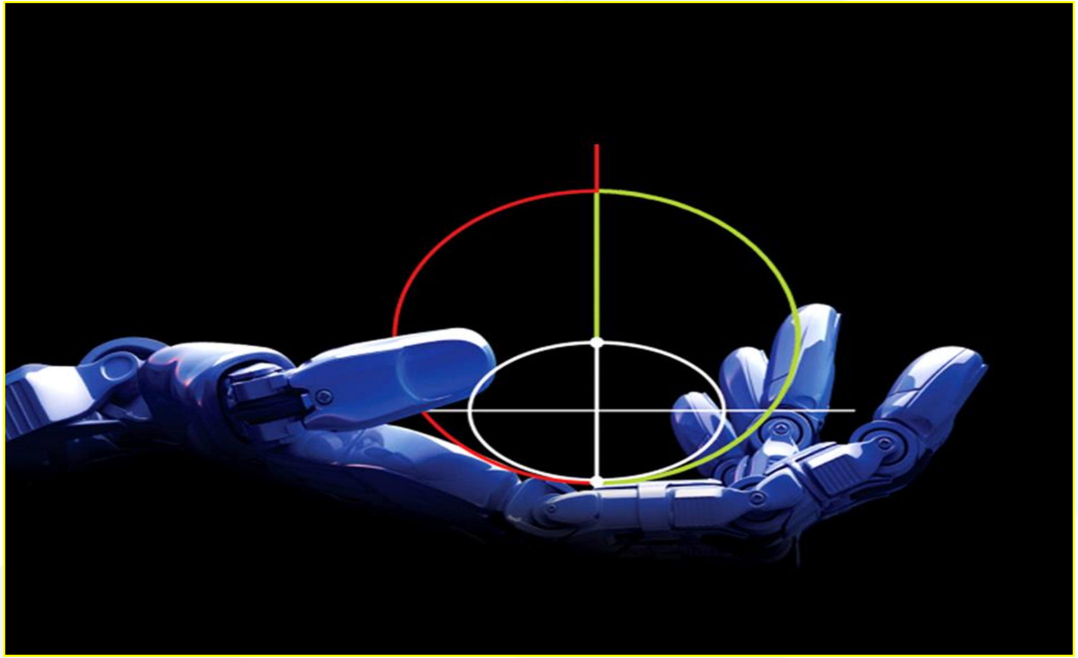
جامعة المنارة

كلية الهندسة

قسم ميكاترونكس

مقرر النمذجة و المحاكاة

Time Domain Analysis



العام الدراسي 2023-2024

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Time Domain Analysis of First-Order System

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots\ldots + a_{n-1} y' + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \ldots\ldots + b_{m-1} x' + b_m x$$

- Where *x* is the input of the system and *y* is the output of the system.

Laplace Transformation

$$L\left| f(t) \right| = F(s) = \int_0^\infty f(t) e^{-st} dt$$

$$\text{Transfer function} = G(s) = \left. \frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]} \right|_{\text{zero initial conditions}}$$

$$G(s) = \frac{L[output]}{L[input]} = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots\ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1} s + a_n}$$

$$a\frac{dx_o}{dt}+bx_o=cx_i(t)$$

$$(as+b)X_0(s)=cX_i(s)$$

$$G(s)=\frac{X_o}{X_i}(s)=\frac{c}{as+b}$$

$$G(s)=\frac{\frac{c}{b}}{1+\frac{a}{b}s}$$

$$G(s)=\frac{K}{1+Ts}$$

K: steady-state gain constant

T: time constant (seconds)

Steady-state gain

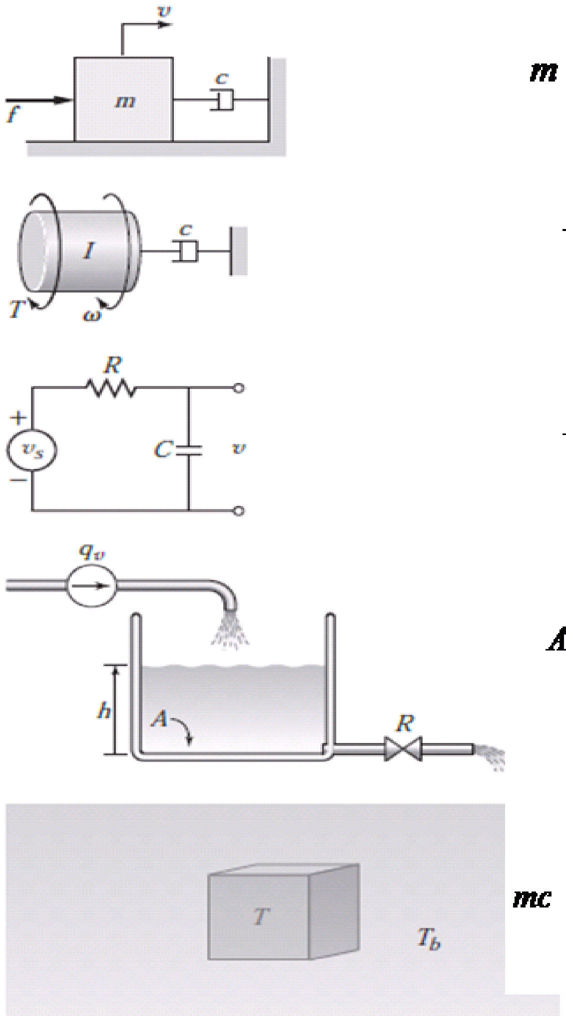
The steady-state of a **TF** can be used to calculate the steady-state change in an output due to a steady-state change in the input. For example, suppose we know two steady states for an input, ***u***, and an output, ***y***. Then we can calculate the steady-state gain, ***K***, from:

$$K = \frac{y_2 - y_1}{u_2 - u_1}$$

Time constant

In brief, the time constant relates to the analytical solution for the unit step response of a first order differential equation, and is the time taken for the output to reach **63%** of the steady-state value

Examples of First-Order Systems



$$m\frac{dv}{dt} + cv = f$$

$$T = \frac{m}{c}$$

$$I\frac{dw}{dt} + cw = T$$

$$T = \frac{I}{c}$$

$$RC\frac{dv}{dt} + v = v_s$$

$$T = RC$$

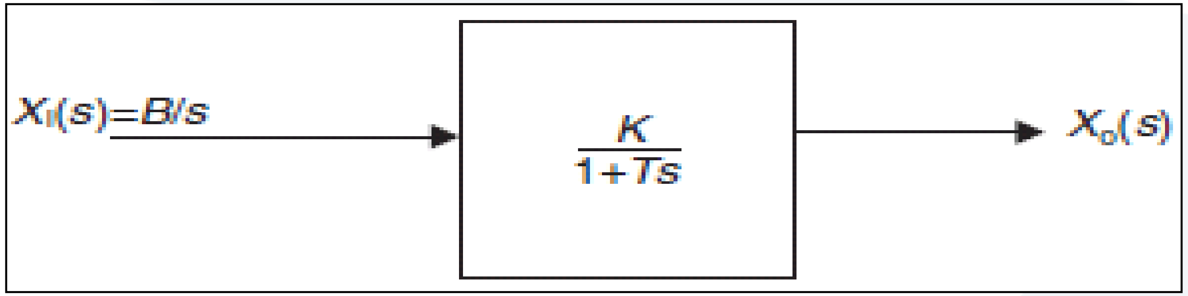
$$AR\frac{dh}{dt} + \rho g h = Rq_v$$

$$T = \frac{AR}{\rho g}$$

$$mc_pR\frac{dT}{dt} + T = T_b$$

$$T = mc_pR$$

Step Response of First-Order System



$$X_o(s) = \frac{BK}{s(1+Ts)} = BK\frac{\frac{1}{T}}{s(s+\frac{1}{T})}$$

Time function $f(t)$	Laplace transform $\mathcal{L}[f(t)] = F(s)$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$

$$x_o(t) = BK(1 - e^{-\frac{t}{T}})$$

B=1(unit step)

