tau/2}{\tau}\right)\right]$$

$$W(f) = A\,\mathrm{sinc}(f\tau)e^{j2\pi f(\tau/2)} - A\,\mathrm{sinc}(f\tau)e^{-j2\pi f(\tau/2)} = 2jA\,\mathrm{sinc}(f\tau)\sin(\pi f\tau)$$

$$W(f) = (j2\pi f)X(f) \Rightarrow X(f) = \frac{W(f)}{j2\pi f} = \frac{2jA\,\mathrm{sinc}(f\tau)\sin(\pi f\tau)}{j2\pi f} = A\tau\mathrm{sinc}^2(f\tau)$$

Differentiation in the frequency domain

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad (-jt)^n x(t) \xleftrightarrow{\mathcal{F}} \frac{d^n}{d\omega^n}[X(\omega)]$$

Convolution property

$$x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) \quad \text{and} \quad x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega)$$

$$\Rightarrow \quad x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(\omega)X_2(\omega)$$

Multiplication of two signals

$$x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) \quad \text{and} \quad x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega)$$

$$\Rightarrow x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega), \quad x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f) * X_2(f)$$

- Example 18: Transform of a truncated sinusoidal signal

$$x(t) = \begin{cases} \cos(2\pi f_0 t), & |t| < \tau \\ 0, & |t| > \tau \end{cases}$$

$$x(t) = x_1(t)x_2(t), \quad x_1(t) = \cos(2\pi f_0 t), \quad x_2(t) = \Pi\left(\frac{t}{2\tau}\right)$$

$$X_1(f) = \frac{1}{2}\delta(f + f_0) + \frac{1}{2}\delta(f - f_0), \quad X_2(f) = 2\tau\,\mathrm{sinc}(2\tau f)$$

$$X(f) = X_1(f) * X_2(f) = \tau\,\mathrm{sinc}\left(2\tau(f + f_0)\right) + \tau\,\mathrm{sinc}\left(2\tau(f - f_0)\right)$$

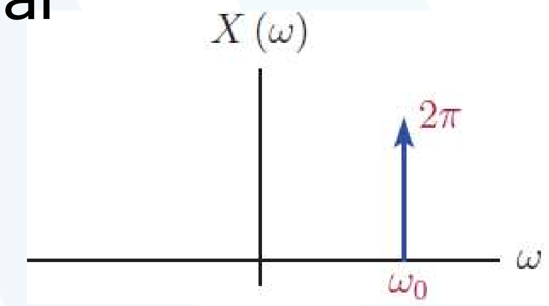
Integration $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Applying Fourier transform to periodic signals

- **Example 19:** Fourier transform of complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

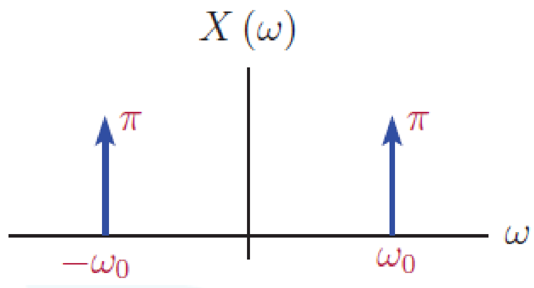
$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \quad \Rightarrow \quad \mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$$



- Example 20: Fourier transform of sinusoidal signal

$$x(t) = \cos(\omega_0 t)$$

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \Rightarrow \mathcal{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

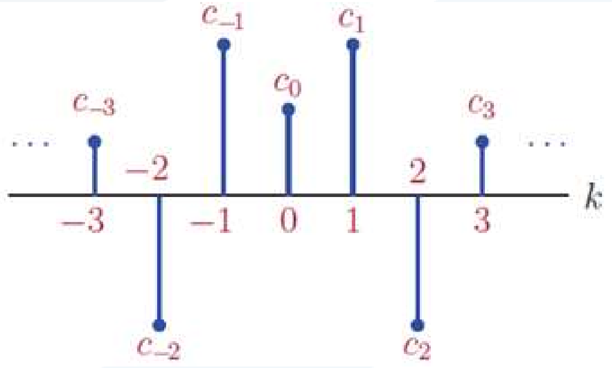


- In general for a periodic CT signal $\tilde{x}(t)$ that has an EFS representation:

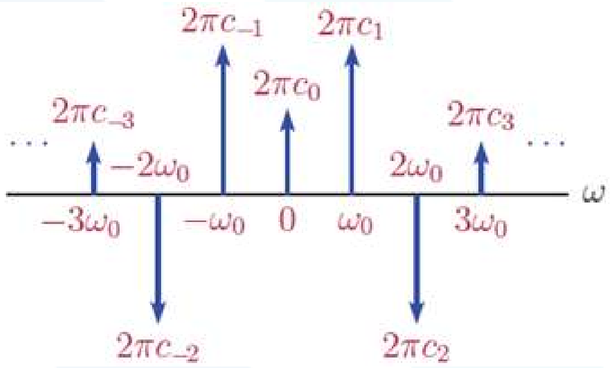
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow X(\omega) = \int_{-\infty}^{\infty} \tilde{x}(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k \left[\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \right]$$

$$= \sum_{k=-\infty}^{\infty} c_k \left[2\pi\delta(\omega - k\omega_0) \right]$$



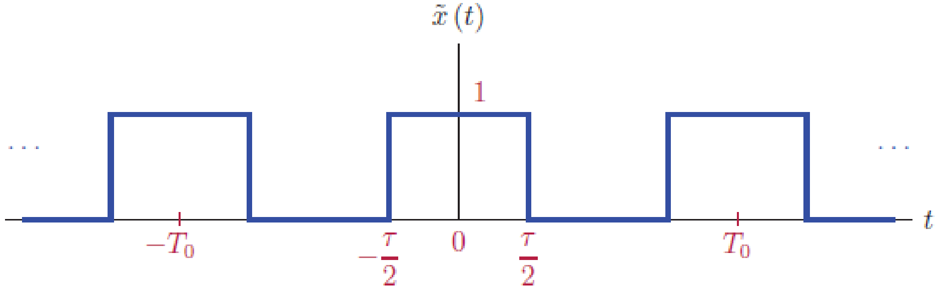
EFS coefficients for a signal



Fourier transform obtained

- Example 21: Fourier transform of periodic pulse train

  Determine the FT of the periodic pulse train with duty cycle $d = \tau / T_0$



$$c_k = d\,\mathrm{sinc}(kd)$$

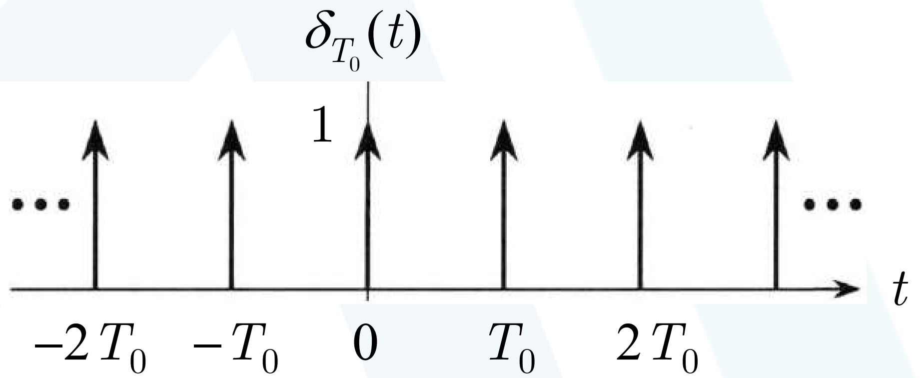
$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi d\,\mathrm{sinc}(kd)\delta(\omega - k\omega_0)$$

$\omega_0 = 1/T_0$ is the fundamental radian frequency.

