

## Time Domain Analysis





## Contents

**Time Domain Response of First-Order System**

**Time Domain Response of Second-Order System**

## Time Domain Analysis of First-Order System

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \dots + b_{m-1} x' + b_m x$$

- Where  $x$  is the input of the system and  $y$  is the output of the system.

Laplace Transformation

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Transfer function =  $G(s) = \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}$  | zero initial conditions

$$G(s) = \frac{L[\text{output}]}{L[\text{input}]} = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

$$a \frac{dx_o}{dt} + bx_o = cx_i(t)$$

$$(as + b)X_o(s) = cX_i(s)$$

$$G(s) = \frac{X_o}{X_i}(s) = \frac{c}{as + b}$$

$$G(s) = \frac{\frac{c}{b}}{1 + \frac{a}{b}s}$$

$$G(s) = \frac{K}{1 + Ts}$$

**$K$** : steady-state gain constant

**$T$** : time constant (seconds)

### *Steady-state gain*

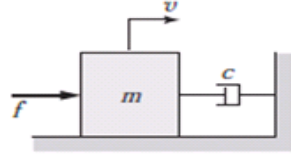
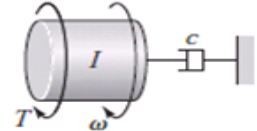
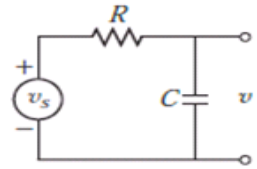
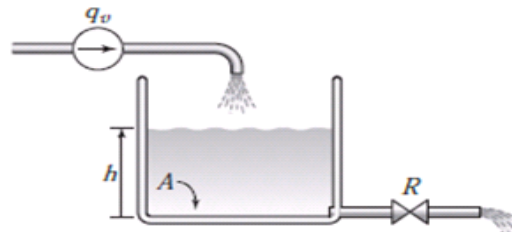
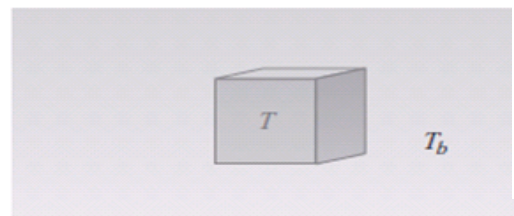
The steady-state of a TF can be used to calculate the steady-state change in an output due to a steady-state change in the input. For example, suppose we know two steady states for an input,  $u$ , and an output,  $y$ . Then we can calculate the steady-state gain,  $K$ , from:

$$K = \frac{y_2 - y_1}{u_2 - u_1}$$

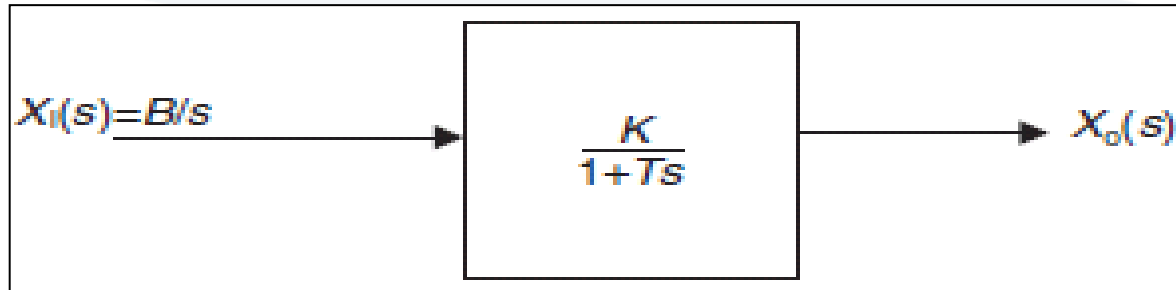
### *Time constant*

In brief, the time constant relates to the analytical solution for the unit step response of a first order differential equation, and is the time taken for the output to reach **63%** of the steady-state value

## Examples of First-Order Systems

	$m \frac{dv}{dt} + cv = f$ $T = \frac{m}{c}$
	$I \frac{d\omega}{dt} + c\omega = T$ $T = \frac{I}{c}$
	$RC \frac{dv}{dt} + v = v_s$ $T = RC$
	$AR \frac{dh}{dt} + \rho gh = Rq_v$ $T = \frac{AR}{\rho g}$
	$mc_p R \frac{dT}{dt} + T = T_b$ $T = mc_p R$

## Step Response of First-Order System



$$X_o(s) = \frac{BK}{s(1+Ts)} = BK \frac{\frac{1}{T}}{s(s + \frac{1}{T})}$$

$$x_o(t) = BK(1 - e^{-\frac{t}{T}})$$

$B=1$ (unit step)

