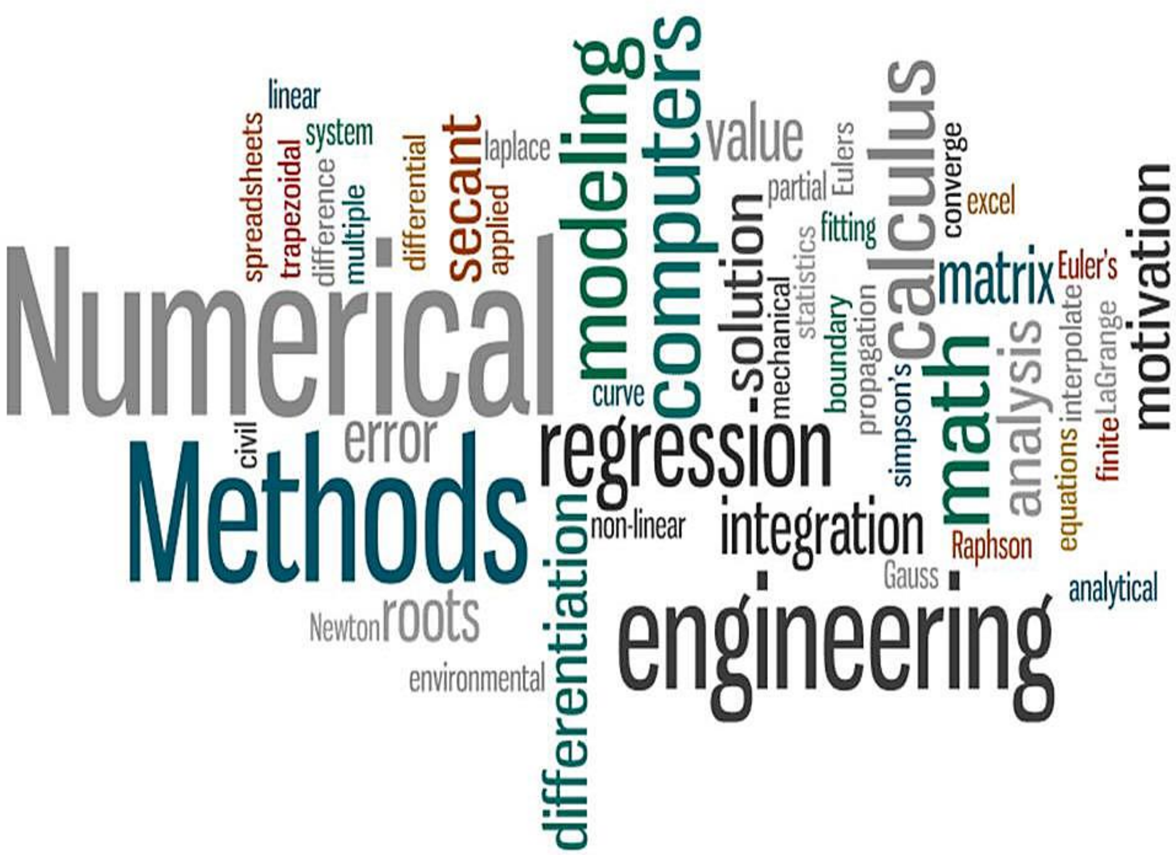
# Numerical Analysis and Programming



# Roots Of Equations
## Part-03

# Nonlinear Equation Solvers

- Graphical Solutions
- Numerical Solutions
  - Bracketing Methods
    - Bisection Method
    - False Position Method
  - Open Methods
    - Fixed Point Iteration
    - Newton Raphson Method
    - Secant Method
- Analytical Solutions

# Newton Raphson Method

- Given an initial guess of the root $x_0$, Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.
- **Based on Taylor series expansion:**

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of $x_{i+1}$ when $f(x_{i+1}) = 0$