- Example 22: Fourier transform of periodic impulse train

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

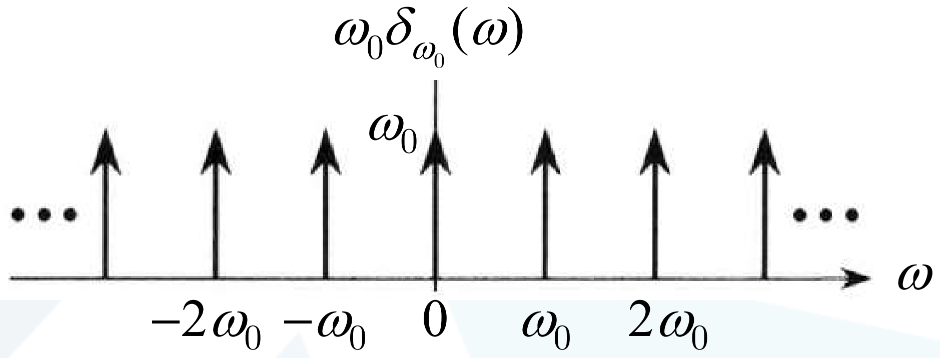
$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$



$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0}$$

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

$$\mathcal{F}\{\delta_{T_0}(t)\} = \frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \omega_0 \delta_{\omega_0}(\omega)$$



4. Energy and Power in the Frequency Domain

Parseval’s theorem

- For a periodic power signal $\tilde{x}(t)$ with period T_0 and EFS coefficients $\{c_k\}$:

$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

- For a non-periodic energy signal $x(t)$ with a Fourier transform $X(f)$:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Energy and power spectral density

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_0)$$ power spectral density of the signal $x(t)$

$$\int_{-\infty}^{\infty} S_x(f) \, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \, d\omega = \sum_{k=-\infty}^{\infty} |c_k|^2$$

$$P_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} S_x(f) \, df$$

$$G_x(f) = |X(f)|^2$$ energy spectral density of the signal $\tilde{x}(t)$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} G_x(f) \, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_x(\omega) \, d\omega$$

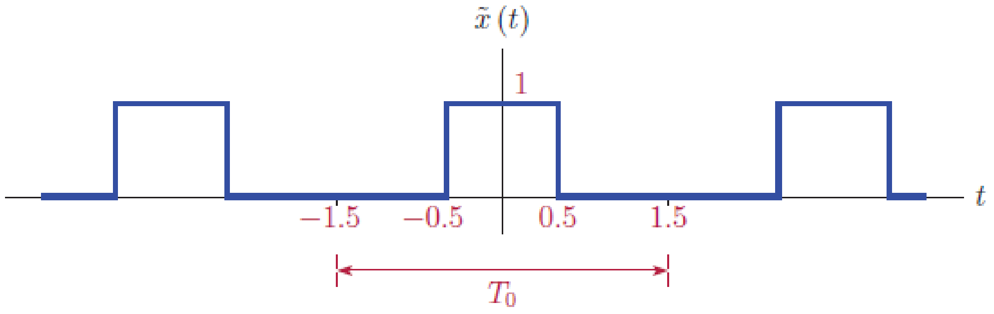
$$E_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} G_x(f) \, df$$

- Example 23: Power spectral density of a periodic pulse train

Determine the power spectral density for the periodic pulse train $\tilde{x}(t)$. Also find the total power, the DC power, the power in the first three harmonics, and the power above 1 Hz.

$$c_k = \tfrac{1}{3}\,\text{sinc}(k/3)$$

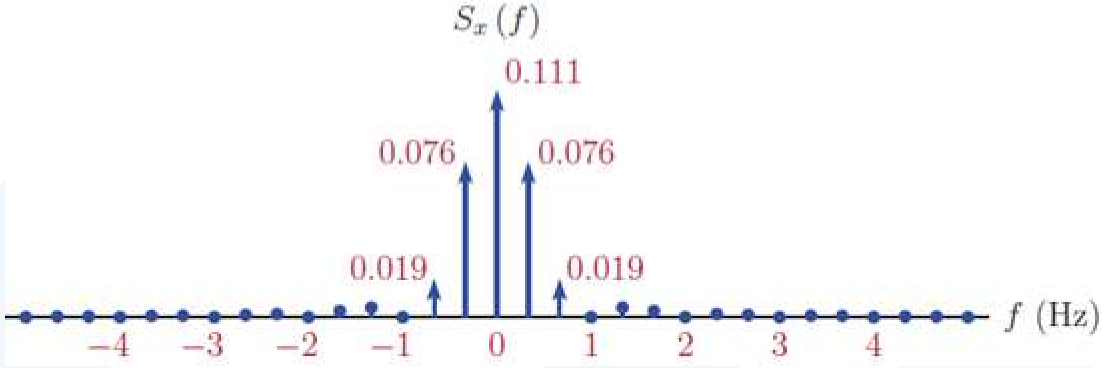
$$S_x(f) = \sum_{k=-\infty}^{\infty} \left|\frac{1}{3}\,\text{sinc}(k/3)\right|^2 \delta(f - k/3)$$



The total power in the signal $x(t)$:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2\,dt = \frac{1}{3}\int_{-0.5}^{0.5} (1)^2\,dt = \frac{1}{3}$$

$$P_{dc} = |c_0|^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} \approx 0.1111$$



$$P_1 = |c_{-1}|^2 + |c_1|^2 = \frac{3}{2\pi^2} \approx 0.1520, \quad P_2 = |c_{-2}|^2 + |c_2|^2 = \frac{3}{8\pi^2} \approx 0.0380, \quad P_3 = 0$$

The third harmonic is at frequency $f = 1$ Hz. Thus, the power above 1 Hz:

$$P_{hf} = P_x - P_{dc} - P_1 - P_2 - P_3 = 0.3333 - 0.1111 - 0.1520 - 0.0380 - 0 = 0.0322$$