$K=1$ (unity gain)

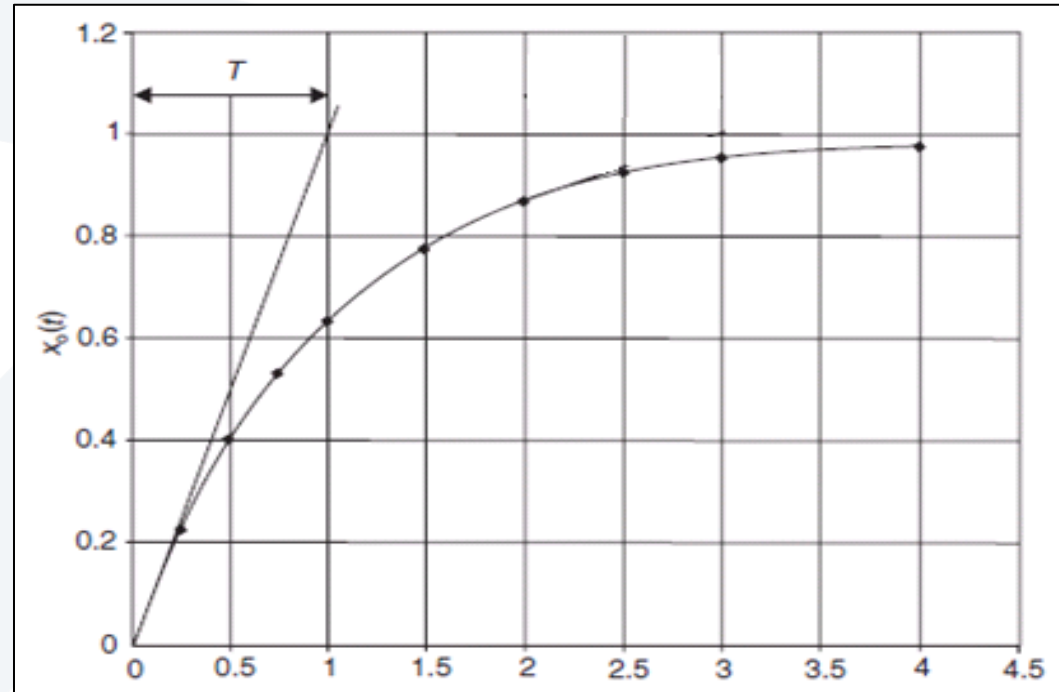
$$x_o(t) = 1 - e^{-\frac{t}{T}}$$

Time function $f(t)$	Laplace transform $\mathcal{L}\{f(t)\} = F(s)$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$

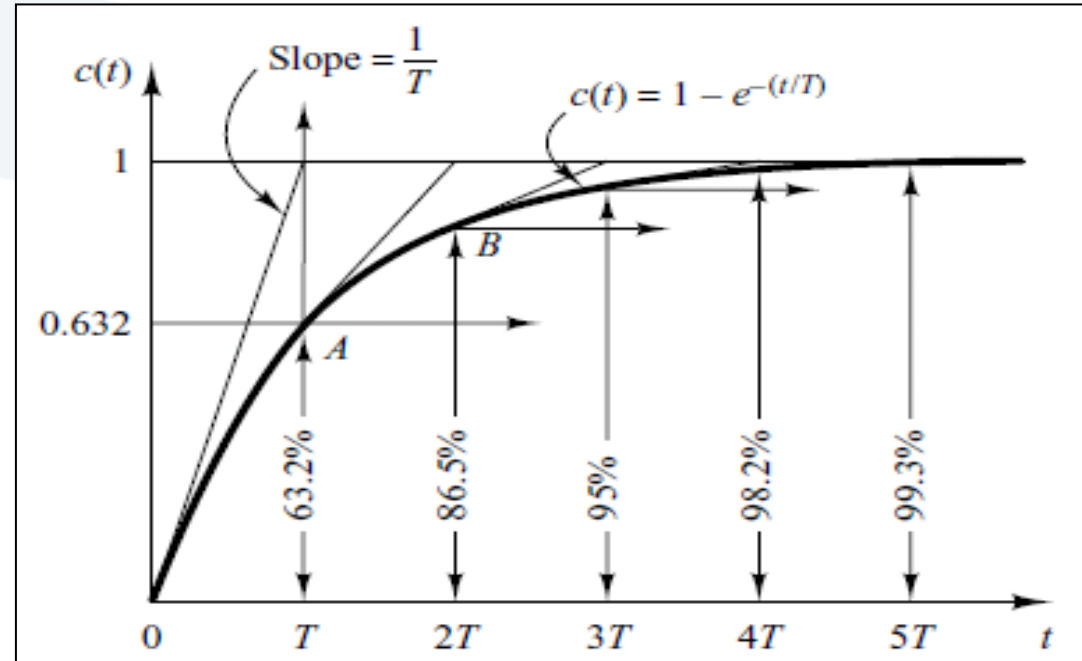
The system time constant is the intersection of the slope at  $t=0$  with the final value line