Rearranging,

Solve for

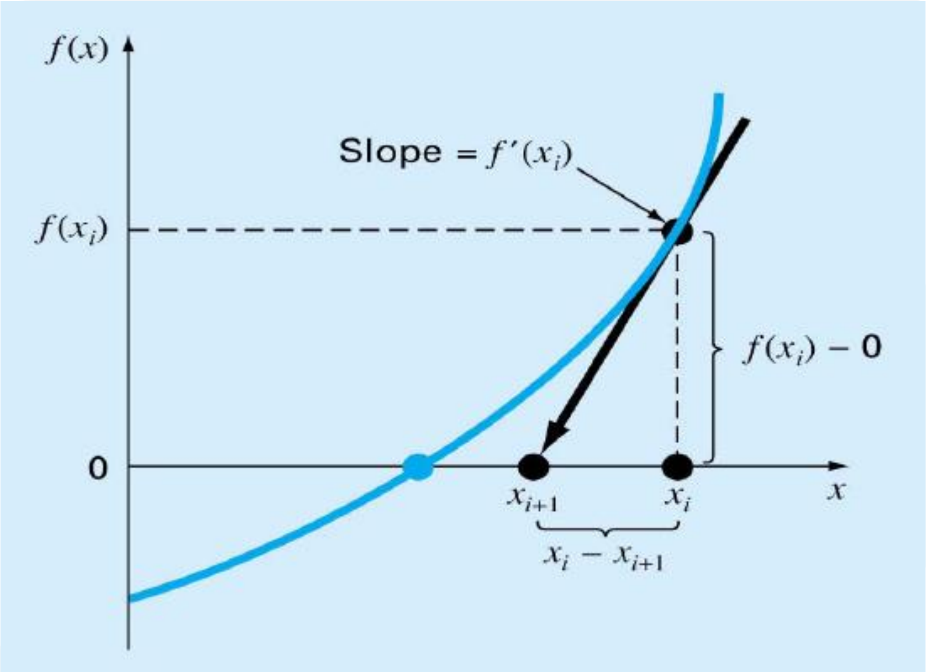
$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton-Raphson formula

# Newton Raphson Method

- **Graphical Depiction:** If the initial guess at the root is $x_i$, then a tangent to the function of $x_i$ that is $f'(x_i)$ is extrapolated down to the x-axis to provide an estimate of the root at $x_{i+1}$.




A convenient method for functions whose derivatives can be evaluated analytically.

# Newton Raphson Method

- **Example:** Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$, employing an initial guess of $x_0 = 0$
- **Solution:** The first derivative of the function can be evaluated as: $f'(x) = -e^{-x} - 1$ which can be substituted along with the original function: $x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{e^{-x_i} - 1}$

  Starting with an initial guess of $x_0 = 0$, the iterative equation can be applied to compute:

| $i$ | $x_i$ | $\varepsilon_t$ (%) | $E_t$ |
|---|---|---|---|
| 0 | 0.000000000 | 100 | 0.567143290 |
| 1 | 0.500000000 | 11.8 | 0.067143290 |
| 2 | 0.566311003 | 0.147 | 0.000832287 |
| 3 | 0.567143165 | 0.0000221 | 0.000000125 |
| 4 | 0.567143290 | $< 10^{-8}$ | 0.000000000 |

| $i$ | $x_i$ | $f(x_i)$ | $f'(x_i)$ | $\frac{f(x_i)}{f'(x_i)}$ | $\varepsilon_t$ (%) | $Et$ |
|---|---|---|---|---|---|---|
| 0 | 0.0000 | 0.50000 | - | - | 100 | |
| 1 | 0.50000 | −0.6321 | −1.3679 | 0.4621 | | |
| 2 | 0.5379 | 0.0461 | −1.5840 | −0.0291 | | |
| 3 | 0.5670 | 0.0002 | −1.5672 | −0.0002 | | |
| 4 | 0.5671 | 0.0000 | −1.5671 | −0.0000 | | |

# Pitfalls of Newton Raphson Method

*Assumptions:* $f(x),\ f'(x),\ x_0$ *are available,*

$$f'(x_0) \neq 0$$

*Newton's Method new estimate:*

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem:

$f'(x_i)$ is not available,

or difficult to obtain analytically.

It may not be convenient for functions whose derivatives cannot be evaluated analytically.

# The Secant Method

- A slight variation of Newton's method for functions whose **derivatives are difficult to evaluate**. For these cases the derivative can be approximated by a backward finite divided difference.