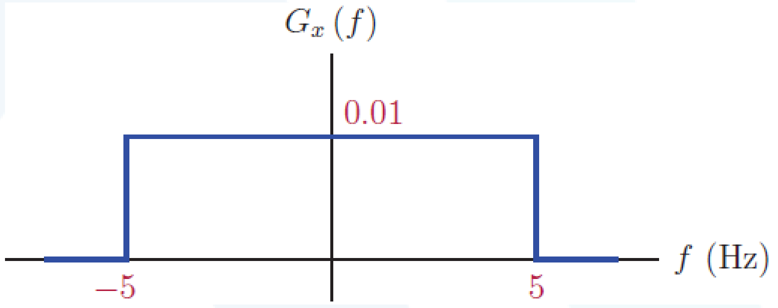
- Example 24: Energy spectral density of the sinc function

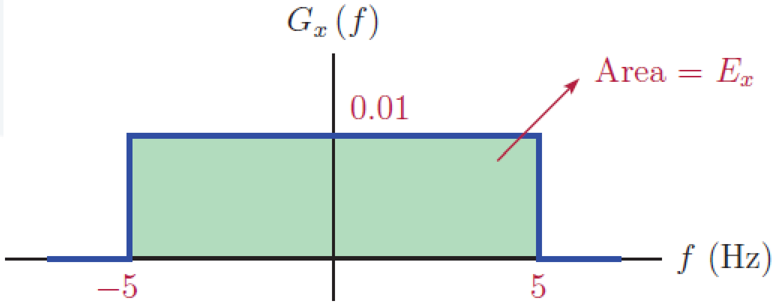
  Determine the energy spectral density of $x(t) = \mathrm{sinc}(10t)$. Afterwards, compute the total energy, and the energy in the sinc pulse at frequencies up to 3 Hz.

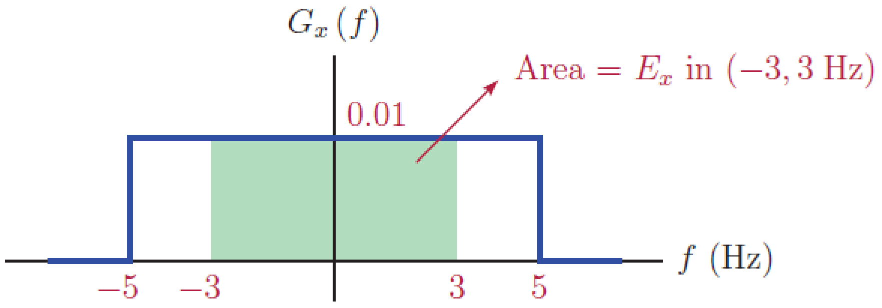
$$X(f) = \frac{1}{10}\Pi\left(\frac{f}{10}\right), \quad G_x(f) = |X(f)|^2 = \frac{1}{100}\Pi\left(\frac{f}{10}\right)$$

$$E_x = \int_{-\infty}^{\infty} G_x(f)\,df = \int_{-5}^{5} \frac{1}{100}\,df = 0.1$$

$$E_x \text{ in } (-3, 3 \text{ Hz}) = \int_{-3}^{3} G_x(f)\,df = \int_{-3}^{3} \frac{1}{100}\,df = 0.06$$







Autocorrelation

- For an energy signal $x(t)$ the autocorrelation function is defined as:

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

- For a periodic power signal $\tilde{x}(t)$ with period T_0, the corresponding definition of the autocorrelation function is:

$$\tilde{r}_{xx}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\tilde{x}(t+\tau)dt$$

- The energy spectral density is the FT of the autocorrelation function:

$$\mathcal{F}\{r_{xx}(\tau)\} = G_x(f)$$

- The power spectral density is the FT of the autocorrelation function:

$$\mathcal{F}\{\tilde{r}_{xx}(\tau)\} = S_x(f)$$

- Example 25: Power spectral density of a sinusoidal signal revisited

$$\tilde{x}(t) = 5\cos(200\pi t)$$

$$\tilde{r}_{xx}(\tau) = \frac{1}{0.01}\int_{-0.005}^{0.005} 25\cos(200\pi t)\cos(200[t+\tau])dt = \frac{25}{2}\cos(200\pi\tau)$$

$$S_x(f) = F\{\tilde{r}_{xx}(\tau)\} = \frac{25}{4}\delta(f+100) + \frac{25}{4}\delta(f-100)$$

Properties of the autocorrelation function

- $r_{xx}(0) \geq |r_{xx}(\tau)|$ for all τ

- $r_{xx}(-\tau) = r_{xx}(\tau)$ for all τ, that is, the autocorrelation function has even symmetry.
- If the signal $x(t)$ is periodic with period T, then its autocorrelation function $\tilde{r}_{xx}(\tau)$ is also periodic with the same period.

5. Transfer Function Concept

- In time-domain analysis of systems we have relied on two distinct description forms for CTLTI systems:
  1. A linear constant-coefficient differential equation that describes the relationship between the input and the output signals.
  2. An impulse response which can be used with the convolution operation for determining the response of the system to an arbitrary input signal.

- The concept of Transfer function will be introduced as the third method for describing the characteristics of a system.

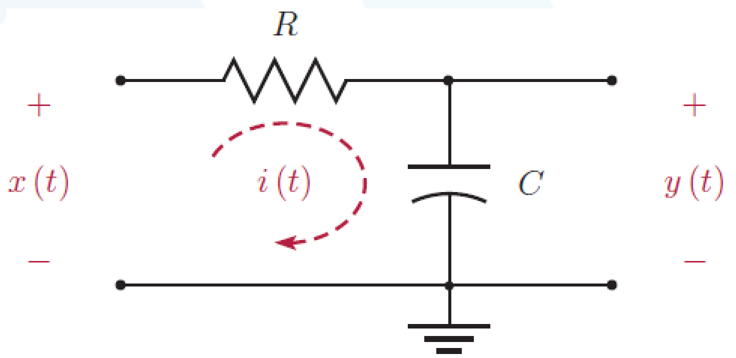
$$H(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

- Note: The transfer function concept is valid for LTI systems only.
- In general, $H(\omega)$ is a complex function of ω, $H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$.
- Example 26: Transfer function for the simple RC circuit