$$\frac{dx_o}{dt} = 0 - \left(-\frac{1}{T}\right)e^{-\frac{t}{T}} = \frac{1}{T}e^{-\frac{t}{T}}$$

$$\frac{dx_o}{dt} \Big|_{t=0} = \frac{1}{T}$$







In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At  $t = 3T, 4T$ , and  $5T$ , the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for  $t \geq 4T$ , the response remains within 2% of the final value.

## Time Domain Analysis of Second-Order System

$$a \frac{d^2 x_o}{dt^2} + b \frac{dx_o}{dt} + cx_o = ex_i(t)$$

$$(as^2 + bs + c)X_o(s) = eX_i(s)$$

$$G(s) = \frac{X_o}{X_i}(s) = \frac{e}{as^2 + bs + c}$$

$$G(s) = \frac{\frac{e}{c}}{\frac{a}{c}s^2 + \frac{b}{c}s + 1}$$

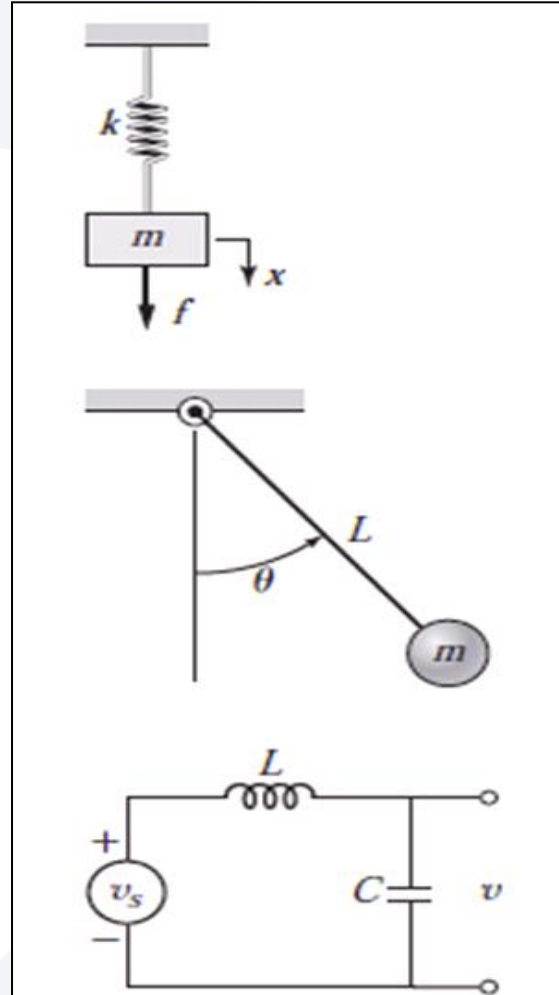
$$G(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$K$ : steady-state gain constant

$\omega_n$ : undamped natural frequency(rad/s)

$\zeta$ : damping ratio



$$m \frac{d^2 x}{dt^2} + kx = f(t)$$

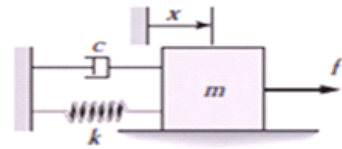
$$\omega_n = \sqrt{\frac{k}{m}}$$

$$L \frac{d^2 \theta}{dt^2} + g\theta = 0$$

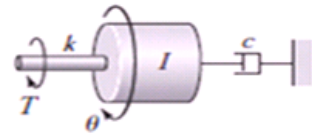
$$\omega_n = \sqrt{\frac{g}{L}}$$

$$LC \frac{d^2 v}{dt^2} + v = v_s$$

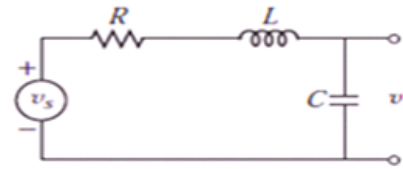
$$\omega_n = \frac{1}{\sqrt{LC}}$$



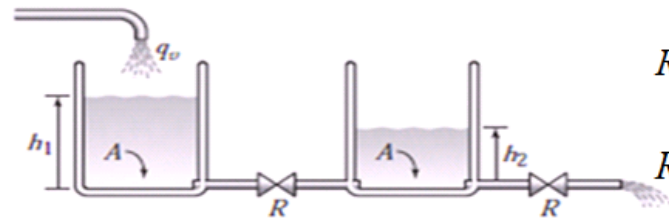
$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = f$$