# The Secant Method - Derivation

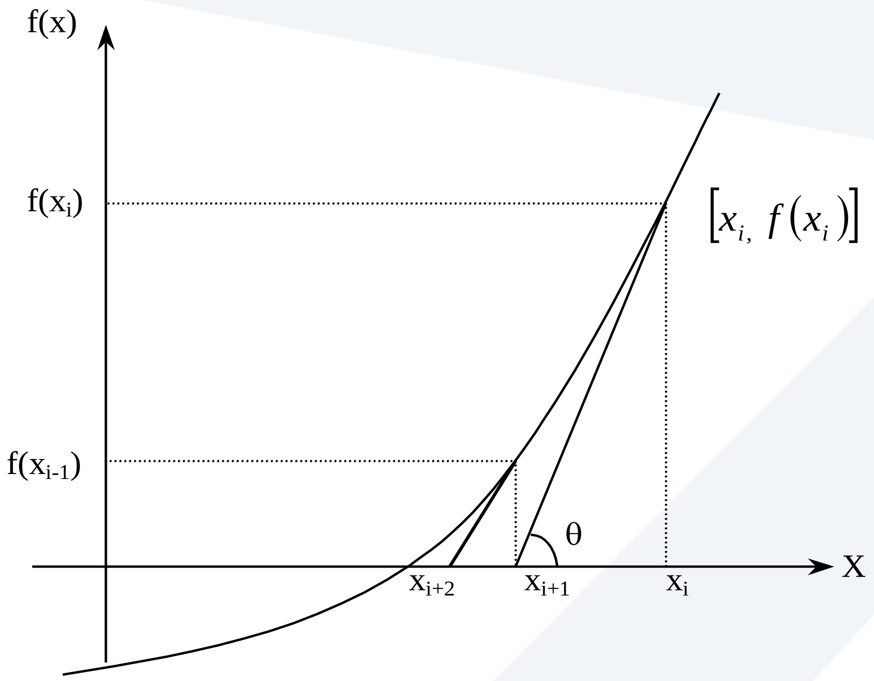



**Figure 1 Geometrical illustration of the Newton-Raphson method.**

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

Approximate the derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \qquad (2)$$

Substituting Equation (2) into Equation (1) gives the Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# The Secant Method - Derivation

The secant method can also be derived from geometry:

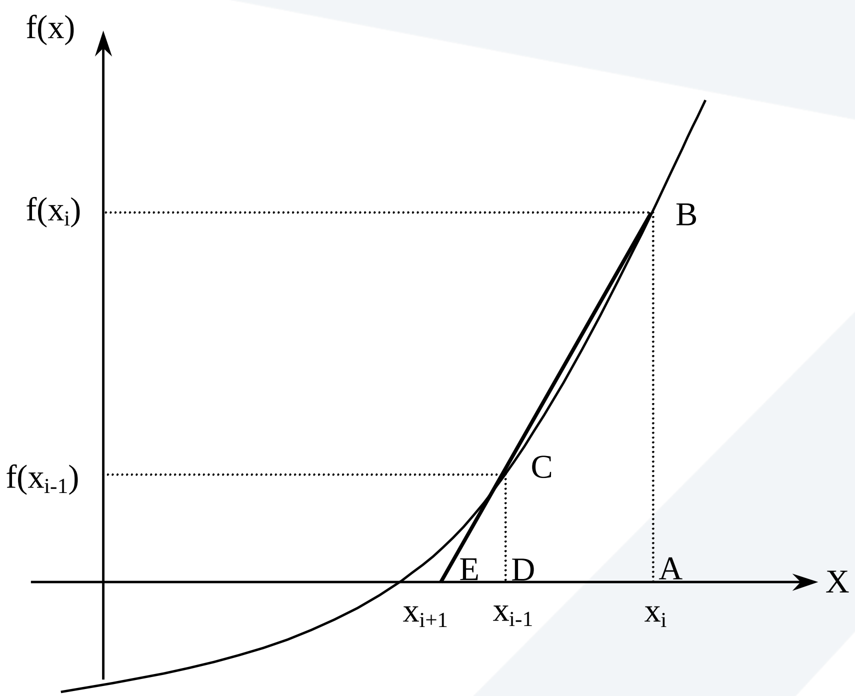



**Figure 2 Geometrical representation of the Secant method.**

The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

# The Secant Method

- **Example:** Use the secant method to estimate the root of ***f*(*x*) = *e*$^{-x}$ – x,** start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$
- **Solution:**

Solution. Recall that the true root is 0.56714329. . . .