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t)$$

$$H(\omega) = \int_{0}^{\infty} \frac{1}{RC}e^{-t/RC}e^{-j\omega t}dt = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_c)},$$

$$\omega_c = \frac{1}{RC}$$



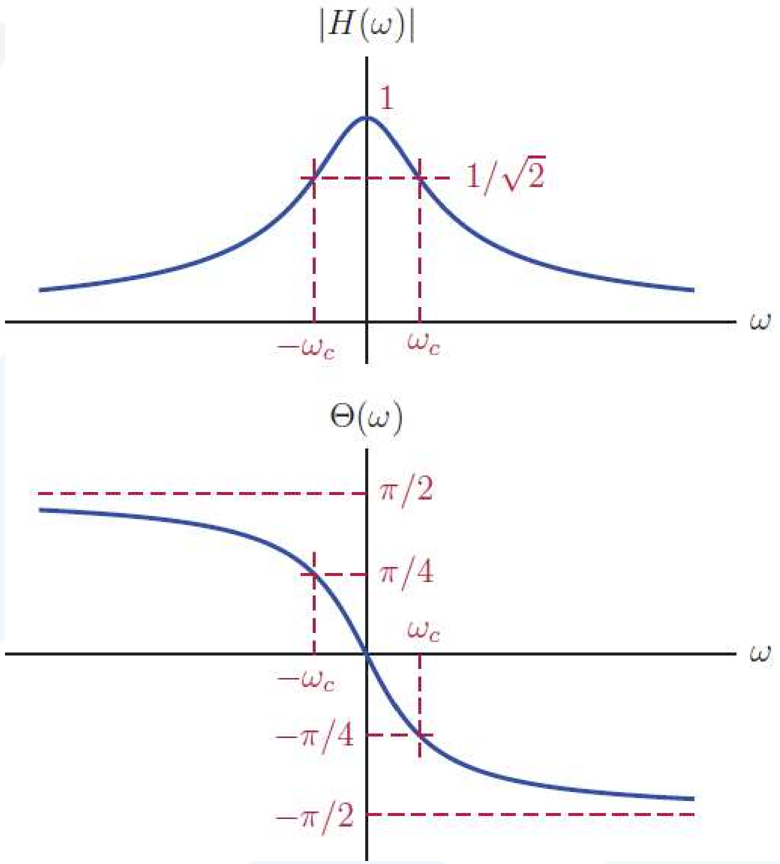
$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}, \qquad \Theta(\omega) = -\tan^{-1}(\omega/\omega_c)$$

$$H(\omega_c) = \frac{1}{1 + j}, \qquad |H(\omega_c)| = \frac{1}{\sqrt{2}}$$

- ω_c represents the frequency at which the magnitude of the transfer function is 3 decibels below its peak value at $\omega = 0$,

$$20\log_{10}\frac{|H(\omega_c)|}{|H(0)|} = 20\log_{10}\frac{1}{\sqrt{2}} \approx -3\,\text{dB}$$

- The frequency ω_c is often referred to as the 3 dB cutoff frequency of the system.



Obtaining the transfer function from the differential equation

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = H(\omega)X(\omega) \Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$\frac{d^k y(t)}{dt^k} \xleftrightarrow{\mathcal{F}} (j\omega)^k\, Y(\omega), \quad \frac{d^k x(t)}{dt^k} \xleftrightarrow{\mathcal{F}} (j\omega)^k\, X(\omega), \qquad k = 0, 1, \cdots$$

- Example 27: Transfer function from the differential equation

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 26y(t) = x(t)$$

$$(j\omega)^2\, Y(\omega) + 2(j\omega)\, Y(\omega) + 26Y(\omega) = X(\omega)$$

$$[(26 - \omega^2) + j2\omega]\, Y(\omega) = X(\omega) \Rightarrow H(\omega) = \frac{1}{(26 - \omega^2) + j2\omega}$$

6. CTLTI Systems with Periodic Input Signals

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Response of a CTLTI system to complex exponential signal

$$\tilde{x}(t) = e^{j\omega_0 t}$$

$$y(t) = h(t) * \tilde{x}(t) = \int_{-\infty}^{\infty} h(\tau)\,\tilde{x}(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,e^{j\omega_0 (t-\tau)}\,d\tau$$

$$= e^{j\omega_0 t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega_0 \tau}\,d\tau = e^{j\omega_0 t} H(\omega_0) = |H(\omega_0)|\,e^{j[\omega_0 t + \Theta(\omega_0)]}$$

- That is, $e^{j\omega t}$ is an eigenfunction of a LTI system and $H(\omega)$ is the corresponding eigenvalue. We refer to H as the frequency response of the system.

Response of a CTLTI system to sinusoidal signal

$$\tilde{x}(t) = \cos(\omega_0 t)$$

$$\tilde{x}(t) = \cos(\omega_0 t) = \tfrac{1}{2} e^{j\omega_0 t} + \tfrac{1}{2} e^{-j\omega_0 t}$$

$$y(t) = \tfrac{1}{2} e^{j\omega_0 t} H(\omega_0) + \tfrac{1}{2} e^{-j\omega_0 t} H(-\omega_0)$$

$$= \tfrac{1}{2} e^{j\omega_0 t} \left|H(\omega_0)\right| e^{j\Theta(\omega_0)} + \tfrac{1}{2} e^{-j\omega_0 t} \left|H(-\omega_0)\right| e^{-j\Theta(\omega_0)}$$

If the impulse response $h(t)$ is real-valued:

$$\left|H(-\omega_0)\right| = \left|H(\omega_0)\right|, \qquad \Theta(-\omega_0) = -\Theta(\omega_0)$$

$$y(t) = \frac{1}{2}\left|H(\omega_0)\right| e^{j[\omega_0 t + \Theta(\omega_0)]} + \frac{1}{2}\left|H(\omega_0)\right| e^{-j[\omega_0 t + \Theta(\omega_0)]} = \left|H(\omega_0)\right| \cos(\omega_0 t + \Theta(\omega_0))$$