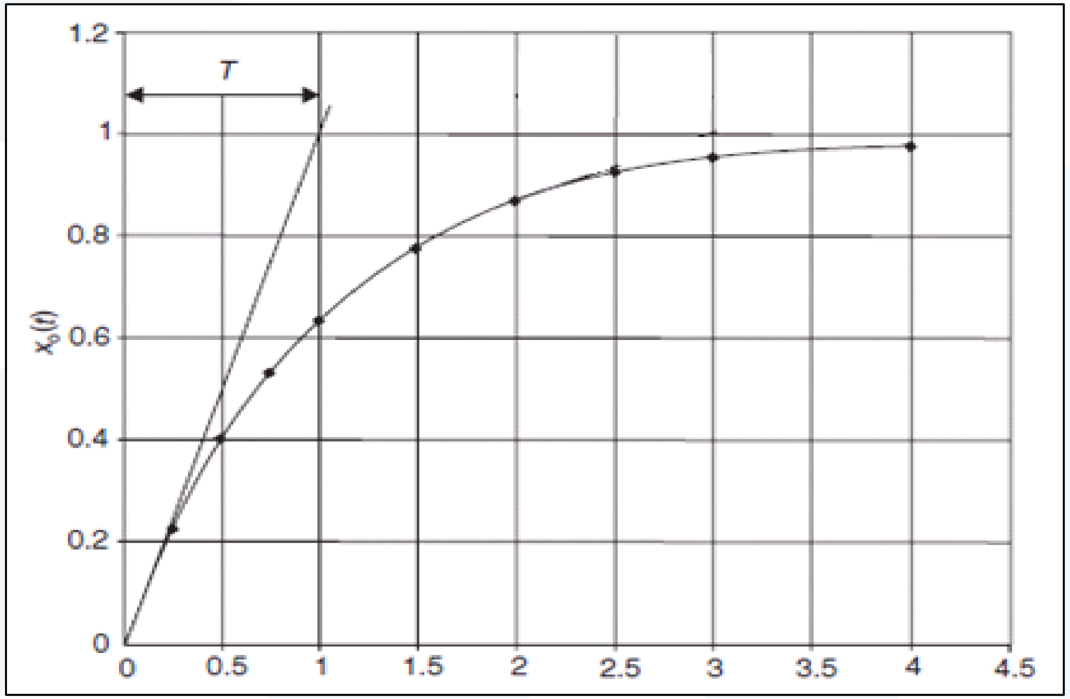
K=1(unity gain)

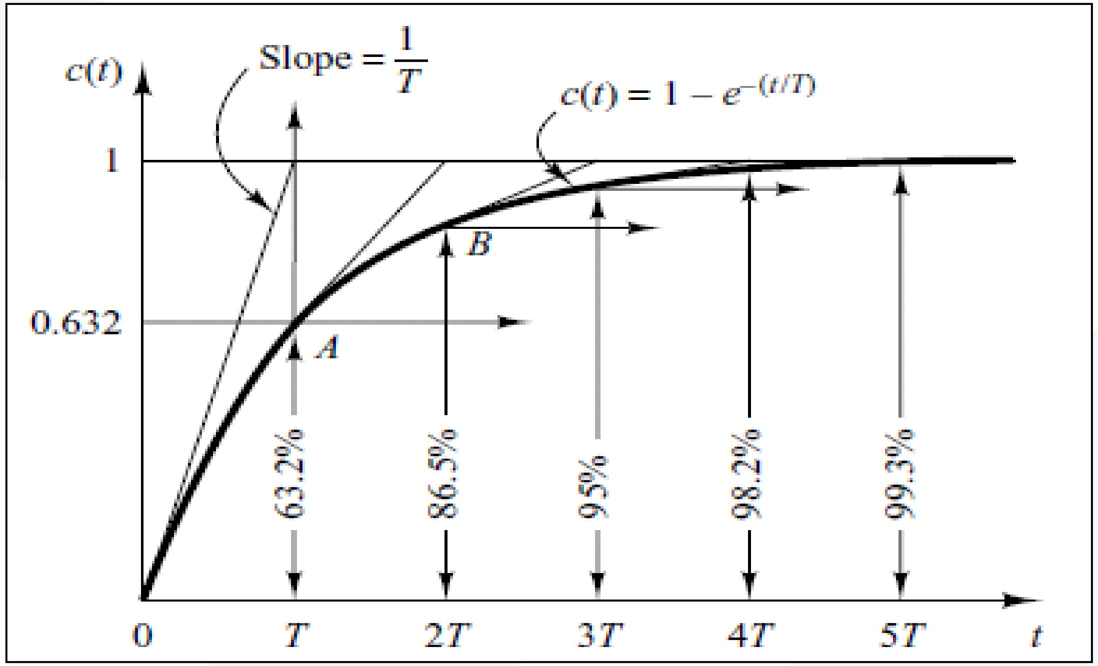
$$x_o(t) = 1 - e^{-\frac{t}{T}}$$

The system time constant is the intersection of the slope at t=0 with the final value line

$$\frac{dx_o}{dt} = 0 - \left(-\frac{1}{T}\right)e^{-\frac{t}{T}} = \frac{1}{T}e^{-\frac{t}{T}}$$

$$\left.\frac{dx_o}{dt}\right|_{t=0} = \frac{1}{T}$$





In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At $t = 3T, 4T$, and $5T$, the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for $t \geq 4T$, the response remains within 2% of the final value.

Example

من أجل السيارة المبينة في الشكل :

m=100

k=50

١- أوجد تابع التحويل بين قوة الدفع و السرعة مع حساب ثابت كسب الحالة المستقرة K و الثابت الزمني T و استفد من تابع التحويل لإيجاد الاستجابة الخطوية للسيارة

٢- صمم نموذج محاكاة باستخدام Simulink للتأكد من أن ثابت الكسب الحالة المستقرة يحقق العلاقة التالية :

$$K = \frac{y_2 - y_1}{u_2 - u_1}$$

٣- صمم نماذج محاكاة برمجية لتحليل تأثير تغير كل من قوة الدفع F و كتلة السيارة m و ثابت الإعاقة k على سرعة السيارة v

٤- صمم نموذج محاكاة باستخدام Simulink لسرعتي عربتين بكتلتين مختلفتين مع ثبات باقي المتغيرات و حلل النتائج من خلال محاكاة المسافات المقطوعة