First iteration:

$$x_{-1} = 0 \qquad f(x_{-1}) = 1.00000$$

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 1 - \frac{-0.63212(0 - 1)}{1 - (-0.63212)} = 0.61270 \qquad \varepsilon_t = 8.0\%$$

Second iteration:

$$x_0 = 1 \qquad f(x_0) = -0.63212$$

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

(Note that both estimates are now on the same side of the root.)

$$x_2 = 0.61270 - \frac{-0.07081(1 - 0.61270)}{-0.63212 - (-0.07081)} = 0.56384 \qquad \varepsilon_t = 0.58\%$$

Third iteration:

$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

$$x_2 = 0.56384 \qquad f(x_2) = 0.00518$$

$$x_3 = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - (-0.00518)} = 0.56717 \qquad \varepsilon_t = 0.0048\%$$

# Homework

**Problem Statement:** Determine the highest real root of:

$$f(x) = -6 + 17.5x - 11.6x^2 + 2.1x^3$$

(5 digits)

a) Graphically.

b) Fixed Point iteration method (Five iterations, $x_0$=3).

c) Newton Raphson method (Five iterations, $x_0$=3).

d) Secant method (Five iterations, $x_{-1}$=3 , $x_0$=4).

Compute the approximate percent relative errors for your solutions.

# Linear Algebraic Equations

## Part-01

# Noncomputer Methods for Solving System of Equations

- For small number of equations (**$n \leq 3$**) linear equations can be solved readily by simple techniques such as "**method of elimination**."
- Linear algebra provides the tools to solve such systems of linear equations.
- There are many ways to solve a system of linear equations:
  - Graphical Methods.
  - Cramer's Rule.
  - Method of Elimination.

For $n \leq 3$

- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.

# Graphical Methods

- Consider a set of two equations:

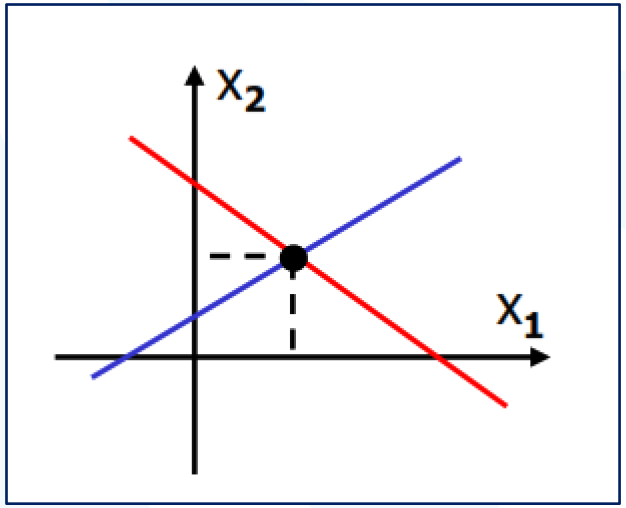
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Plot these on the Cartesian coordinate system with axes $x_1$ and $x_2$.

**Solve for $(x_2)$**

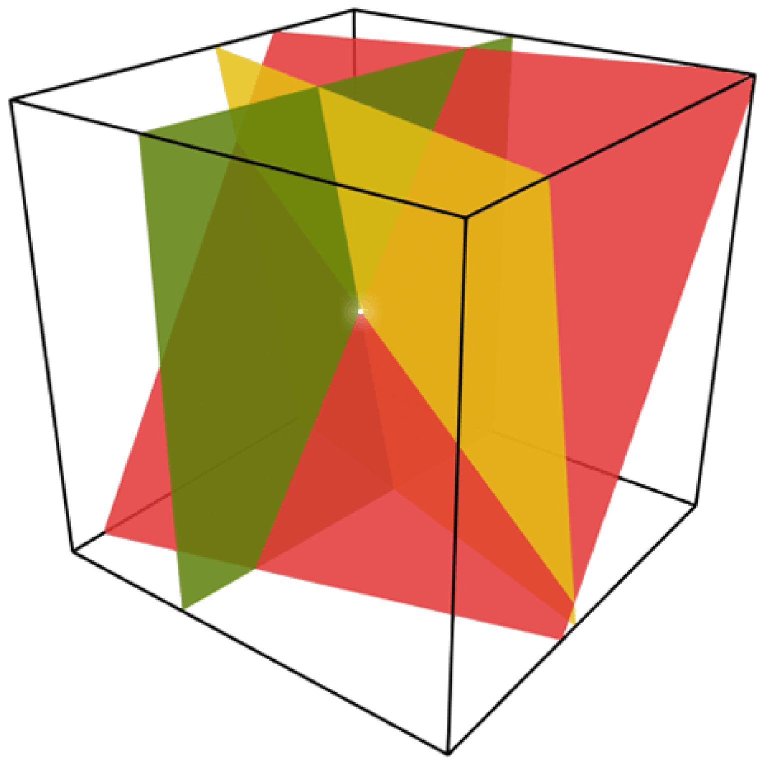
$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



# Graphical Methods

- For n=3, each equation will be a plane on a 3D coordinate system. Solution is the point where these planes intersect.