$$I \frac{d^2 \theta}{dt^2} + c \frac{d\theta}{dt} + k\theta = T$$

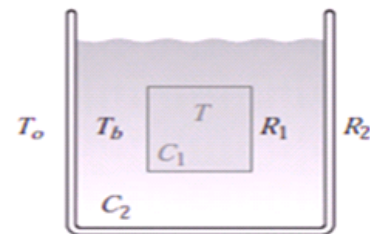


$$LC \frac{d^2 v}{dt^2} + RC \frac{dv}{dt} + v = v_s$$



$$RA \frac{dh_1}{dt} + \rho g (h_1 - h_2) = Rq_v$$

$$RA \frac{dh_2}{dt} + \rho g (h_2 - h_1) + \rho gh_2 = 0$$



$$R_1 C_1 \frac{dT}{dt} + T = T_b$$

$$R_1 R_2 C_2 \frac{dT_b}{dt} + (R_1 + R_2) T_b = R_2 T + R_1 T_o$$

## Roots of the Characteristic Equation

The time response of any system has two components:

- (a) Transient response
- (b) Steady-state response

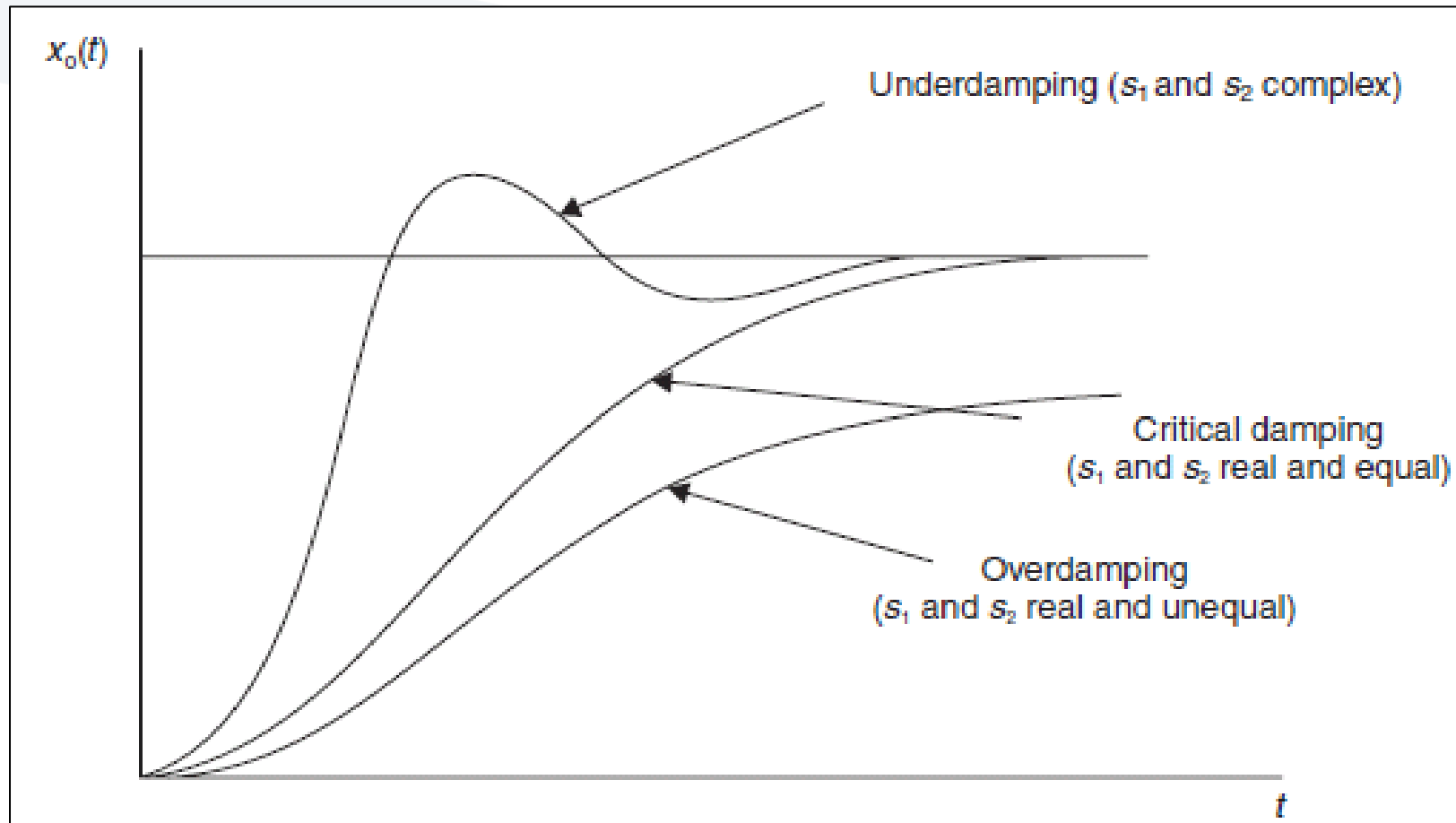
$$(as^2 + bs + c)X_0(s) = 0$$

$$(as^2 + bs + c) = 0$$

*Characteristic Equation*

$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<i>Discriminant</i>	<i>Roots</i>	<i>Transient response type</i>
$b^2 > 4ac$	$s_1$ and $s_2$ real and unequal	Overdamped Transient Response