حل الطلب الأول

$$m\frac{dv}{dt} + kv = F$$

$$a\frac{dx_o}{dt} + bx_o = cx_i(t)$$

a=m b=k c=1

$$G(s) = \frac{\frac{c}{b}}{1+\frac{a}{b}s} \qquad G(s) = \frac{K}{1+Ts} \qquad K = \frac{1}{k} = \frac{1}{50} = 0.02 \qquad T = \frac{m}{k} = \frac{100}{50} = 2$$

$$x_o(t) = BK(1-e^{-\frac{t}{T}}) \qquad x_i(t) = B = F$$

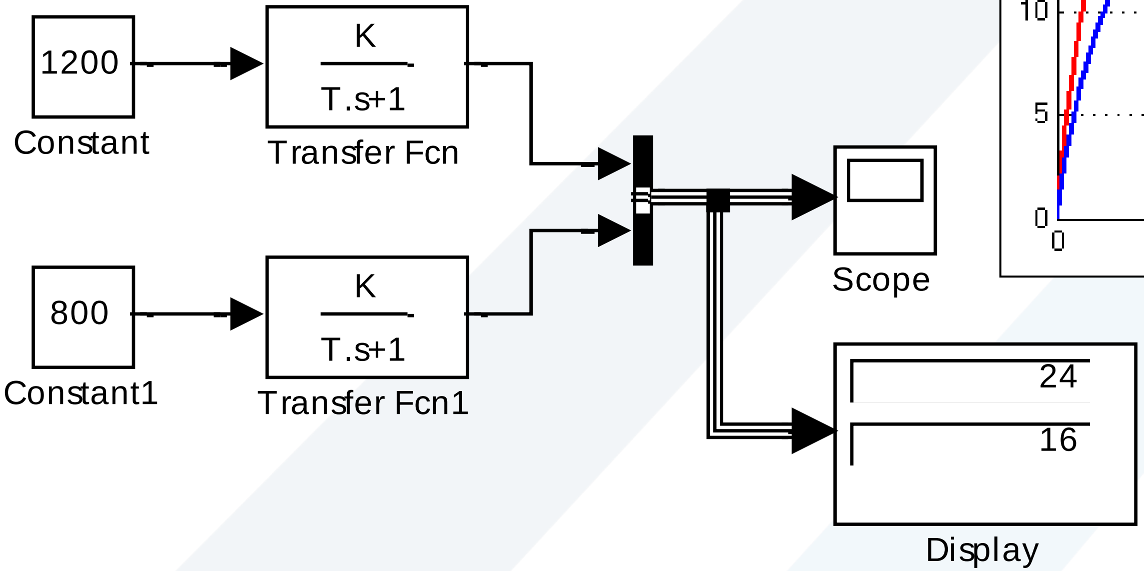
$$v = \frac{F}{k}(1-e^{-\frac{k}{m}t}) \qquad v = 0.02F(1-e^{-0.5t})$$

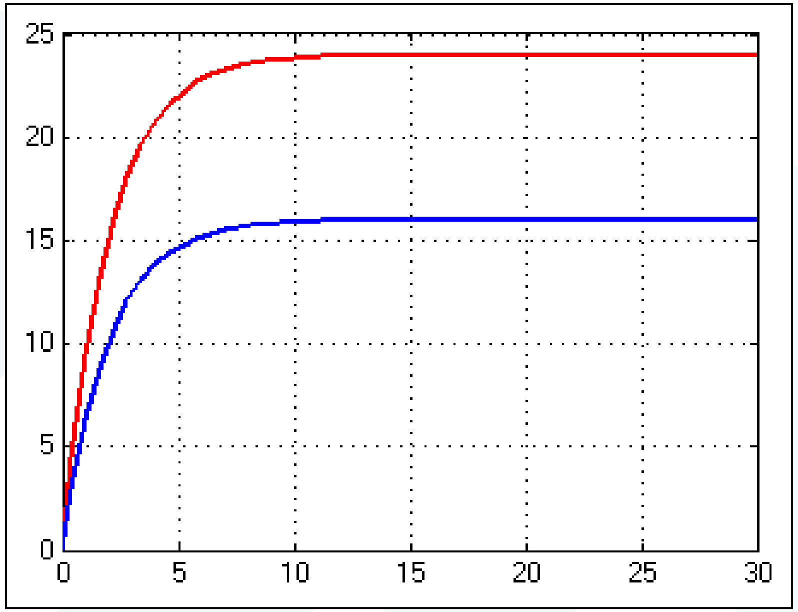
حل الطلب الثاني

m=100

k=50

$$K = \frac{y_2 - y_1}{u_2 - u_1}$$





$$K = \frac{24-16}{1200-800} = 0.02$$

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حل الطلب الثالث

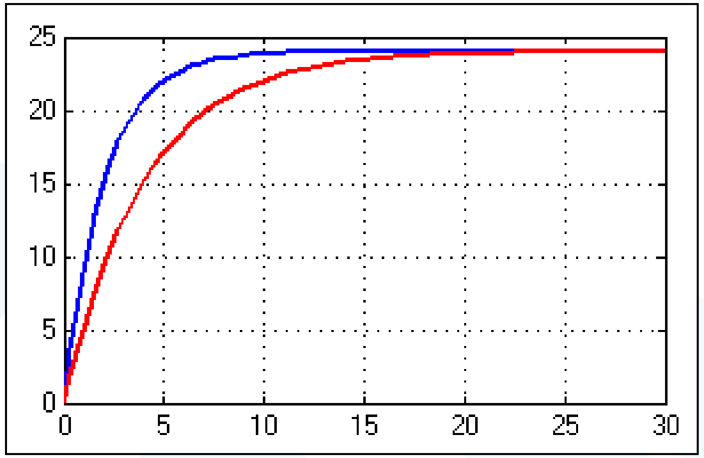
```
% v1=F1.K1(1-exp(-t/T1))
m1=100;
k1=50;
F1=1000;
K1=1/k1;
T1=m1/k1;
Num1 = [F1*K1];
Den1=[T1 1];
step (Num1, Den1)
grid
title('original case')
% v2=F2.K2(1-exp(-t/T2))
m2=100;
k2=50;
F2=1200;
K2=1/k2;
T2=m2/k2;
Num2 = [F2*K2];
Den2=[T2 1];
figure
step (Num2, Den2)
grid
title('modified case')
```

```
% v1=F1.K1(1-exp(-t/T1))
m1=100;
k1=50;
F1=1000;
K1=1/k1;
T1=m1/k1;
Num1 = [F1*K1];
Den1=[T1 1];
step (Num1, Den1)
grid
title('original case')
% v2=F2.K2(1-exp(-t/T2))
m2=200;
k2=50;
F2=1000;
K2=1/k2;
T2=m2/k2;
Num2 = [F2*K2];
Den2=[T2 1];
figure
step (Num2, Den2)
grid
title('modified case')
```

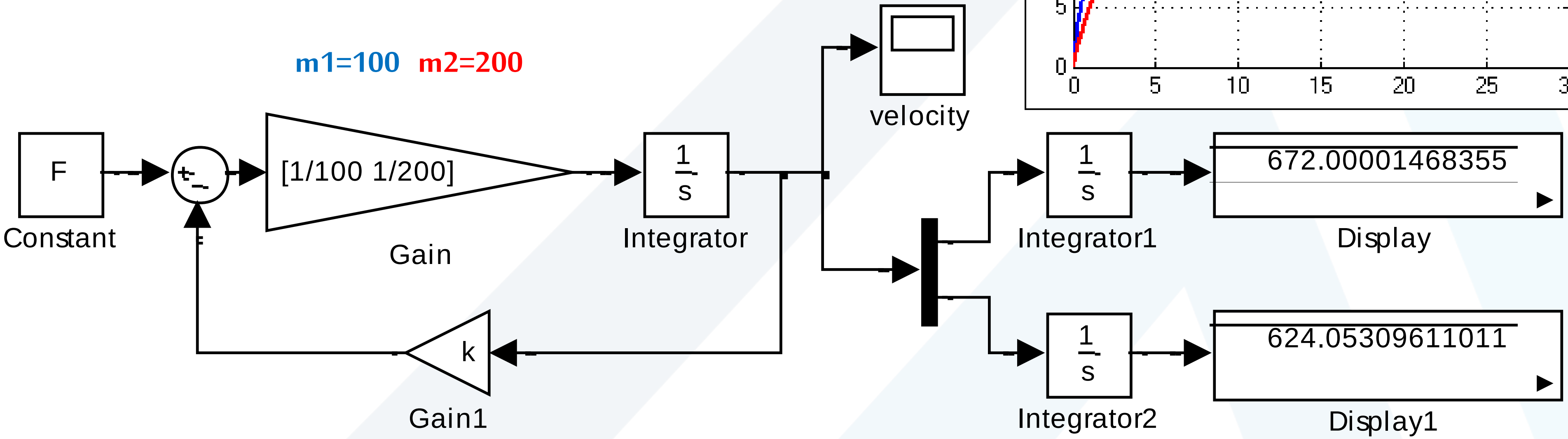
```
% v1=F1.K1(1-exp(-t/T1))
m1=100;
k1=50;
F1=1000;
K1=1/k1;
T1=m1/k1;
Num1 = [F1*K1];
Den1=[T1 1];
step (Num1, Den1)
grid
title('original case')
% v2=F2.K2(1-exp(-t/T2))
m2=100;
k2=60;
F2=1000;
K2=1/k2;
T2=m2/k2;
Num2 = [F2*K2];
Den2=[T2 1];
figure
step (Num2, Den2)
grid
title('modified case')
```