- Example 28: Steady-state response of RC circuit for single-tone input

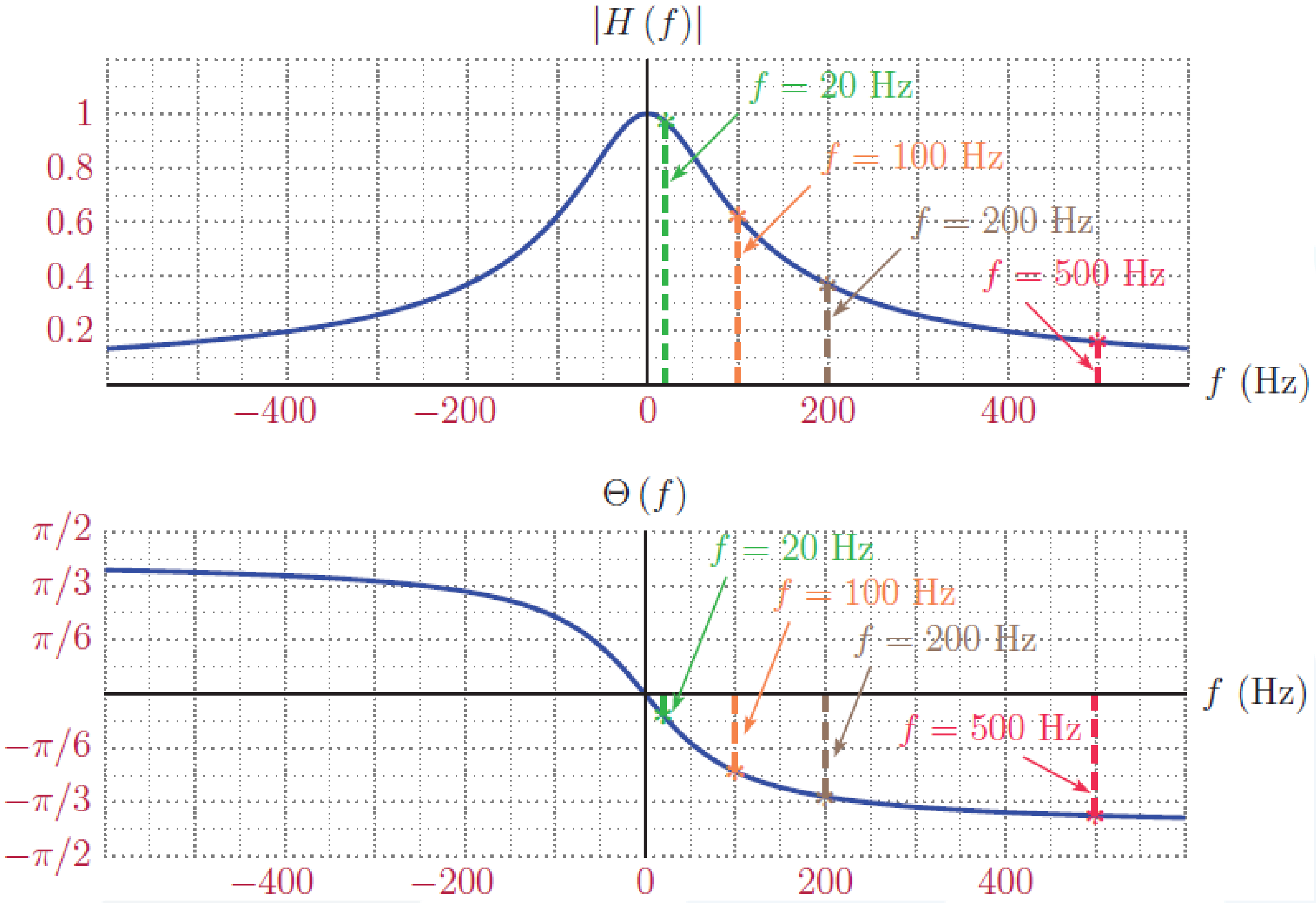
Let the component values be chosen to yield a 3 dB cutoff frequency of $\omega_c = 160\pi$ rad/s, or equivalently $f_c = 80$ Hz. Let the input signal be in the form $\tilde{x}(t) = 5\cos(2\pi f t)$.

Compute the steady-state output signal for the cases $f_1 = 20$ Hz, $f_2 = 100$ Hz, $f_3 = 200$ Hz, and $f_4 = 500$ Hz.

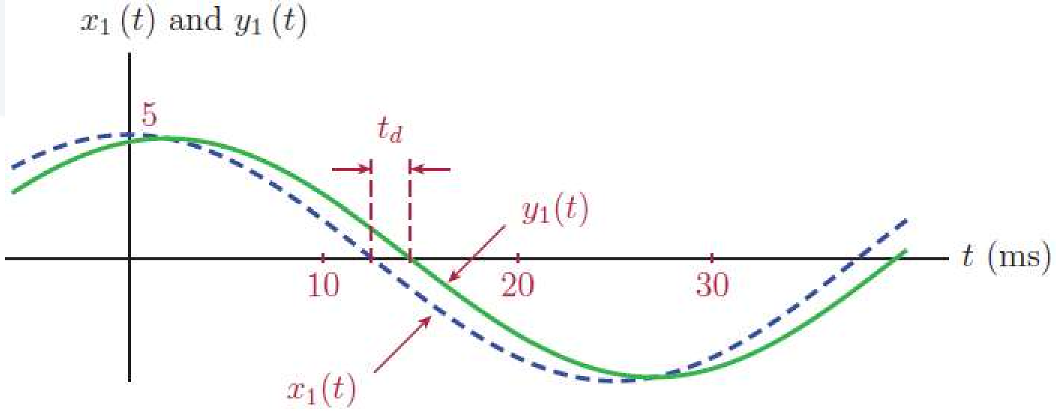
$$H(f) = \frac{1}{1 + j(f/80)} \Rightarrow |H(f)| = \frac{1}{\sqrt{1 + (f/80)^2}}, \qquad \Theta(f) = -\tan^{-1}(f/80)$$

$$|H(20)| = \frac{1}{\sqrt{1 + (20/80)^2}} = 0.9701, \qquad \Theta(20) = -\tan^{-1}(20/80) = -0.245 \text{ rad}$$

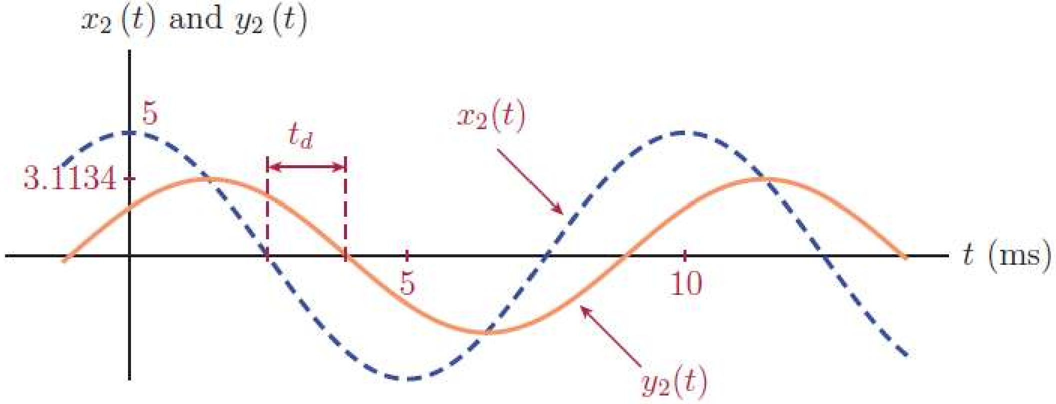
$$y_1(t) = 5(0.9701)\cos(40\pi t - 0.245) = 4.8507\cos(40\pi(t - 0.0019))$$