$b^2 = 4ac$	$s_1$ and $s_2$ real and equal	Critically Damped Transient Response
$b^2 < 4ac$	$s_1$ and $s_2$ complex conjugate of the form: $s_1, s_2 = -\sigma \pm j\omega$	Underdamped Transient Response



## Critical Damping and Damping Ratio

### *Critical damping*

When the damping coefficient  $C$  of a second-order system has its critical value  $C_c$ , the system, when disturbed, will reach its steady-state value in the minimum time without overshoot. This is when the roots of the Characteristic Equation have equal negative real roots.

### *Damping ratio*

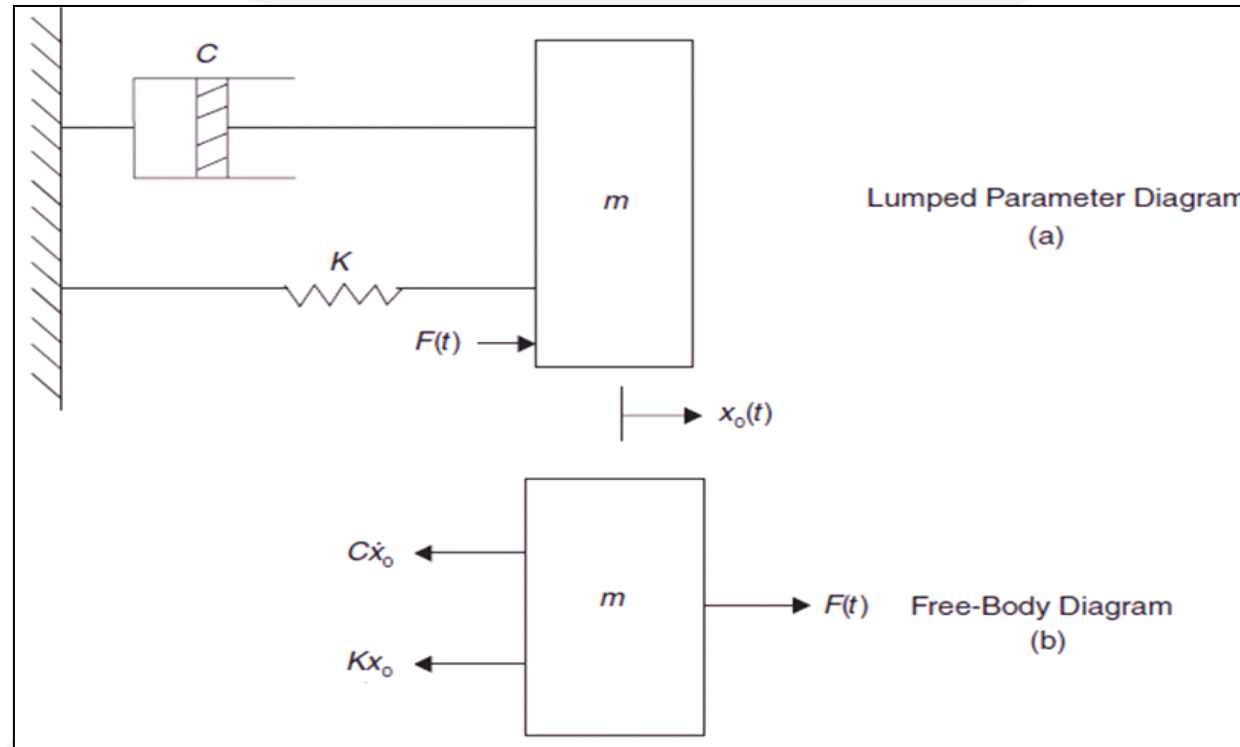
The ratio of the damping coefficient  $C$  in a second-order system compared with the value of the damping coefficient  $C_c$  required for critical damping is called the Damping Ratio  $\zeta$

$$\zeta = \frac{C}{C_c}$$

$\zeta = 0$	No damping
$\zeta < 1$	Underdamping
$\zeta = 1$	Critical damping
$\zeta > 1$	Overdamping