- For n>3, graphical solution is not practical.

# Graphical Methods

- For example : solve

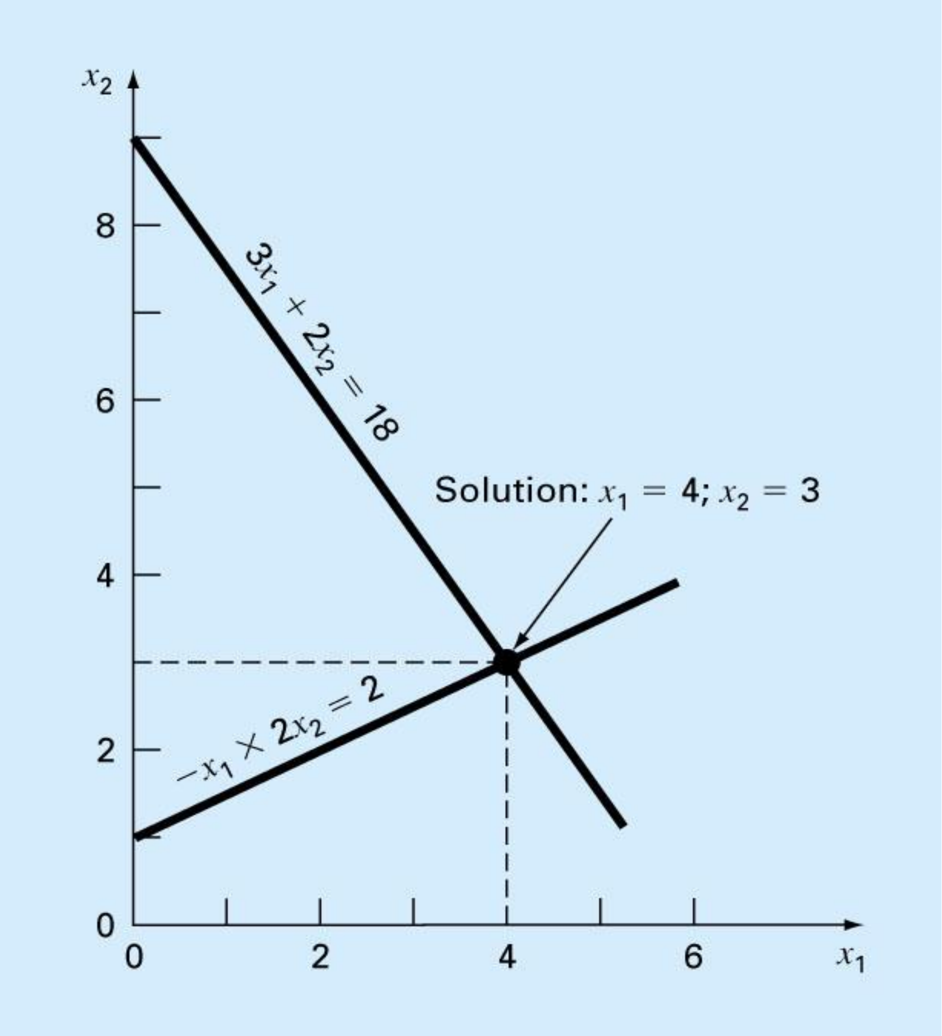
$$3x_1 + 2x_2 = 18$$

$$-x_1 + 2x_2 = 2$$

- The solution is the intersection of the two lines at $x_1=4$ and $x_2=3$.
- This result can be checked by substituting these values into the original equations to yield

$$3(4)+2(3)=18$$

$$-(4)+2(3)=2$$



# Cramer's Rule

- Cramer's rule is another solution technique that is best suited to small numbers of equations.
- This rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of coefficients of the unknown in question by the constants $b_1, b_2, \dots, b_n$.