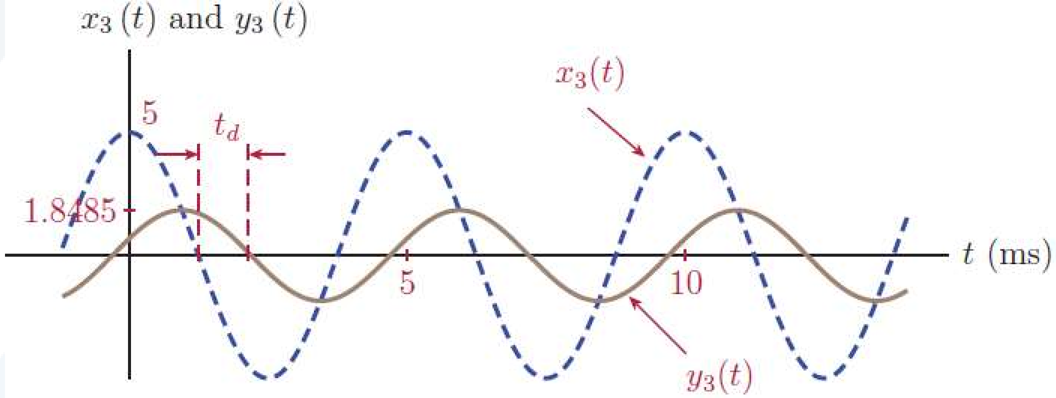


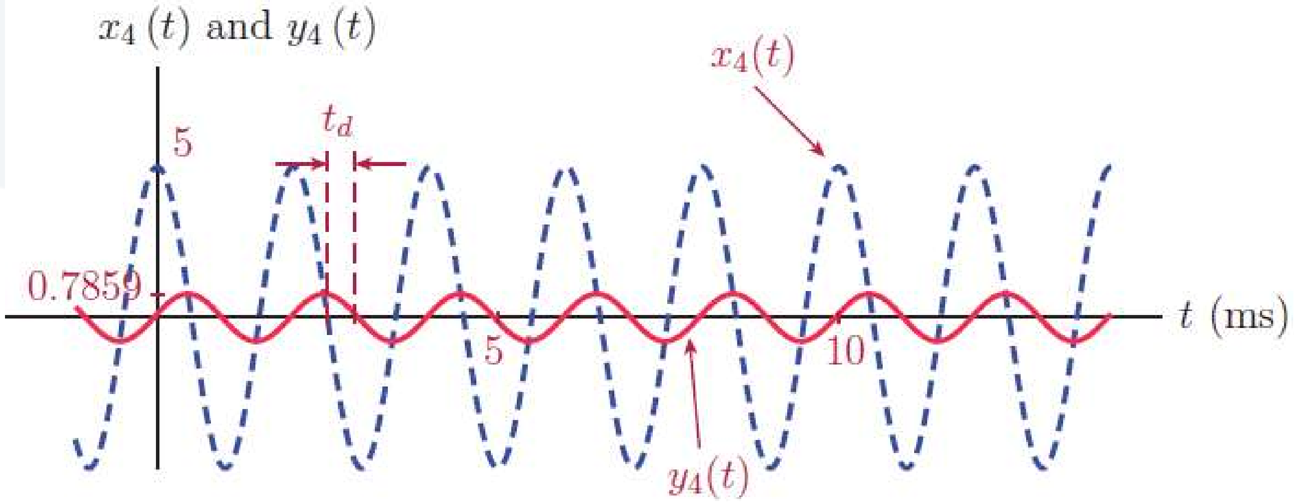
The phase shift of –0.245 radians translates to a time-delay of about $t_d = 1.9$ ms



f (Hz)	$\|H(f)\|$	$\Theta(f)$ (rad)	t_d (ms)
20	0.9701	−0.2450	1.95
100	0.6247	−0.8961	1.43
200	0.3714	−1.1903	0.94
500	0.1580	−1.4121	0.45







Response of a CTLTI system to periodic input signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

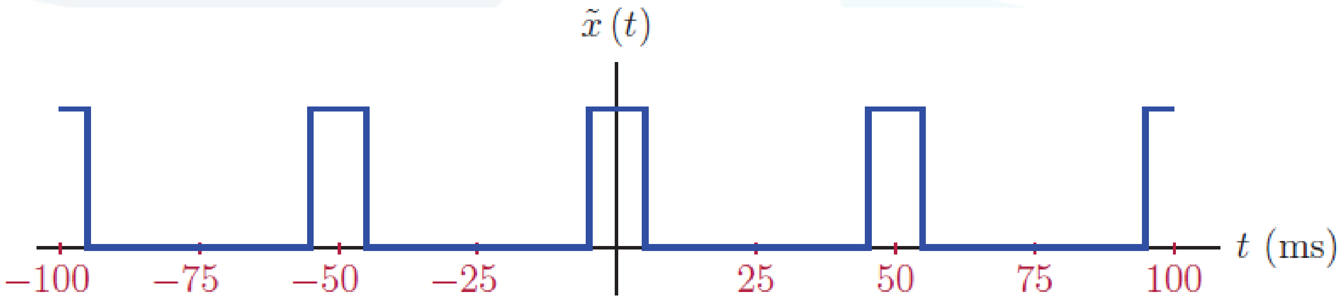
$$T\{\tilde{x}(t)\} = T\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} T\left\{c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} c_k T\left\{e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$$

- Example 29: RC circuit with pulse-train input

Let the input signal be a pulse train with period $T_0 = 50$ ms and duty cycle $d = 0.2$. Determine the output signal in steady state.

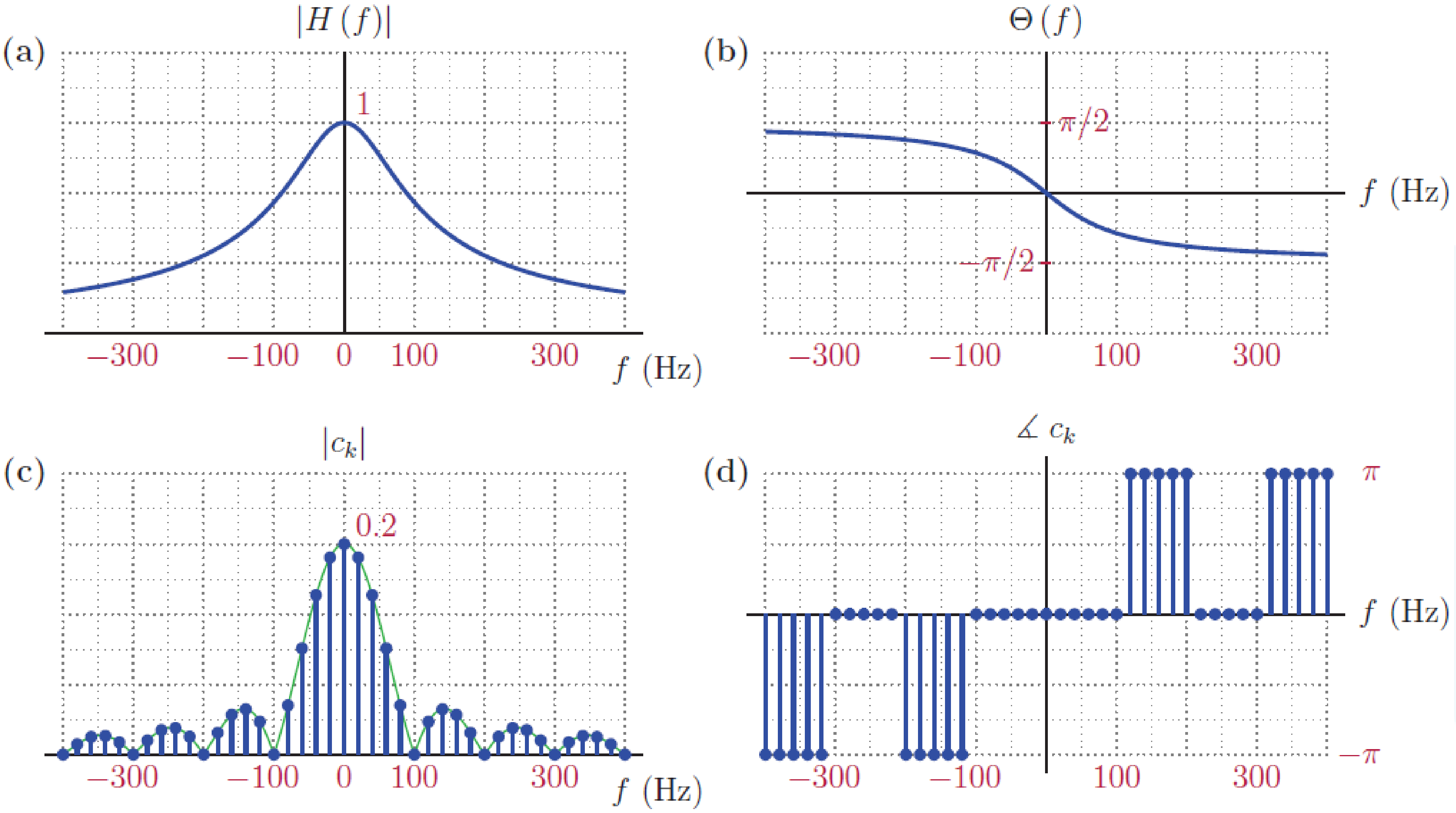
$$c_k = 0.2\,\mathrm{sinc}(0.2k) \Rightarrow$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} 0.2\,\mathrm{sinc}(0.2k)\,e^{j40\pi kt}$$