## EXAMPLE

Find the value of the critical damping coefficient  $C_c$  in terms of  $K$  and  $m$  for the spring–mass–damper system shown in Figure





From Newton's second law

$$\sum Fx = m\ddot{x}_o$$

From the free-body diagram

$$F(t) - Kx_o(t) - C\dot{x}_o(t) = m\ddot{x}_o(t)$$

Taking Laplace transforms, zero initial conditions

$$F(s) - KX_o(s) - CsX_o(s) = ms^2X_o(s)$$

or

$$(ms^2 + Cs + K)X_o(s) = F(s)$$

Characteristic Equation is

$$ms^2 + Cs + K = 0$$

$$\text{i.e. } s^2 + \frac{C}{m} + \frac{K}{m} = 0$$

and the roots are

$$s_1, s_2 = \frac{1}{2} \left\{ -\frac{C}{m} \pm \sqrt{\left(\frac{C}{m}\right)^2 - 4\frac{K}{m}} \right\}$$

For critical damping, the discriminant is zero, hence the roots become

$$s_1 = s_2 = -\frac{C_c}{2m}$$

Also, for critical damping

$$\frac{C_c^2}{m^2} = \frac{4K}{m}$$

$$C_c^2 = \frac{4Km^2}{m}$$

giving

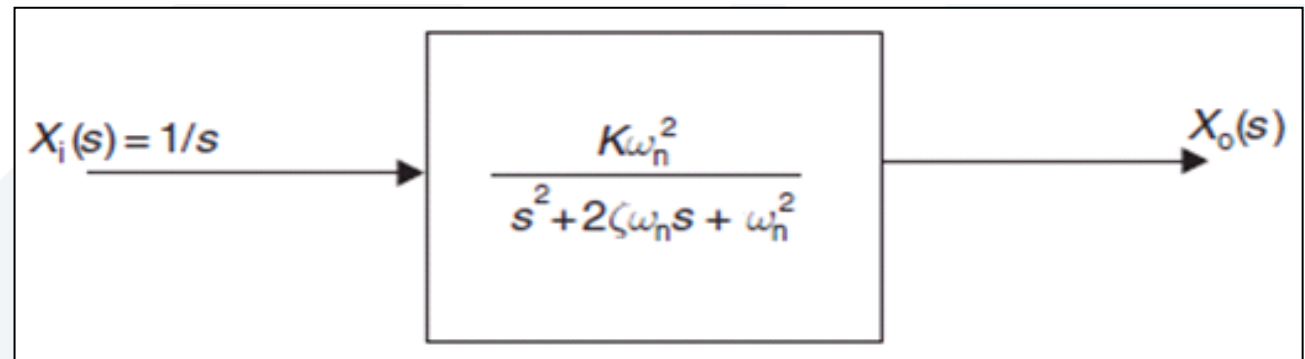
$$C_c = 2\sqrt{Km}$$

## Generalized Second-Order System Response to a unit Step Input

Consider a second-order system whose steady-state gain is  $K$ , undamped natural frequency is  $\omega_n$  and whose damping ratio is  $\zeta$ , where  $\zeta < 1$ . For a unit step input, the block diagram is as shown in Figure

$$X_o(s) = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$X_o(s) = \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$



multiply by  $s(s^2 + 2\zeta\omega_n s + \omega_n^2)$

$$K\omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs^2 + Cs$$

### Equating coefficients

$$(s^2) : 0 = A + B$$

$$(s^1) : 0 = 2\zeta\omega_n A + C$$

$$(s^0) : K\omega_n^2 = \omega_n^2 A$$

$$A = K, \quad B = -K \quad \text{and} \quad C = -2\zeta\omega_n K$$

$$X_o(s) = K \left[ \frac{1}{s} - \left\{ \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\} \right]$$

### Completing the square

$$X_o(s) = K \left[ \frac{1}{s} - \left\{ \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2} \right\} \right] = K \left[ \frac{1}{s} - \left\{ \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_n \sqrt{1 - \zeta^2})^2} \right\} \right]$$

The terms in the brackets { } can be written in the standard forms

$$\text{Term (1)} = \frac{-s}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2}$$

$$\text{Term (2)} = -\left\{ \frac{2\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}} \right\} \left\{ \frac{\omega_n\sqrt{1 - \zeta^2}}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1 - \zeta^2})^2} \right\}$$

Inverse transform

$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$

$$x_o(t) = K \left[ 1 - e^{-\zeta\omega_n t} \left\{ \cos(\omega_n\sqrt{1 - \zeta^2})t + \left( \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \sin(\omega_n\sqrt{1 - \zeta^2})t \right\} \right]$$

When  $\zeta = 0$

$$\begin{aligned}x_o(t) &= K[1 - e^0\{\cos\omega_n t + 0\}] \\ &= K[1 - \cos\omega_n t]\end{aligned}$$

From equation it can be seen that when there is no damping, a step input will cause the system to oscillate continuously at  $\omega_n$  (rad/s).