For a 3x3 system

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

# Cramer's Rule- Example

- Example: Cramer's Rule can be used to solve the system

$$x_1 + x_2 = 3$$
$$x_1 + 2x_2 = 5$$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \qquad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems.

To solve N by N system requires $(N+1)(N-1)N!$ multiplications.

To solve a 30 by 30 system, $2.38 \times 10^{35}$ multiplications are needed.

It can be used if the determinants are computed in efficient way

# Cramer's Rule- Example

- Example:

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 + 2\,x_2 - 2\,x_3 &= 0 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{6}{2} = 3$$

$$\{x\} = \begin{Bmatrix} 2 \\ 2 \\ 3 \end{Bmatrix}$$

# The Elimination of Unknowns





- The elimination of unknowns by combining equations is an algebraic approach that can be illustrated for a set of two equations:

$$a_{11}x_1+a_{12}x_2=b_1 \qquad (1)$$

$$a_{21}x_1+a_{22}x_2=b_2 \qquad (2)$$

- Eq. (1) might be multiplied by $a_{21}$ and Eq. (2) by $a_{11}$ to give

$$a_{11}a_{21}x_1+a_{12}a_{21}x_2=b_1a_{21} \qquad (3)$$

$$a_{21}a_{11}x_1+a_{22}a_{11}x_2=b_2a_{11} \qquad (4)$$

Subtracting Eq. (3) from Eq. (4) will, therefore, eliminate the $x_1$ term from the equations to yield

$$a_{22}a_{11}x_2-a_{12}a_{21}x_2=b_2a_{11}-b_1a_{21}$$

# The Elimination of Unknows

- Which can be solved for:

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

- $x_2$ can then be substituted into Eq. (1), which can be solved for:

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

# The Elimination of Unknows - Example

- Use the elimination to solve:

$$3x_1 + 2x_2 = 18$$
$$- x_1 + 2x_2 = 2$$

- The elimination of unknowns can be extended to systems with more than two or three equations. However, the numerous calculations that are required for larger systems make the method extremely tedious to implement by hand. However, as described in the next section, the technique can be formalized and readily programmed for the computer.

# Naïve Gauss Elimination

- In the previous section, the elimination of unknowns was used to solve a pair of simultaneous equations. The procedure consisted of two steps:

1. The equations were manipulated to eliminate one of the unknowns from the equations. The result of this elimination step was that we had one equation with one unknown.

2. Consequently, this equation could be solved directly and the result back-substituted into one of the original equations to solve for the remaining unknown.

- This basic approach can be extended to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back-substitute. Gauss elimination is the most basic of these schemes.

# Naïve Gauss Elimination

- The method consists of two steps:
  - **Forward Elimination**: the system is reduced to upper triangular form. A sequence of elementary operations is used.
  - **Backward Substitution**: Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

# Naïve Gauss Elimination

The two phases of Gauss elimination:

Forword elimination and back substitution.

The primes indicate the number of times that the coefficients and constants have been modified.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3/a''_{33}$$

$$x_2 = (c'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

# Naïve Gauss Elimination - Example

## Example

Use Gauss elimination to solve

$$x_1 - 2x_2 + 2x_3 = 1$$

$$2x_1 + x_2 - 3x_3 = -3$$

$$-3x_1 + x_2 - x_3 = 4$$

Carry six significant figures during the computation.

| | | | | |
|---|---|---|---|---|
| 1 | -2 | 2 | | 1 |
| 2 | 1 | -3 | | -3 |
| -3 | 1 | -1 | | 4 |
| | | | | |
| 1 | -2 | 2 | | 1 |
| 0 | 5 | -7 | | -5 |
| 0 | -5 | 5 | | 7 |
| | | | | |
| 1 | -2 | 2 | | 1 |
| 0 | 5 | -7 | | -5 |
| 0 | 0 | -2 | | 2 |
| | | | | |
| | | | x3 | -1 |
| | | | x2 | -2,4 |
| | | | x1 | -1,8 |

# Naïve Gauss Elimination - Homework

**Example:** Solve the following system using Naïve Gauss Elimination.

$$
\begin{aligned}
6x_1 - 2x_2 + 2x_3 + 4x_4 &= 16 \\
12x_1 - 8x_2 + 6x_3 + 10x_4 &= 26 \\
3x_1 - 13x_2 + 9x_3 + 3x_4 &= -19 \\
-6x_1 + 4x_2 + x_3 - 18x_4 &= -34
\end{aligned}
$$