حل الطلب الرابع

تصل السيارتان إلى نفس السرعة لكن تصل السيارة ذات الكتلة الأقل (المنحني الأزرق) إلى السرعة النهائية بزمن أقل لأن الثابت الزمني في هذه الحالة سيكون أقل
وعلى اعتبار أن المسافة هي تكامل السرعة بالنسبة للزمن وحيث أن المساحة تحت المنحني الأزرق أكبر من المساحة تحت المنحني الأحمر هذا يعني أن السيارة ذات الكتلة الأقل ستقطع مسافة أكبر من السيارة ذات الكتلة الأكبر
وهو ما توضحه المحاكاة عند إجراء تكامل لسرعتي السيارتين



m1=100 m2=200



السيارة ذات الكتلة الأقل قطعت مسافة أكبر رغم أن السيارتين وصلتا إلى السرعة النهائية ذاتها

Time Domain Analysis of Second-Order System

$$a\frac{d^2x_o}{dt^2}+b\frac{dx_o}{dt}+cx_o=ex_i(t)$$

$$(as^2+bs+c)X_0(s)=eX_i(s)$$

$$G(s)=\frac{X_o}{X_i}(s)=\frac{e}{as^2+bs+c}$$

$$G(s)=\frac{\frac{e}{c}}{\frac{a}{c}s^2+\frac{b}{c}s+1}$$

$$G(s)=\frac{K}{\frac{1}{w_n^2}s^2+\frac{2\zeta}{w_n}s+1}$$

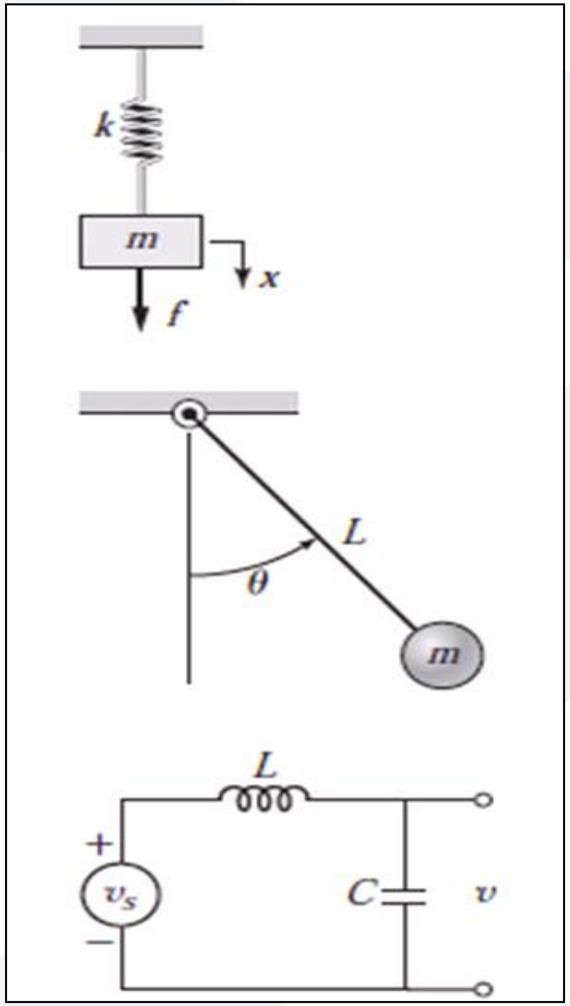
$$G(s)=\frac{Kw_n^2}{s^2+2\zeta w_n s+w_n^2}$$

K: steady-state gain constant

w_n: undamped natural frequency(rad/s)

ζ : damping ratio

جَامعة المَنارة



$$m\frac{d^2x}{dt^2} + kx = f(t)$$

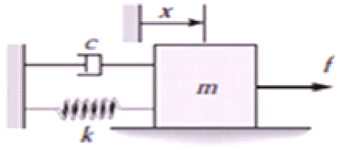
$$\omega_n = \sqrt{\frac{k}{m}}$$

$$L\frac{d^2\theta}{dt^2} + g\theta = 0$$

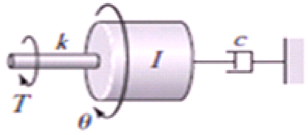
$$\omega_n = \sqrt{\frac{g}{L}}$$

$$LC\frac{d^2v}{dt^2} + v = v_s$$

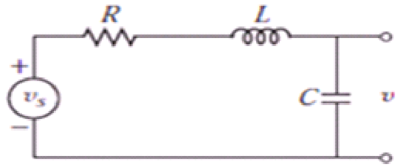
$$\omega_n = \frac{1}{\sqrt{LC}}$$



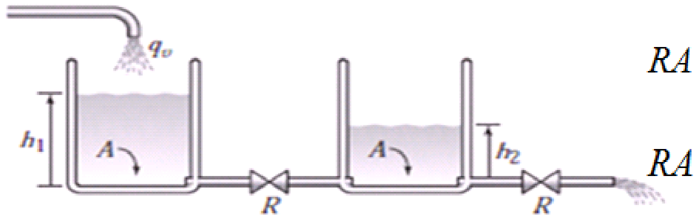
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$



$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = T$$

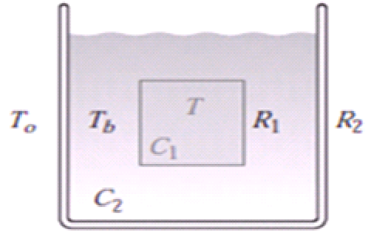


$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$



$$RA\frac{dh_1}{dt} + \rho g(h_1 - h_2) = Rq_v$$

$$RA\frac{dh_2}{dt} + \rho g(h_2 - h_1) + \rho g h_2 = 0$$



$$R_1C_1\frac{dT}{dt} + T = T_b$$

$$R_1R_2C_2\frac{dT_b}{dt} + (R_1 + R_2)T_b = R_2T + R_1T_o$$

Roots of the Characteristic Equation

The time response of any system has two components:

(a) **Transient response**

(b) **Steady-state response**

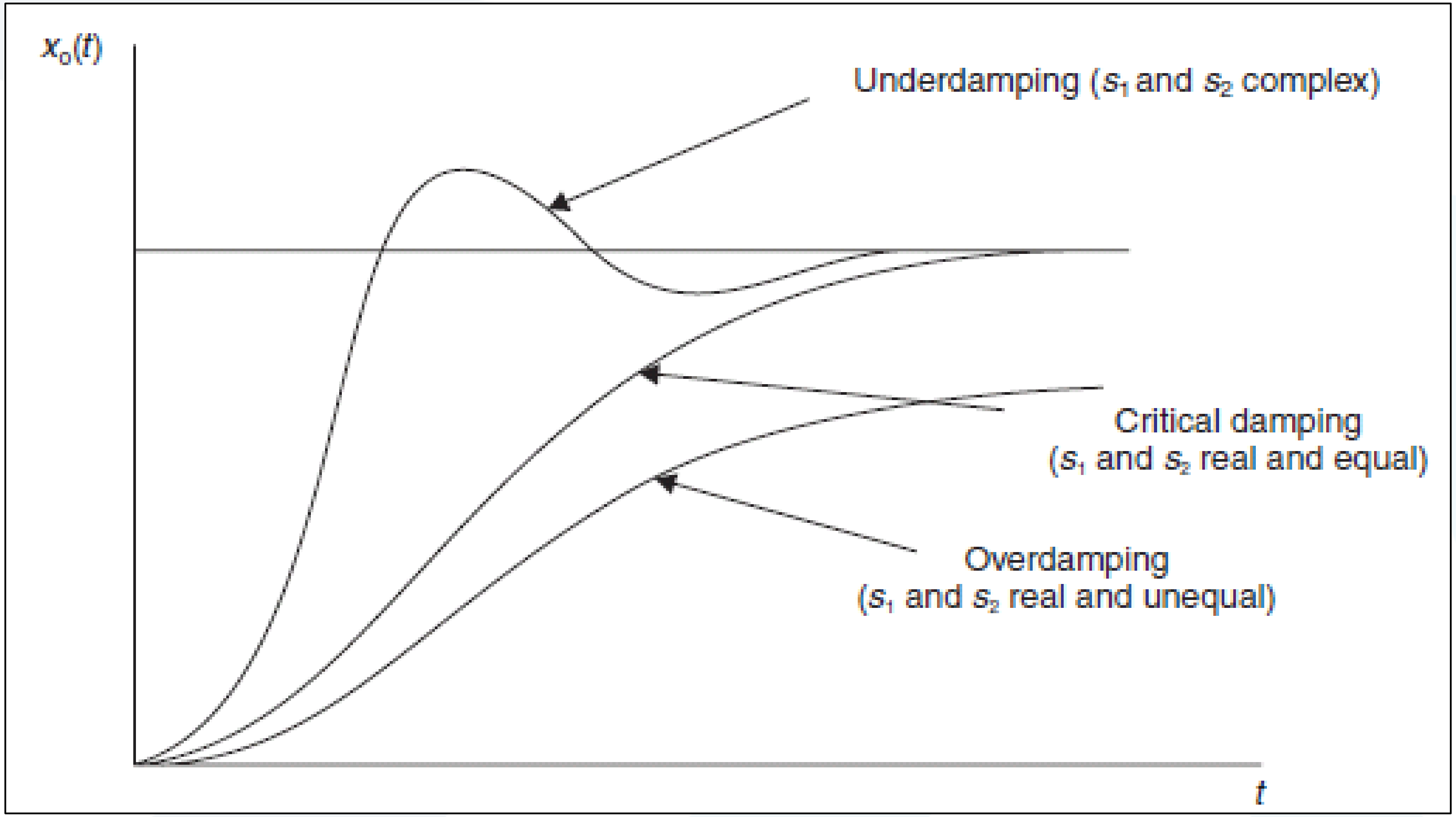
$$(as^2 + bs + c)X_0(s) = 0$$

$$(as^2 + bs + c) = 0$$

Characteristic Equation

$$s_1, s_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant	*Roots*	*Transient response type*
$b^2 > 4ac$	s_1 and s_2 real and unequal	Overdamped Transient Response
$b^2 = 4ac$	s_1 and s_2 real and equal	Critically Damped Transient Response
$b^2 < 4ac$	s_1 and s_2 complex conjugate of the form: $s_1, s_2 = -\sigma \pm j\omega$	Underdamped Transient Response


$x_o(t)$
Underdamping (s_1 and s_2 complex)
Critical damping
(s_1 and s_2 real and equal)
Overdamping
(s_1 and s_2 real and unequal)
t

Critical Damping and Damping Ratio

Critical damping

When the damping coefficient C of a second-order system has its critical value C_c, the system, when disturbed, will reach its steady-state value in the minimum time without overshoot. This is when the roots of the Characteristic Equation have equal negative real roots.

Damping ratio

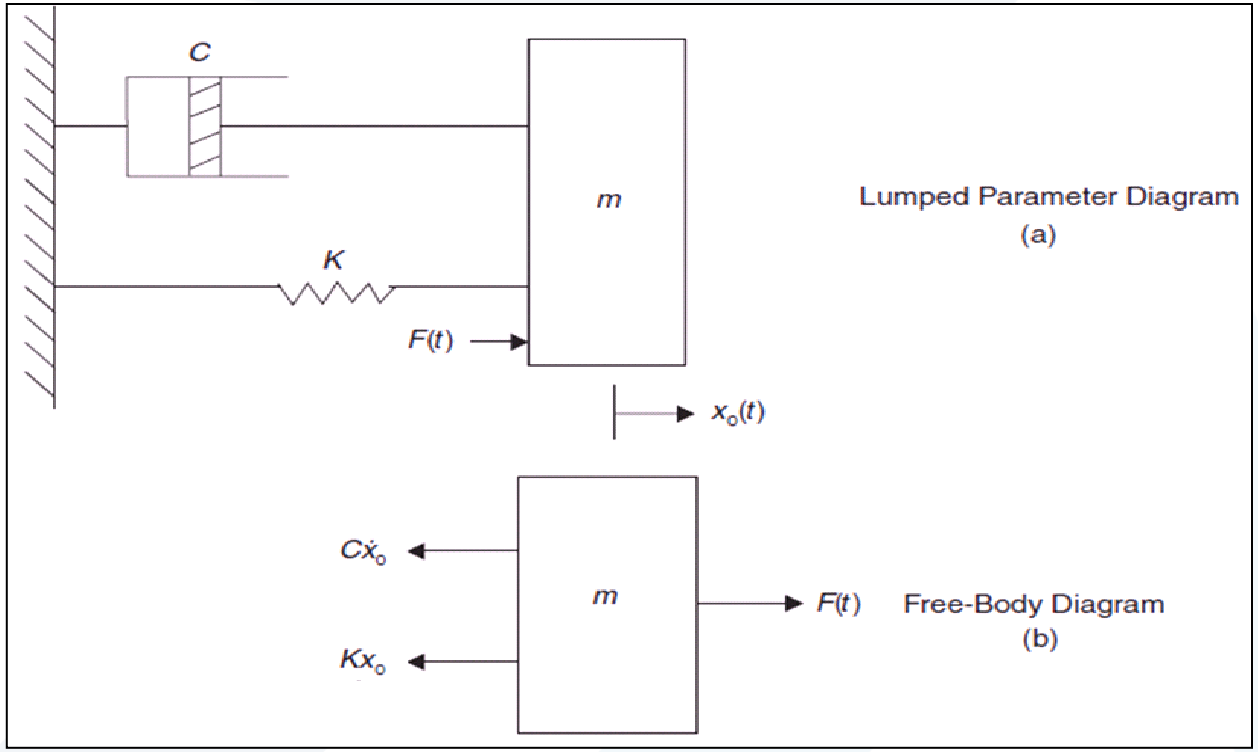
The ratio of the damping coefficient C in a second-order system compared with the value of the damping coefficient C_c required for critical damping is called the Damping Ratio ζ

$$\zeta = \frac{C}{C_c}$$

$\zeta = 0$	No damping
$\zeta < 1$	Underdamping
$\zeta = 1$	Critical damping
$\zeta > 1$	Overdamping