***Damped natural frequency  $\omega_d$***

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

where  $\omega_d$  is called the damped natural frequency.

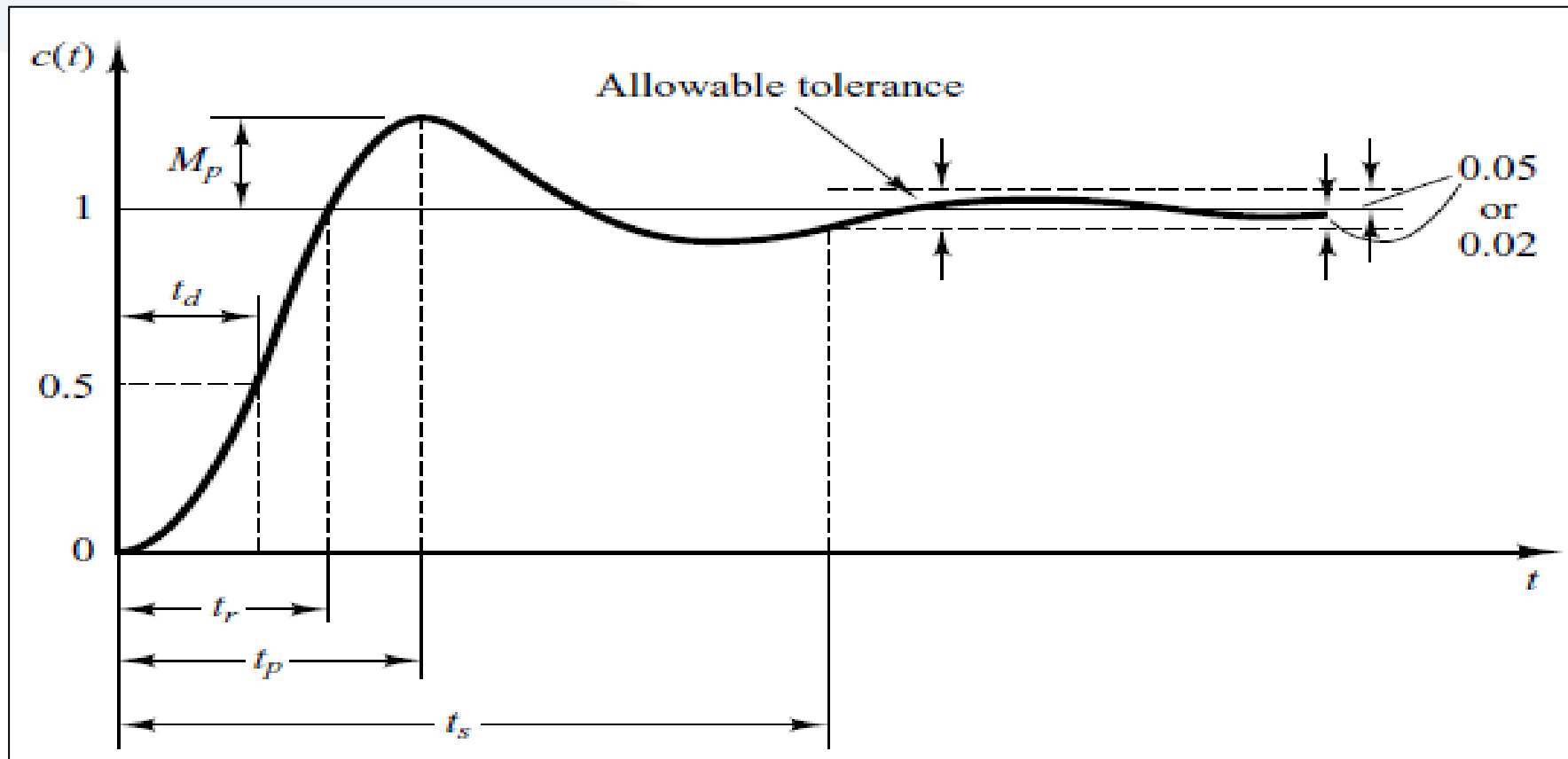
$$\begin{aligned}x_o(t) &= K \left[ 1 - e^{-\zeta\omega_n t} \left\{ \cos\omega_d t + \left( \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \sin\omega_d t \right\} \right] \\ &= K \left[ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_d t + \phi) \right] \quad \tan\phi = \frac{\sqrt{1 - \zeta^2}}{\zeta}\end{aligned}$$