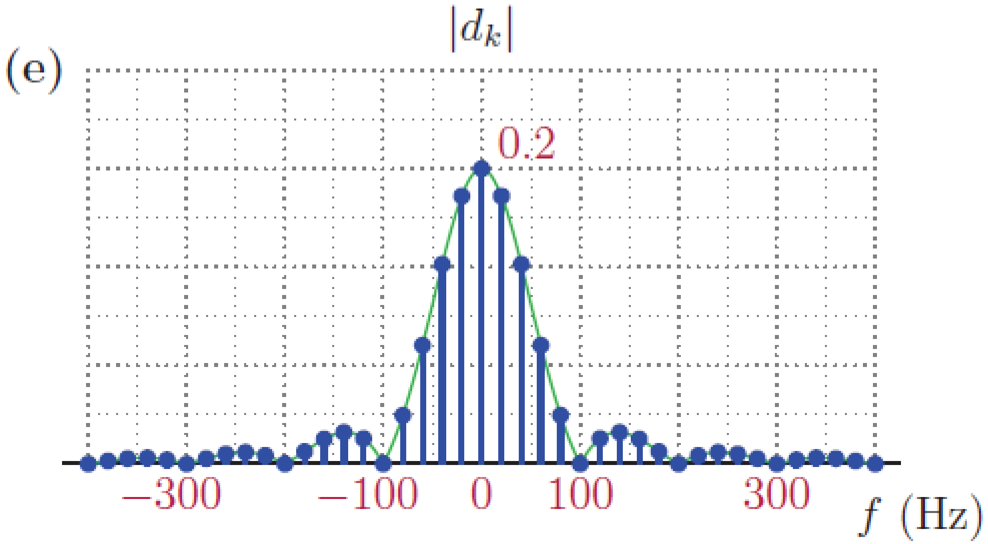


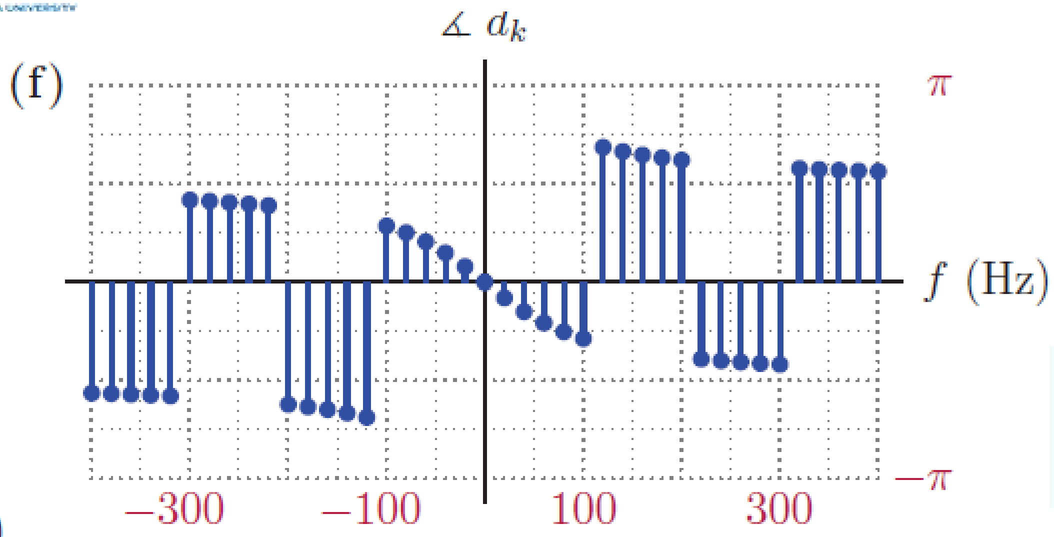
$$\tilde{y}(t) = \sum_{k=-\infty}^{\infty} \frac{c_k}{1 + j(20k/80)}\,e^{j40\pi kt} = \sum_{k=-\infty}^{\infty} d_k e^{j40\pi kt}$$

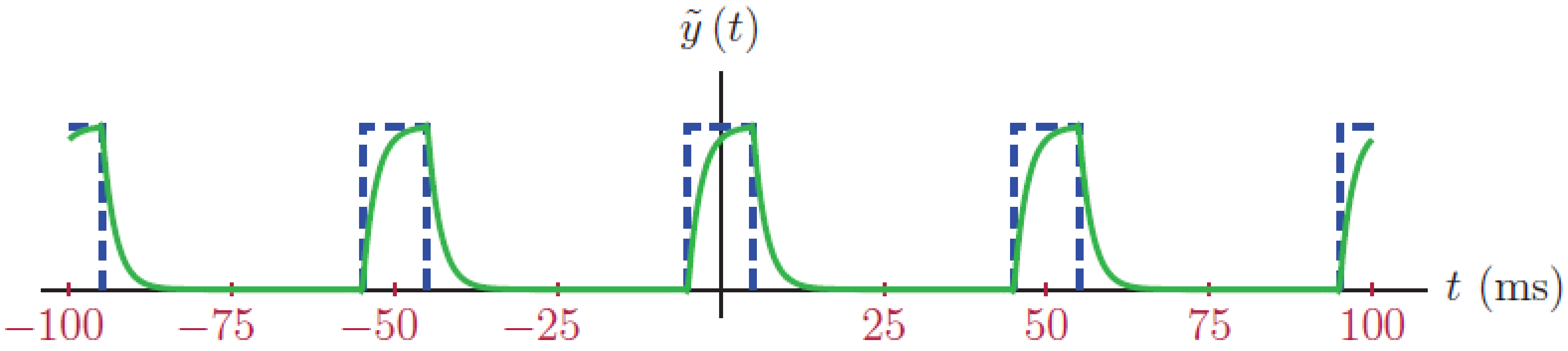
$$|d_k| = |c_k|\,|H(kf_0)| = \frac{|c_k|}{\sqrt{1 + (20k/80)^2}}, \qquad \measuredangle d_k = \measuredangle c_k - \tan^{-1}(20k/80)$$