EXAMPLE

Find the value of the critical damping coefficient C_c in terms of K and m for the spring–mass–damper system shown in Figure



From Newton's second law

$$\sum Fx = m\ddot{x}_o$$

From the free-body diagram

$$F(t) - Kx_o(t) - C\dot{x}_o(t) = m\ddot{x}_o(t)$$

Taking Laplace transforms, zero initial conditions

$$F(s) - KX_o(s) - CsX_o(s) = ms^2X_o(s)$$

or

$$(ms^2 + Cs + K)X_o(s) = F(s)$$

Characteristic Equation is $ms^2 + Cs + K = 0$

$$\text{i.e.} \quad s^2 + \frac{C}{m} + \frac{K}{m} = 0$$

and the roots are

$$s_1, s_2 = \frac{1}{2}\left\{-\frac{C}{m} \pm \sqrt{\left(\frac{C}{m}\right)^2 - 4\frac{K}{m}}\right\}$$

For critical damping, the discriminant is zero, hence the roots become

$$s_1 = s_2 = -\frac{C_c}{2m}$$

Also, for critical damping

$$\frac{C_c^2}{m^2} = \frac{4K}{m}$$

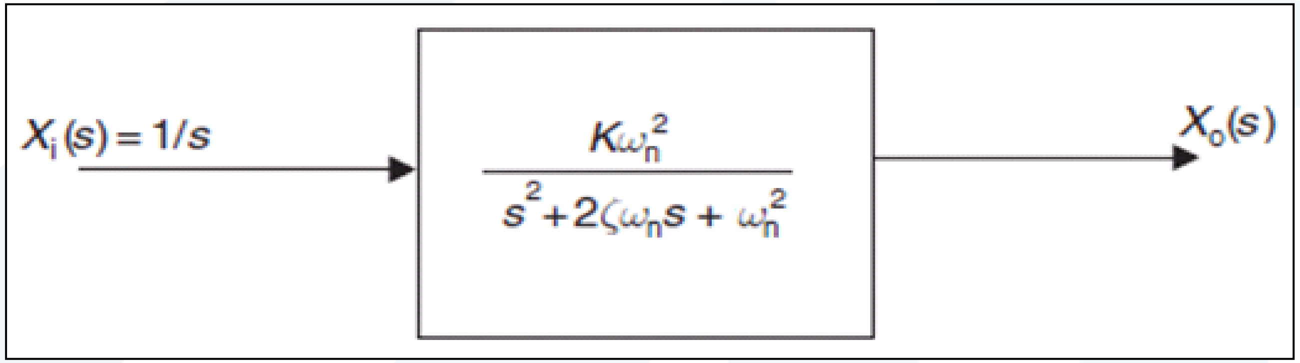
$$C_c^2 = \frac{4Km^2}{m}$$

giving

$$C_c = 2\sqrt{Km}$$

Generalized Second-Order System Response to a unit Step Input

Consider a second-order system whose steady-state gain is K, undamped natural frequency is ω_n and whose damping ratio is ζ, where $\zeta < 1$. For a unit step input, the block diagram is as shown in Figure



$$X_o(s) = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$X_o(s) = \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

multiply by $s(s^2 + 2\zeta\omega_n s + \omega_n^2)$

$$K\omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs^2 + Cs$$

Equating coefficients

$$(s^2): \quad 0 = A + B$$

$$(s^1): \quad 0 = 2\zeta\omega_n A + C$$

$$(s^0): \quad K\omega_n^2 = \omega_n^2 A$$

$$A = K, \quad B = -K \quad \text{and} \quad C = -2\zeta\omega_n K$$

$$X_o(s) = K\left[\frac{1}{s} - \left\{\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\}\right]$$

Completing the square

$$X_o(s) = K\left[\frac{1}{s} - \left\{\frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2 - \zeta^2\omega_n^2}\right\}\right] = K\left[\frac{1}{s} - \left\{\frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1 - \zeta^2}\right)^2}\right\}\right]$$

The terms in the brackets { } can be written in the standard forms

$$\text{Term (1)} = \frac{-s}{(s+\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

$$\text{Term (2)} = -\left\{\frac{2\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\right\}\left\{\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}\right\}$$

$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse transform

$$x_o(t) = K\left[1 - e^{-\zeta\omega_n t}\left\{\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t + \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\right]$$

When $\zeta = 0$

$$x_o(t) = K[1 - e^0\{\cos\omega_n t + 0\}]$$
$$= K[1 - \cos\omega_n t]$$

From equation it can be seen that when there is no damping, a step input will cause the system to oscillate continuously at ω_n (rad/s).

Damped natural frequency ω_d

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

where ω_d is called the damped natural frequency.

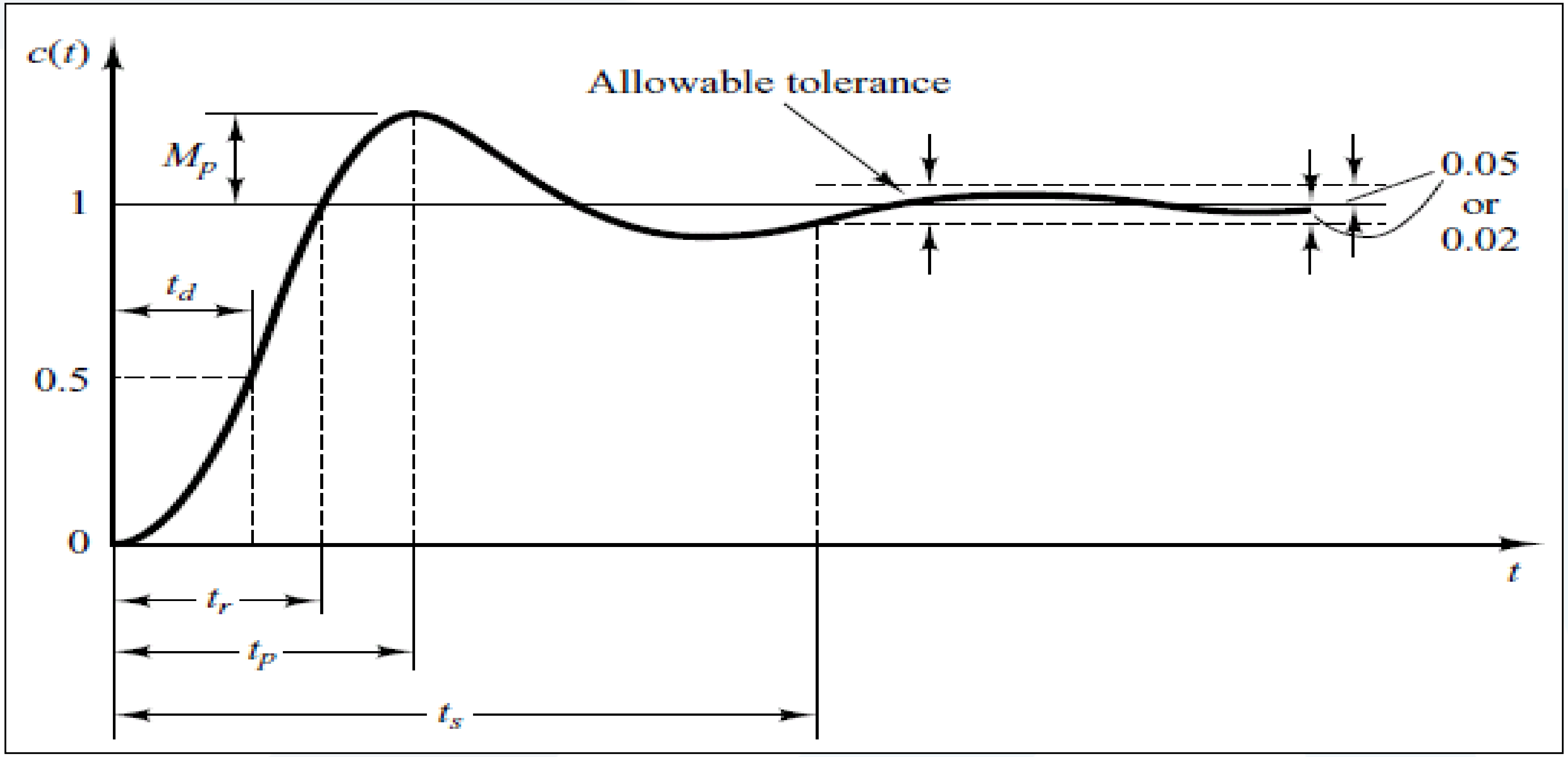
$$x_o(t) = K\left[1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\sin\omega_d t\right\}\right]$$
$$= K\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right] \qquad \tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Definitions of Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