## Definitions of Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time,  $t_d$
2. Rise time,  $t_r$
3. Peak time,  $t_p$
4. Maximum overshoot,  $M_p$
5. Settling time,  $t_s$

These specifications are defined in what follows and are shown graphically in Figure





1. Delay time,  $t_d$ : The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time,  $t_r$ : The rise time is the time required for the response to rise from 10% to 90%, or 0% to 100% of its final value.
3. Peak time,  $t_p$ : The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot,  $M_p$ : The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

5. Settling time,  $t_s$ : The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.

## Second-Order System and Transient-Response Specifications

*Peak time  $t_p$ :* Referring to Equation , we may obtain the peak time by differentiating  $c(t)$  with respect to time and letting this derivative equal zero. Since

$$\begin{aligned} \frac{dc}{dt} = & \zeta \omega_n e^{-\zeta \omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right) \\ & + e^{-\zeta \omega_n t} \left( \omega_d \sin \omega_d t - \frac{\zeta \omega_d}{\sqrt{1 - \zeta^2}} \cos \omega_d t \right) \end{aligned}$$

and the cosine terms in this last equation cancel each other,  $dc/dt$ , evaluated at  $t = t_p$ , can be simplified to

$$\left. \frac{dc}{dt} \right|_{t=t_p} = (\sin \omega_d t_p) \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin \omega_d t_p = 0$$

or

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \dots$$

Since the peak time corresponds to the first peak overshoot,  $\omega_d t_p = \pi$ . Hence

$$t_p = \frac{\pi}{\omega_d}$$

**Maximum overshoot  $M_p$ :** The maximum overshoot occurs at the peak time or at  $t = t_p = \pi/\omega_d$ . Assuming that the final value of the output is unity,  $M_p$  is obtained from Equation as

$$\begin{aligned}
 M_p &= c(t_p) - 1 \\
 &= -e^{-\zeta\omega_n(\pi/\omega_d)} \left( \cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \\
 &= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \quad \text{where } \sigma \text{ is called the } \textit{attenuation}
 \end{aligned}$$

The maximum percent overshoot is  $e^{-(\sigma/\omega_d)\pi} \times 100\%$ .

If the final value  $c(\infty)$  of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

*Settling time  $t_s$ :* For convenience in comparing the responses of systems, we commonly define the settling time  $t_s$  to be

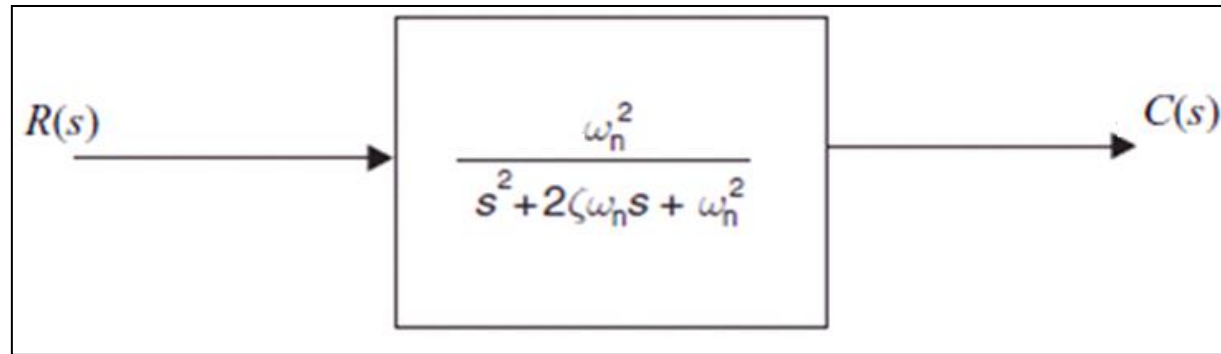
$$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad (2\% \text{ criterion})$$

or

$$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad (5\% \text{ criterion})$$

## EXAMPLE

Consider the system shown in Figure , where  $\zeta = 0.6$  and  $\omega_n = 5$  rad/sec. Let us obtain the peak time  $t_p$ , maximum overshoot  $M_p$ , and settling time  $t_s$  when the system is subjected to a unit-step input.



From the given values of  $\zeta$  and  $\omega_n$ , we obtain  $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4$  and  $\sigma = \zeta\omega_n = 3$ .

*Peak time  $t_p$ :* The peak time is  $t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785$  sec



*Maximum overshoot  $M_p$ :* The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

*Settling time  $t_s$ :*

For the 2% criterion, the settling time is  $t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33$  sec

For the 5% criterion,  $t_s = \frac{3}{\sigma} = \frac{3}{3} = 1$  sec

انتهت المحاضرة