$|H(f)|$
(a)
1
-300
-100
0
100
300
f (Hz)
$\Theta(f)$
(b)
$\pi/2$
$-\pi/2$
f (Hz)
-300
-100
100
300
$|c_k|$
(c)
0.2
-300
-100
0
100
300
f (Hz)
$\measuredangle\, c_k$
(d)
π
f (Hz)
$-\pi$
-300
-100
100
300


(e)
|d_k|
0.2
−300
−100
0
100
300
f (Hz)


(f)
∡ d_k
π
−π
f (Hz)
−300
−100
100
300


ỹ (t)
t (ms)
−100
−75
−50
−25
25
50
75
100

7. CTLTI Systems with Non-Periodic Input Signals

$$y(t) = h(t) * x(t) \Rightarrow Y(\omega) = H(\omega)X(\omega)$$

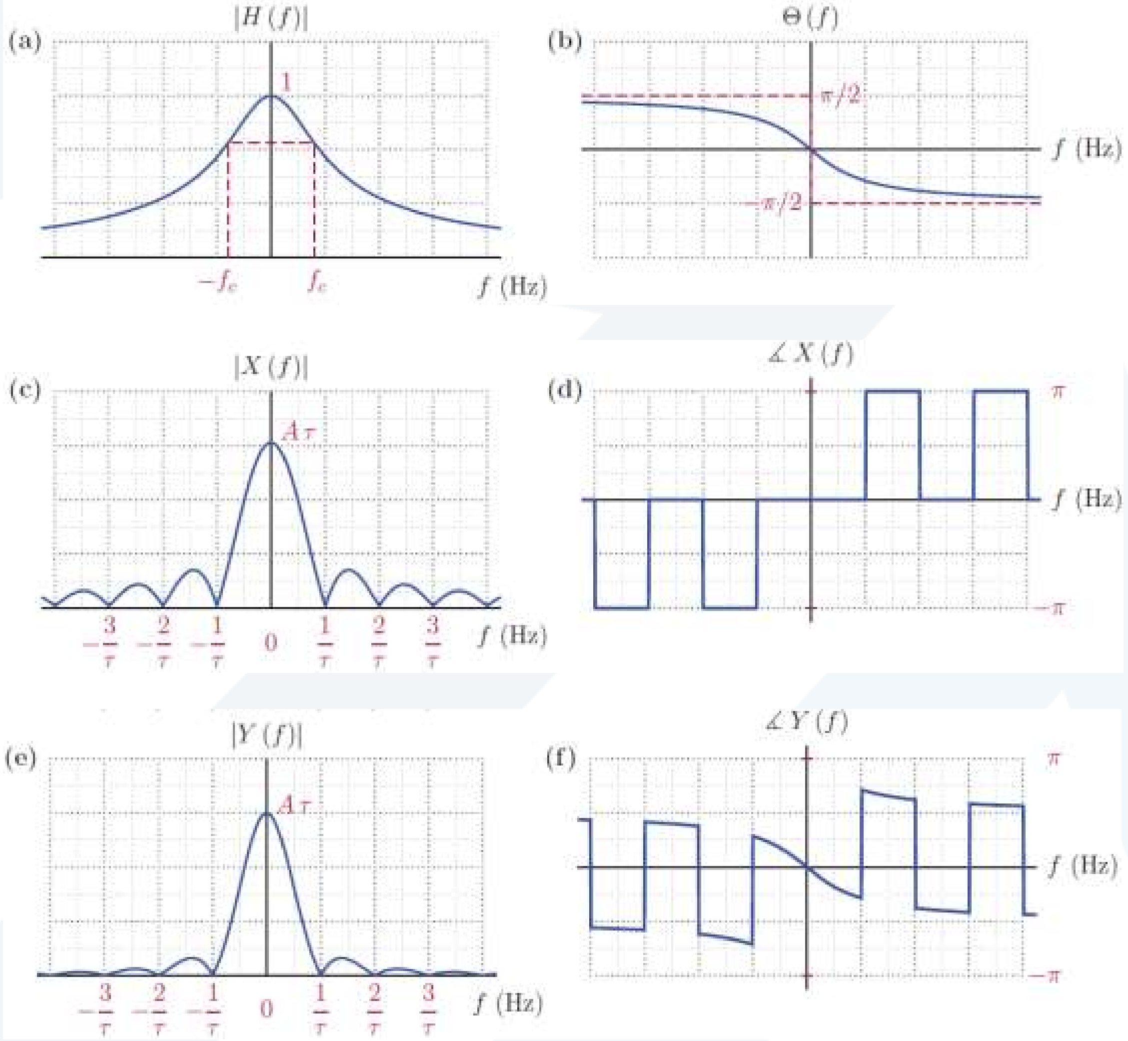
$$|Y(\omega)| = |H(\omega)||X(\omega)|, \quad \measuredangle Y(\omega) = \measuredangle X(\omega) + \Theta(\omega)$$

- **Example 30:** Pulse response of RC circuit revisited

  Consider again the RC circuit. Let $f_c = 1/RC = 80$ Hz. Determine the FT of the response of the system to the unit-pulse input signal $x(t) = \Pi(t)$.

$$H(f) = \frac{1}{1 + j(f/f_c)}, \quad X(f) = \mathrm{sinc}(f), \quad Y(f) = \frac{1}{1 + j(f/80)}\mathrm{sinc}(f)$$

$$|Y(f)| = \frac{1}{\sqrt{1 + (f/80)^2}}|\mathrm{sinc}(f)|, \quad \measuredangle Y(f) = -\tan^{-1}(f/80) + \measuredangle[\mathrm{sinc}(f)]$$


(a)
|H (f)|
1
−f_c
f_c
f (Hz)
(b)
Θ (f)
π/2
−π/2
f (Hz)
(c)
|X (f)|
Aτ
−3/τ
−2/τ
−1/τ
0
1/τ
2/τ
3/τ
f (Hz)
(d)
∡X (f)
π
−π
f (Hz)
(e)
|Y (f)|
Aτ
−3/τ
−2/τ
−1/τ
0
1/τ
2/τ
3/τ
f (Hz)
(f)
∡Y (f)
π
−π
f (Hz)