These specifications are defined in what follows and are shown graphically in Figure


c(t)
Allowable tolerance
M_p
1
0.05
or
0.02
t_d
0.5
0
t
t_r
t_p
t_s

1. Delay time, t_d: The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time, t_r: The rise time is the time required for the response to rise from 10% to 90%, or 0% to 100% of its final value.
3. Peak time, t_p: The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percent) overshoot, M_p: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

5. Settling time, t_s: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.

Second-Order System and Transient-Response Specifications

Peak time t_p: Referring to Equation , we may obtain the peak time by differentiating $c(t)$ with respect to time and letting this derivative equal zero. Since

$$\frac{dc}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(\omega_d \sin\omega_d t - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos\omega_d t\right)$$

and the cosine terms in this last equation cancel each other, dc/dt, evaluated at $t = t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

This last equation yields the following equation:

$$\sin \omega_d t_p = 0$$

or

$$\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

Since the peak time corresponds to the first peak overshoot, $\omega_d t_p = \pi$. Hence

$$t_p = \frac{\pi}{\omega_d}$$

Maximum overshoot M_p: The maximum overshoot occurs at the peak time or at $t = t_p = \pi/\omega_d$. Assuming that the final value of the output is unity, M_p is obtained from Equation as

$$M_p = c(t_p) - 1$$

$$= -e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$\sigma = \zeta\omega_n$$

$$= e^{-(\sigma/\omega_d)\pi} = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

where σ is called the *attenuation*

The maximum percent overshoot is $e^{-(\sigma/\omega_d)\pi} \times 100\%$.

If the final value $c(\infty)$ of the output is not unity, then we need to use the following equation:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

Settling time t_s: For convenience in comparing the responses of systems, we commonly define the settling time t_s to be

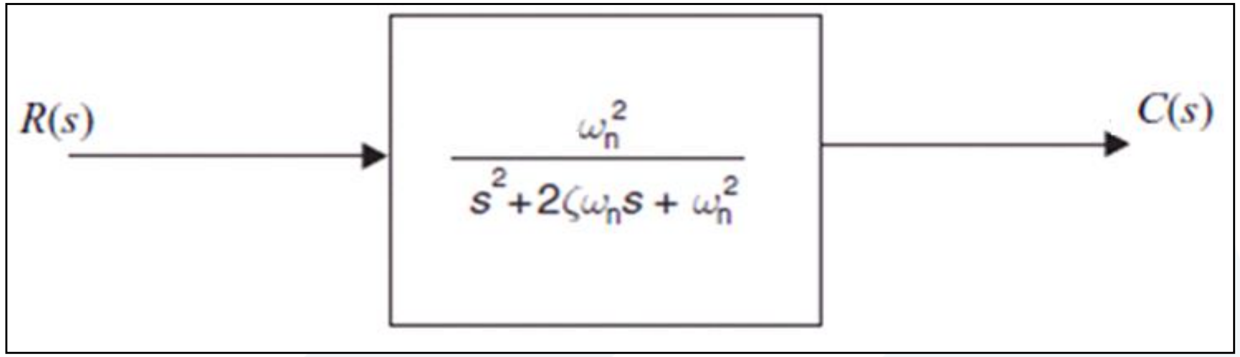
$$t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \qquad (2\% \text{ criterion})$$

or

$$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \qquad (5\% \text{ criterion})$$

EXAMPLE

Consider the system shown in Figure , where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.



From the given values of ζ and ω_n, we obtain $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

Peak time t_p: The peak time is $t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785$ sec

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling time t_s:

For the 2% criterion, the settling time is $t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33$ sec

For the 5% criterion, $t_s = \frac{3}{\sigma} = \frac{3}{3} = 1$ sec

Example

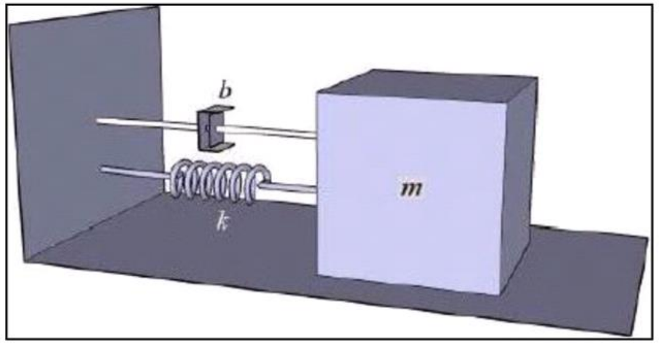
في منظومة كتلة نابض مخمد معطى ما يلي:

m=1;

b=0.1;

k=0.1;

F=2;



١- ارسم منحني الإزاحة باستخدام تعليمة step

٢- أوجد التردد الطبيعي غير المتخامد-نسبة التخامد –ثابت كسب الحالة المستقرة

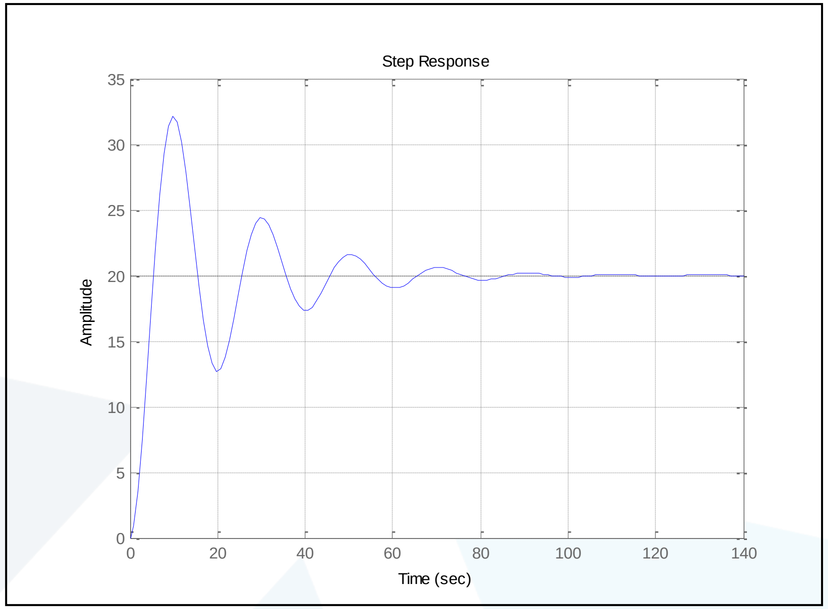
٣- اكتب العلاقة الرياضية للإزاحة بالاستفادة من الطلب ٢ وارسم العلاقة وقارنها مع المنحني في الطلب ١

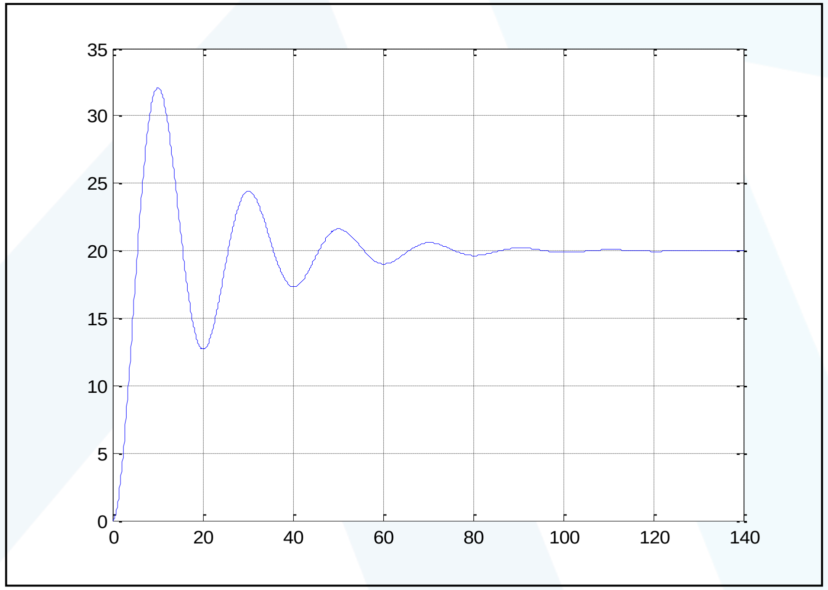
```
clc
clear
m=1;
b=0.1;
k=0.1;
F=2;
Num=[F];
Den=[1 0.1 0.1];
step(Num,Den)
grid
% K.wn^2=Num;
% 2.zeta.wn=0.1;
% wn^2=0.1
wn=sqrt(0.1)
zeta=0.1/(2*wn)
K=1/wn^2
figure
t=0:0.1:140;
c=K*F*(1-exp(-zeta*wn*t).*(cos(wn*sqrt(1-
zeta^2)*t)+(zeta/(sqrt(1-
zeta^2))).*sin(wn*sqrt(1-zeta^2)*t)));
plot(t,c)
grid
```

```
wn =
   0.3162

zeta =
   0.1581

K =
   10
```





Example

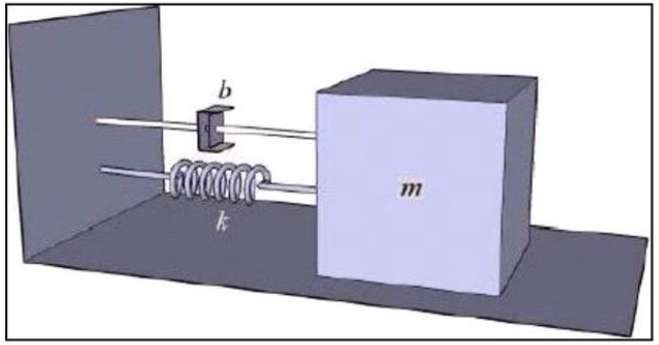
في منظومة كتلة نابض مخمد معطى ما يلي:

wn=5;

ζ=0.6;

K=1;

F=1;



١- اكتب العلاقة الرياضية للإزاحة بالاستفادة من البارامترات المعطاة وارسمها

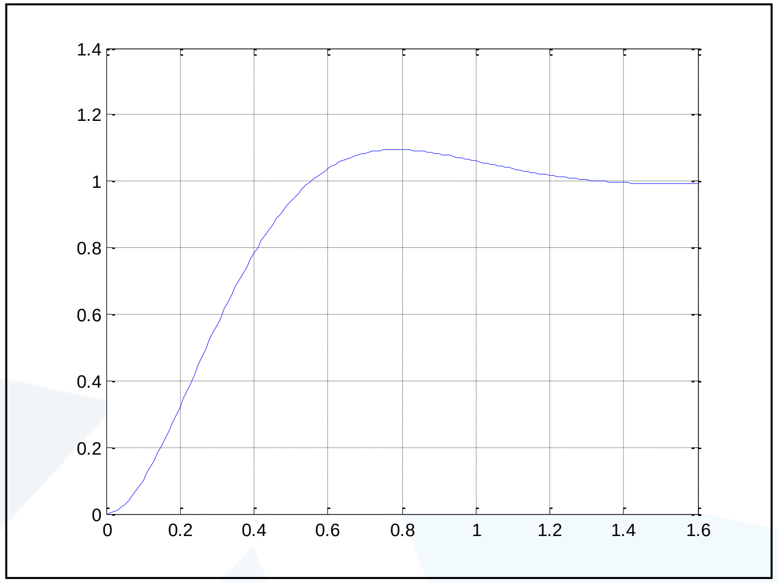
٢- احسب m k b وارسم منحني الإزاحة باستخدام تعليمة step

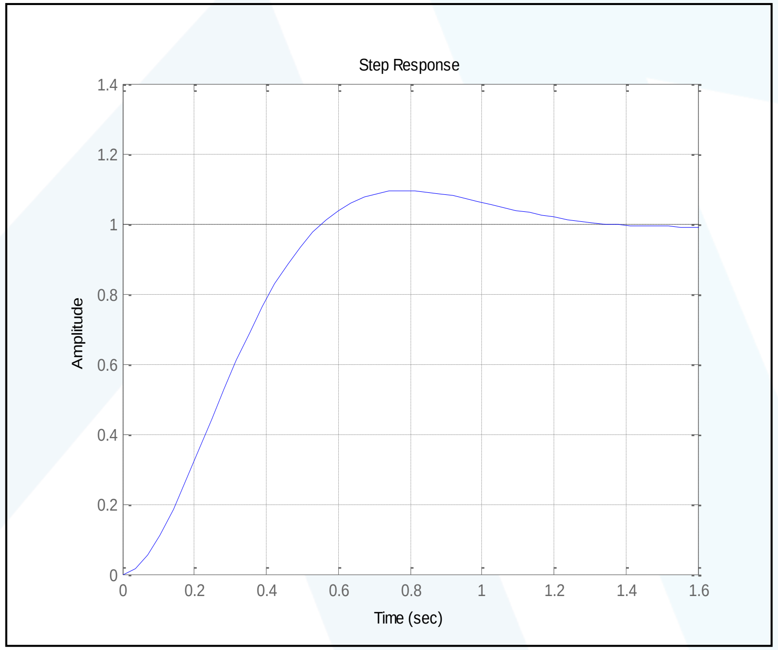
```
clc
clear
wn=5;
zeta=0.6;
K=1;
F=1;
t=0:0.01:1.6;
c=K*F*(1-exp(-zeta*wn*t).*(cos(wn*sqrt(1-
zeta^2)*t)+(zeta/(sqrt(1-zeta^2))).*sin(wn*sqrt(1-
zeta^2)*t)));
plot(t,c)
grid
%K.wn^2/(s^2+2.zeta.wn.s+wn^2)
%1/(m.s^2+b.s+k) or (1/m)/(s^2+(b/m)s+(k/m))
% (1/m)=K*wn^2
m=1/(K*wn^2)
% k/m=wn^2
k=m*wn^2
% (b/m)=2.zeta.wn
b=2*m*zeta*wn
num=[1];
den=[m b k];
figure
step(num,den)
grid
```

```
m =
    0.0400

k =
     1

b =
    0.2400
```





انتهت المحاضرة